Deformation of Concrete

Fall 2010

Dept of Architecture
Seoul National University

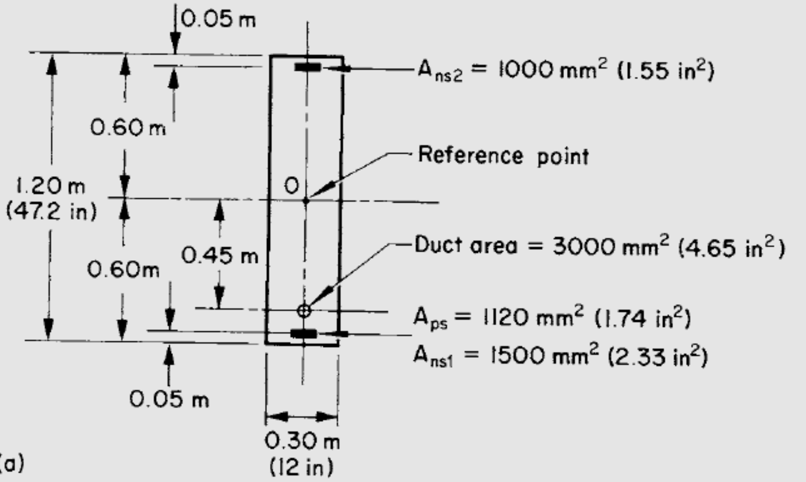
6^{th} week

Examples(Chapter 2)

Example 2.2

Post tension

A prestress force $P = 1400 \times 10^3 N$ (315 kip) and a bending moment $M = 390 \times 10^3$ N-m (3450 kip-in) are applied at age t_0 on the rectangular post-tensioned concrete section shown in Fig. 2.6(a). Calculate the stresses, the axial strain and curvature at age t_0 and t given the following data: $E_c(t_0) = 30.0$ GPa (4350 ksi); $E_{ns} = E_{ps} = 200$ GPa (29×10^3 ksi); uniform free shrinkage value $\varepsilon_{cs}(t, t_0) = -240 \times 10^{-6}$; $\varphi(t, t_0) = 3$; $\chi = 0.8$; reduced relaxation, $\Delta\bar{\sigma}_{pr} = -80$ MPa (-12 ksi). The dimensions of the section and cross-section areas of the reinforcement and the prestress duct are indicated in Fig. 2.6(a).



(a)

(a) Stress and strain at age t_0

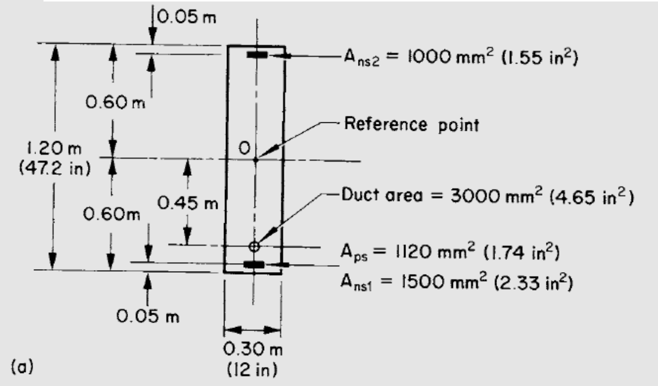
Calculation of the properties of the transformed section at time t_0 is done in Table 2.1. The reference modulus of elasticity, $E_{ref} = E_c(t_0) = 30.0$ GPa. The forces introduced at age t_0 are equivalent to Equation (2.31) is

$$\begin{Bmatrix} N \\ M \end{Bmatrix}_{\text{equivalent}} = \begin{Bmatrix} -1400 \times 10^3 \\ 390 \times 10^3 - 1400 \times 10^3 \times 0.45 \end{Bmatrix} = \begin{Bmatrix} -1400 \times 10^3 \text{ N} \\ -240 \times 10^3 \text{ N-m} \end{Bmatrix}$$

