1 Linear Equations and Matrices

1.1 Vectors and Matrices

- Matrix.
  $\Longrightarrow$ An m by n matrix is a rectangular array of mn real(or complex) numbers arranged in m horizontal rows and n vertical columns.

- Vectors.
  $\Longrightarrow$ A 1 by n or an n by 1 matrix is also called an n-vector and will be denoted by lowercase boldface letters.

- ith component of Ax (strang, p846)

$$= a_{ij}x_1 + a_{ij}x_2 + \ldots + a_{in}x_n$$

$$= \sum_{j=1}^{3} a_{1j}C_{1j}$$

- $(AB)_{ij}$ = (row i of A) $\cdot$ (column j of B)

$$= \sum_{k=1}^{n} a_{ik}b_{kj}$$

  often $AB \neq BA$

$$A(l \times m)\ B(m \times n) = M(l \times n)$$

- $A(BC) = (AB)C$

- $C(A + B) = CA + CB$

- $(A + B)C = AC + BC$

- Identity matrix $Ib = b$

- Elimination matrix E_{ij}

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -l & 0 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} b_3 \\ b_2 \\ b_1 \end{bmatrix}$$

identity marix with rows i j reversed

$\Longrightarrow$ exchanges row i and j when multiplied.

- Augmented matrix

$$[A|b] \quad : \text{elimination acts on whole rows of this matrix.}$$

Q. How can we use the above ideas to solve linear equations ?

Ex.

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 1 \\ 4 & 8 & 9 & 3 \\ 0 & 3 & 2 & 1 \end{array}\right]$$

$\Longrightarrow$ multiply E_{21}

$$\begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 & 1 \\ 4 & 8 & 9 & 3 \\ 0 & 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 3 & 2 & 1 \end{bmatrix}$$

$\Longrightarrow$ multiply P_{32}

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 3 & 2 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$\Longrightarrow$ back substitution

- dot product(inner product) of

$$\vec{v} = \left[\begin{array}{c} v_1 \\ \hline v_2 \end{array}\right], \quad \vec{w} = \left[\begin{array}{c} w_1 \\ \hline w_2 \end{array}\right]$$

$$\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 = \|v\| \|w\| cos\gamma$$

- length (norm) of $\vec{v}$

$$\mathbf{v} = \|v\| = \sqrt{\vec{v} \cdot \mathbb{V}}$$

- unit vector $\vec{u}$

$$\vec{u} \quad : \quad \vec{u} \cdot \vec{u} = 1$$

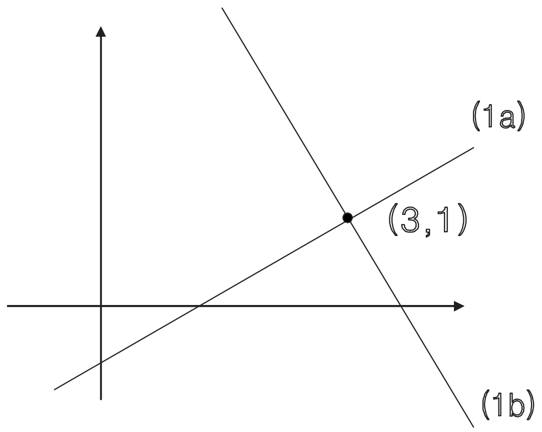
$\frac{\vec{v}}{\|v\|}$ is a unit vector

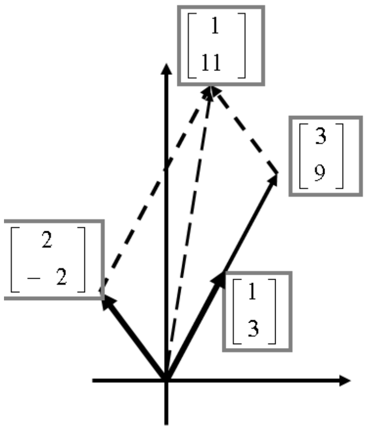
- Schwarz inequality $\quad \|\vec{v} \cdot \vec{w}\| \leq \|\vec{v}\|\|\vec{w}\|$

- Triangle inequality $\quad \|\vec{v} + \vec{w}\| \leq \|\vec{v}\| + \|w\|$

1.2 Linear Equations

$$\begin{pmatrix} x - 2y = 1 & \quad (1a) \\ 3x + 2y = 11 & \quad (1b) \end{pmatrix} \qquad x \begin{bmatrix} 1 \\ 3 \end{bmatrix} + y \begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix}$$





column picture

- Elimination. $\quad$ (strang, p35)

$$\left(\begin{array}{ll} x - 2y = 1 & \quad (1a) \\ 3x + 2y = 11 & \quad (1b) \end{array} \right.$$

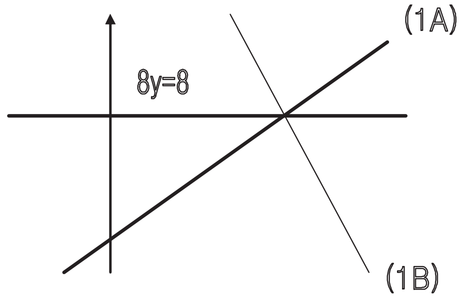
$$\Downarrow \qquad\qquad (1b) - 3 * (1a)$$

$$\left(\begin{array}{ll} x - 2y = 1 & (1a) \\ 8y = 8 & (1b) \end{array} \right. \qquad \begin{bmatrix} 1 & -2 \\ 0 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 8 \end{bmatrix}$$

$\Longrightarrow$ upper triangular system

$\Big($ Last equation : y = 1
substitute into the upper equation : x - 2 = 1

$\Longrightarrow$ back substitution



$$\begin{cases} \text{pivot : first nonzero in the row that does the elimination} \\ \text{Multiplier} = \frac{entry \quad to \quad eliminate}{pivot} \end{cases}$$

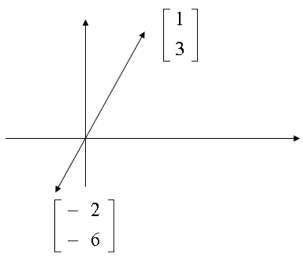
('1' the coefficient of x is the first pivot in the example above)

The pivots are on the diagonal of the triangle after the elimination.

$$\begin{cases} x - 2y = 1 \\ 3x - 6y = 11 \end{cases}$$

$$\Downarrow$$

$$\begin{cases} x - 2y = 1 \\ oy = 8 \quad \text{no solution} \end{cases}$$

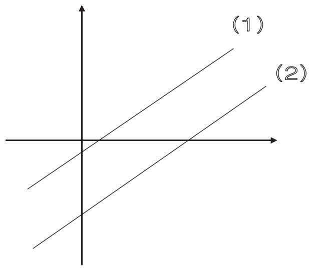
$\Longrightarrow$ 

$$\left(\begin{array}{c} x - 2y = 1 \\ 3x - 6y = 3 \end{array} \right.$$

$$\Downarrow$$

$$\left(\begin{array}{rl} & x - 2y = 1 \\ oy = 0 & \text{infinitely many solutions} \end{array} \right.$$

$\Longrightarrow$



$$\left(\begin{array}{c} 0x + 2y = 4 \\ 3x - 2y = 5 \end{array} \right.$$

$$\Downarrow \text{row exchange}$$

$$\left(\begin{array}{c} 3x - 2y = 5 \\ 2y = 4 \end{array} \right.$$

$\Longrightarrow$