The instantaneous axial strain at O and curvature (Equation (2.32)) is

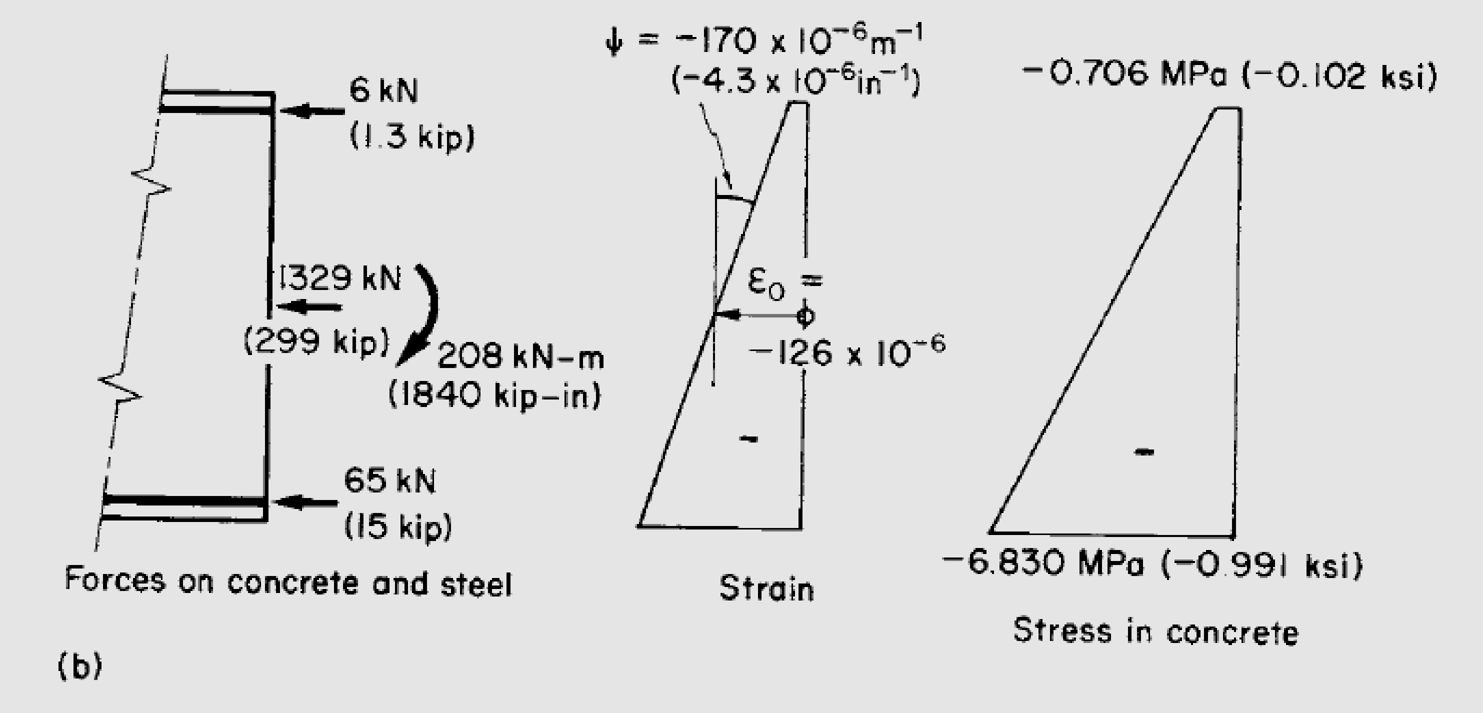
$$\begin{Bmatrix} \varepsilon_O(t_0) \\ \psi(t_0) \end{Bmatrix} = \frac{1}{30 \times 10^9[0.3712 \times 46.88 \times 10^{-3} - (0.208 \times 10^{-3})^2]} \times \begin{bmatrix} 46.88 \times 10^{-3} & -0.208 \times 10^{-3} \\ -0.208 \times 10^{-3} & 0.3712 \end{bmatrix} \begin{Bmatrix} -1400 \times 10^3 \\ -240 \times 10^3 \end{Bmatrix}$$

$$= 10^{-6} \begin{Bmatrix} -126 \\ -170\,\text{m}^{-1} \end{Bmatrix}$$



$$\begin{Bmatrix} \varepsilon_o \\ \psi \end{Bmatrix} = \frac{1}{E_{ref}} \begin{bmatrix} A & B \\ B & I \end{bmatrix}^{-1} \begin{Bmatrix} N \\ M \end{Bmatrix} = \frac{1}{E_{ref}\left(AI - B^2\right)} \begin{bmatrix} I & -B \\ -B & A \end{bmatrix} \begin{Bmatrix} N \\ M \end{Bmatrix}$$

Total effect of section



Section properties for instantaneous deformation

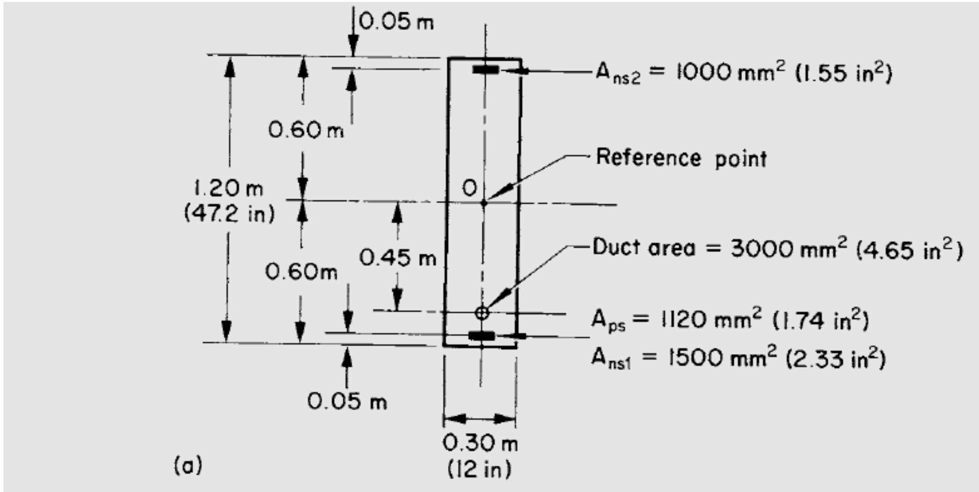


Table 2.1 Calculation of A, B and I of transformed section at time t_0

	Properties of area			Properties of transformed area		
	A (m^2)	B (m^3)	I (m^4)	AE/E_{ref} (m^2)	BE/E_{ref} (m^3)	IE/E_{ref} (m^4)
Concrete	0.3545	-1.625×10^{-3}	41.84×10^{-3}	0.3545	-1.625×10^{-3}	41.84×10^{-3}
Non-prestressed steel	2500×10^{-6}	0.275×10^{-3}	0.756×10^{-3}	0.0167	1.833×10^{-3}	5.04×10^{-3}
Prestressed steel	—	—	—	—	—	—
Properties of transformed section				0.3712 A	0.208×10^{-3} B	46.88×10^{-3} I

(b) Changes in stress and strain due to creep, shrinkage and relaxation

The age-adjusted elasticity modulus of concrete (Equation (1.31)) is

$$\bar{E}_c(t, t_0) = \frac{30 \times 10^9}{1 + 0.8 \times 3} = 8.82\,\text{GPa}.$$

Note: Ac not $\bar{A}_c$

The stress in concrete at the top and bottom fibres when the strain due to creep and shrinkage is artificially restrained (Equations (2.34) and (2.45)) is:

$$(\sigma_{c\ \text{restrained}})_{\text{top}} = -8.82 \times 10^9[3 \times 10^{-6}(-126 + 170 \times 0.6) - 240 \times 10^{-6}]$$
$$= 2.741\,\text{MPa (0.398 ksi)}$$

$$(\sigma_{c\ \text{restrained}})_{\text{bot}} = -8.82 \times 10^9[3 \times 10^{-6}(-126 - 170 \times 0.6) - 240 \times 10^{-6}]$$
$$= 8.145\,\text{MPa (1.181 ksi).}$$

Concrete section only

$$\begin{Bmatrix} \Delta N \\ \Delta M \end{Bmatrix}_{\text{creep}} = -8.82 \times 10^9 \times 3 \begin{bmatrix} 0.3545 & -1.625 \times 10^{-3} \\ -1.625 \times 10^{-3} & 41.84 \times 10^{-3} \end{bmatrix} \times \begin{Bmatrix} -126 \\ -170 \end{Bmatrix} 10^{-6} = 10^6 \begin{Bmatrix} 1.175\,\text{N} \\ 0.1828\,\text{N-m} \end{Bmatrix}$$

$$\begin{Bmatrix} \Delta N \\ \Delta M \end{Bmatrix}_{creep} = -\sum_{i=1}^{m} \left\{ \bar{E}_c \begin{bmatrix} A_c & B_c \\ B_c & I_c \end{bmatrix} \varphi \begin{Bmatrix} \varepsilon_o(t_o) \\ \psi_o(t_o) \end{Bmatrix} \right\}_i$$

$$\begin{Bmatrix} \Delta N \\ \Delta M \end{Bmatrix}_{\text{shrinkage}} = -8.82 \times 10^9(-240 \times 10^{-6}) \begin{Bmatrix} 0.3545 \\ -1.625 \times 10^{-3} \end{Bmatrix} = 10^6 \begin{Bmatrix} 0.750\,\text{N} \\ -0.0034\,\text{N-m} \end{Bmatrix}$$

$$\begin{Bmatrix} \Delta N \\ \Delta M \end{Bmatrix}_{shrinkage} = -\sum_{i=1}^{m} \left\{ \bar{E}_c \begin{bmatrix} A_c & B_c \\ B_c & I_c \end{bmatrix} \begin{Bmatrix} \varepsilon_{cs} \\ 0 \end{Bmatrix} \right\}_i$$

$$\begin{Bmatrix} \Delta N \\ \Delta M \end{Bmatrix}_{\text{relaxation}} = \begin{Bmatrix} 1120 \times 10^{-6}(-80 \times 10^6) \\ 1120 \times 10^{-6} \times 0.45\,(-80 \times 10^6) \end{Bmatrix} = 10^6 \begin{Bmatrix} -0.090\,\text{N} \\ -0.0403\,\text{N-m} \end{Bmatrix}$$

$$\begin{Bmatrix} \Delta N \\ \Delta M \end{Bmatrix}_{relaxation} = \sum \begin{Bmatrix} A_{ps}\Delta\bar{\sigma}_{pr} \\ A_{ps}y_{ps}\Delta\bar{\sigma}_{pr} \end{Bmatrix}_i$$

Total restrained forces

$$\begin{Bmatrix} \Delta N \\ \Delta M \end{Bmatrix} = 10^6 \begin{Bmatrix} 1.175 + 0.750 - 0.090 \\ 0.1828 - 0.0034 - 0.0403 \end{Bmatrix} = 10^6 \begin{Bmatrix} 1.835 \text{ N} \\ 0.139 \text{ N-m} \end{Bmatrix}.$$

Change in strain and curvature

$$\begin{Bmatrix} \Delta \varepsilon_o \\ \Delta \psi \end{Bmatrix} = \frac{1}{\bar{E}_c \left(\bar{A}\bar{I} - \bar{B}^2 \right)} \begin{bmatrix} \bar{I} & -\bar{B} \\ -\bar{B} & \bar{A} \end{bmatrix} \begin{Bmatrix} -\Delta N \\ -\Delta M \end{Bmatrix}$$

Total effect again

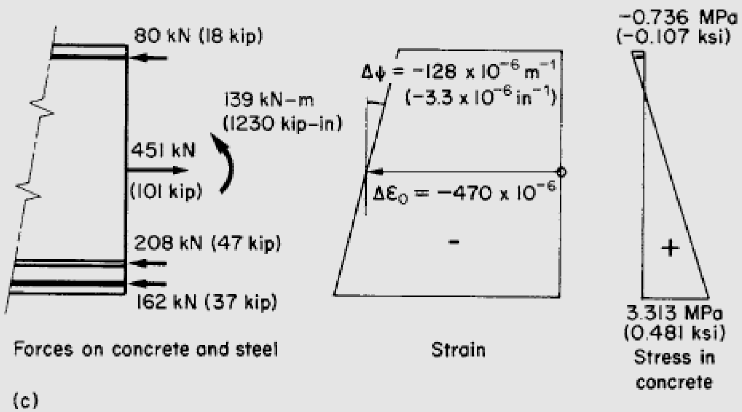
Table 2.2 Calculation of $\bar{A}$, $\bar{B}$ and $\bar{I}$ of the age-adjusted transformed section

	Properties of area			Properties of transformed area		
	A (m^2)	B (m^3)	I (m^4)	AE/E_{ref} (m^2)	BE/E_{ref} (m^3)	IE/E_{ref} (m^4)
Concrete	0.3545	-1.625×10^{-3}	41.84×10^{-3}	0.3545	-1.625×10^{-3}	41.84×10^{-3}
Non-prestressed steel	2500×10^{-6}	0.275×10^{-3}	0.756×10^{-3}	0.0567	6.236×10^{-3}	17.24×10^{-3}
Prestressed steel	1120×10^{-6}	0.504×10^{-3}	0.227×10^{-3}	0.0254	11.429×10^{-3}	5.15×10^{-3}
Properties of age-adjusted transformed section				0.4366 $\bar{A}$	16.040×10^{-3} $\bar{B}$	64.12×10^{-3} $\bar{I}$

$$\begin{Bmatrix} \Delta \varepsilon_O \\ \Delta \psi \end{Bmatrix} = \frac{1}{8.82 \times 10^9 [0.4366 \times 64.12 \times 10^{-3} - (16.04 \times 10^{-3})^2]}$$

$$\times \begin{bmatrix} 64.12 \times 10^{-3} & -16.040 \times 10^{-3} \\ -16.040 \times 10^{-3} & 0.4366 \end{bmatrix} \begin{Bmatrix} -1.835 \\ -0.139 \end{Bmatrix} 10^6$$

$$= 10^{-6} \begin{Bmatrix} -470 \\ -128 \text{ m}^{-1} \end{Bmatrix}$$



$$\Delta\sigma_c = \sigma_{restrained} + \bar{E}_c(t,t_o)\left[\Delta\varepsilon_o + y\Delta\psi\right]$$

$$\Delta\sigma_{\text{ns}} = E_{\text{ns}}\left[\Delta\varepsilon_o + y_{\text{ns}}\Delta\psi\right]$$

$$\Delta\sigma_{\text{ps}} = \Delta\bar{\sigma}_{\text{pr}} + E_{\text{ps}}\left[\Delta\varepsilon_o + y_{\text{ps}}\Delta\psi\right]$$

$$(\Delta\sigma_c)_{\text{top}} = 2.741 \times 10^6 + 8.82[-471 + (-0.6)(-128)]10^3$$

$$= -0.736\,\text{MPa} \quad (-0.107\,\text{ksi})$$

$$(\Delta\sigma_c)_{\text{bot}} = 8.145 \times 10^6 + 8.82[-471 + 0.6(-128)]10^3$$

$$= 3.313\,\text{MPa} \quad (0.481\,\text{ksi})$$

$$\Delta\sigma_{\text{ns2}} = 200[-471 + (-0.55)(-128)]10^3$$

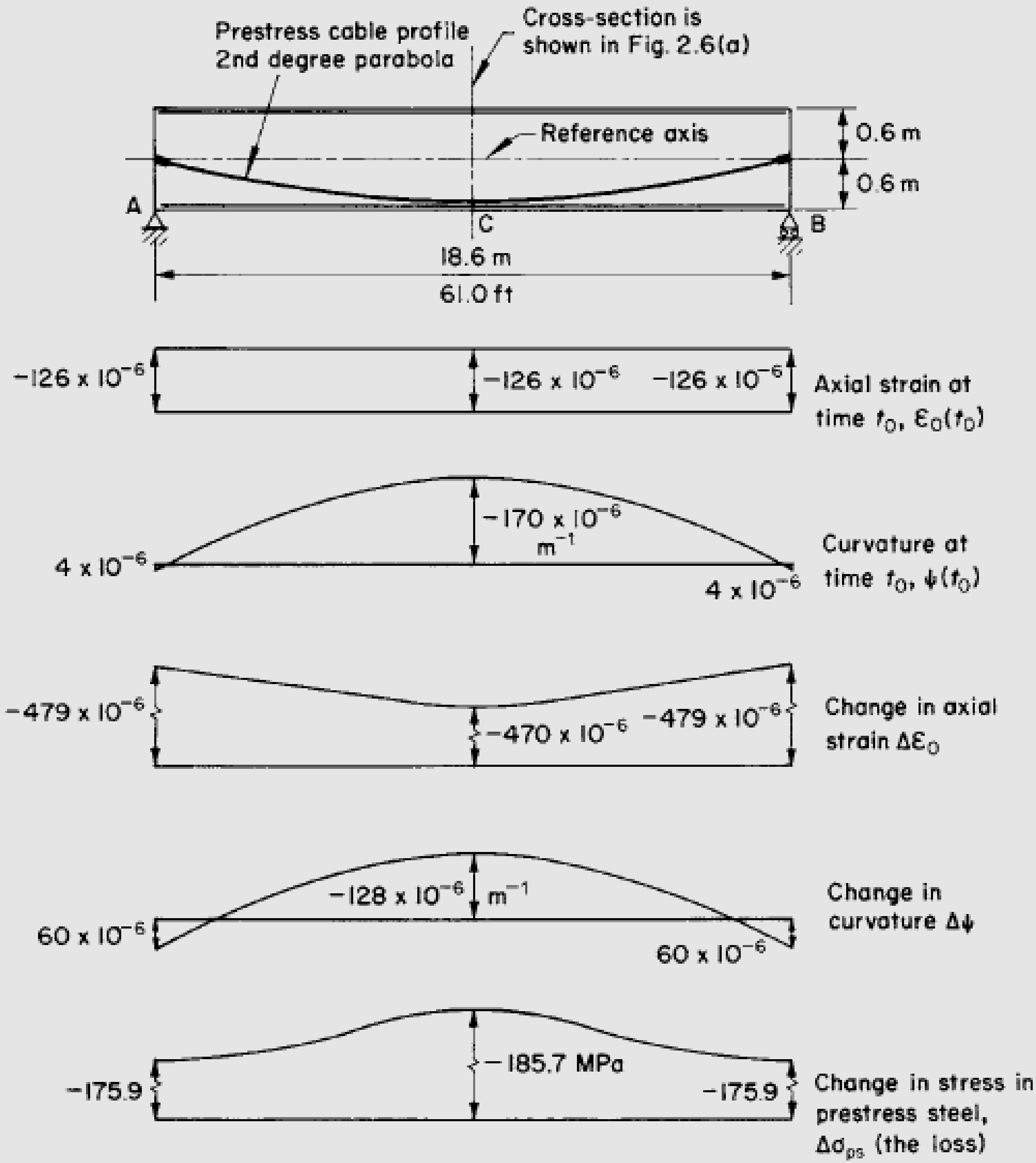
$$= -\,80.1\,\text{MPa} \quad (-11.6\,\text{ksi})$$

$$\Delta\sigma_{\text{ns1}} = 200[-471 + 0.55(-128)]10^3$$

$$= -108.3\,\text{MPa} \quad (-15.7\,\text{ksi})$$

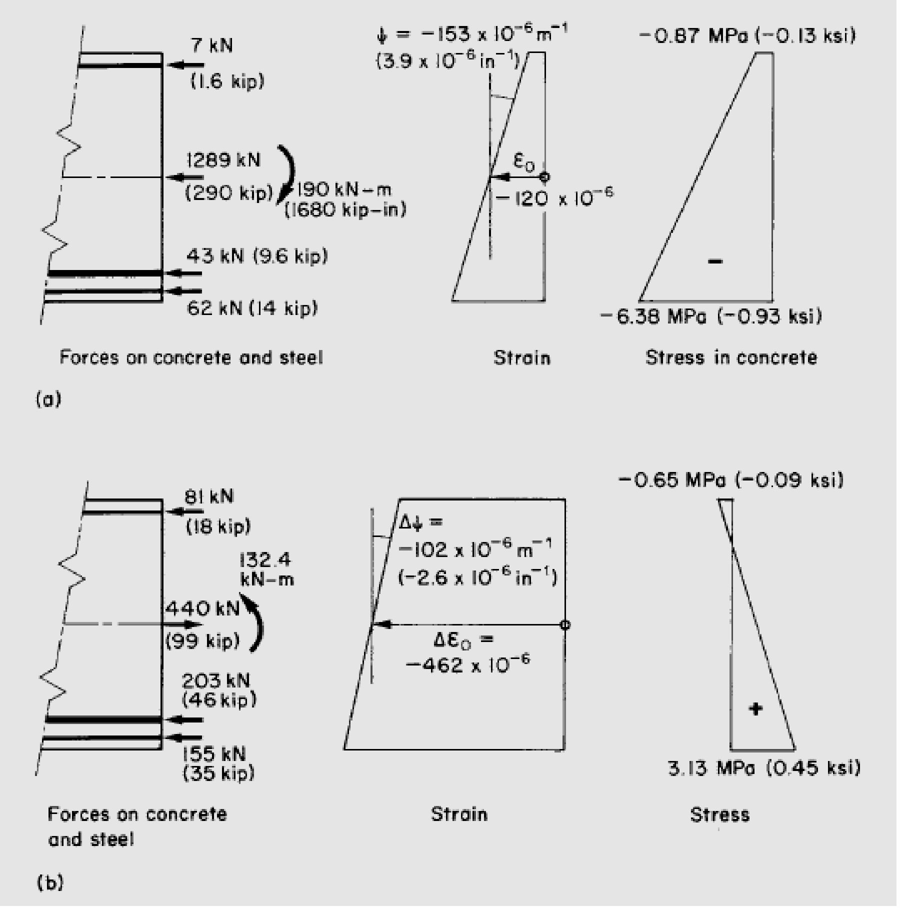
$$\Delta\sigma_{\text{ps}} = -80 \times 10^6 + 200[-471 + 0.45(-128)]10^3$$

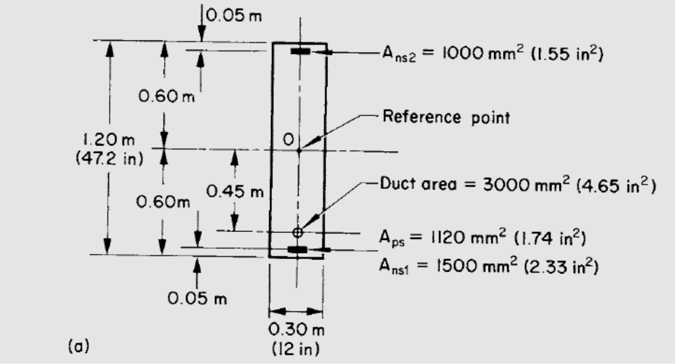
$$= -185.7\,\text{MPa} \quad (-26.9\,\text{ksi})$$


Prestress cable profile
2nd degree parabola
Cross-section is
shown in Fig. 2.6(a)
Reference axis
0.6 m
0.6 m
A
C
B
18.6 m
61.0 ft
−126 x 10⁻⁶
−126 x 10⁻⁶
−126 x 10⁻⁶
Axial strain at
time t₀, ε₀(t₀)
−170 x 10⁻⁶
m⁻¹
4 x 10⁻⁶
4 x 10⁻⁶
Curvature at
time t₀, ψ(t₀)
−479 x 10⁻⁶
−470 x 10⁻⁶
−479 x 10⁻⁶
Change in axial
strain Δε₀
−128 x 10⁻⁶ m⁻¹
60 x 10⁻⁶
60 x 10⁻⁶
Change in
curvature Δψ
−185.7 MPa
−175.9
−175.9
Change in stress in
prestress steel,
Δσps (the loss)

Example 2.3

Pre-tensioning case : The prestress steel must now be included in calculation of the Properties of transformed section at t_0





	A	B	I
concrete	$1.2*0.3-A_{ns1}$ $-A_{ns2}-A_{ps}=0.3564$	$1000*550-1120*450-1500*550$ $=-0.779*10^{-3}$	$43.2-0.756-0.227$ $=42.22*10^{-3}$
Reinforcing	0.0025	$0.275*10^{-3}$	$0.756*10^{-3}$
PS	0.00112	$0.504*10^{-3}$	$0.227*10^{-3}$
n=200/30	A*n	B*n	I*n
Concrete	0.3564	$-0.779*10^{-3}$	$42.22*10^{-3}$
Reinforcing	0.0167	$1.833*10^{-3}$	$5.04*10^{-3}$
PS	0.0075	$3.36*10^{-3}$	$1.513*10^{-3}$
$\sum$	0.3806	$4.464*10^{-3}$	$48.773*10^{-3}$

Strain and curvature at the reference point immediately after prestress transfer

$$E_c\left(t_o\right)$$

$$\begin{Bmatrix} \varepsilon_o\left(t_o\right) \\ \psi\left(t_o\right) \end{Bmatrix} = \frac{1}{30\times10^9}\frac{1}{\left[0.3806\times48.773\times10^{-3}-\left(4.464\times10^{-3}\right)^2\right]}$$

$$\times\begin{bmatrix} 48.773*10^{-3} & -4.464*10^{-3} \\ -4.464*10^{-3} & 0.3806 \end{bmatrix}\begin{Bmatrix} -1400*10^3 \\ -240*10^3 \end{Bmatrix} = \frac{1}{556.292*10^6}\begin{Bmatrix} -67,210 \\ -85,094.4 \end{Bmatrix}$$

$$= \begin{Bmatrix} -120.82*10^{-6} \\ -152.96*10^{-6} \end{Bmatrix}$$

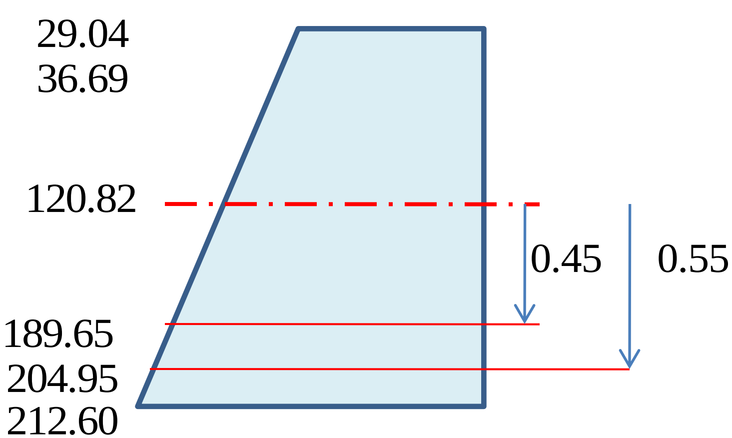
The instantaneous axial strain at O and curvature (Equation (2.32)) is

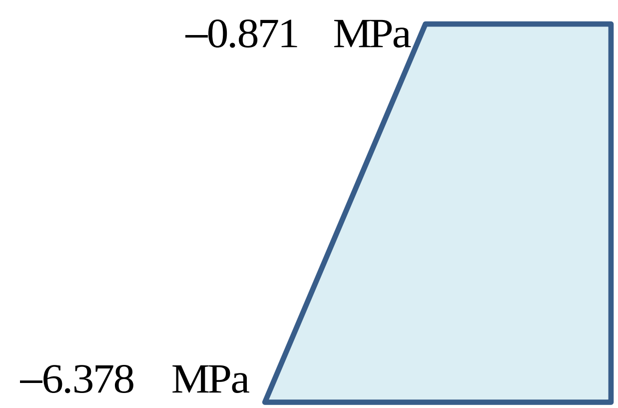
$$\begin{Bmatrix} \varepsilon_O(t_0) \\ \psi(t_0) \end{Bmatrix} = \frac{1}{30\times10^9[0.3712\times46.88\times10^{-3}-(0.208\times10^{-3})^2]}$$

$$\times\begin{bmatrix} 46.88\times10^{-3} & -0.208\times10^{-3} \\ -0.208\times10^{-3} & 0.3712 \end{bmatrix}\begin{Bmatrix} -1400\times10^3 \\ -240\times10^3 \end{Bmatrix}$$

$$= 10^{-6}\begin{Bmatrix} -126 \\ -170\,\mathrm{m}^{-1} \end{Bmatrix}$$

$$\varepsilon_c\left(t_o\right) = -120.82*10^6 + \left(-152.96*10^{-6}\right)y$$





Stress resultants

$$\begin{Bmatrix} N_c \\ M_c \end{Bmatrix} = 30*10^9 \begin{bmatrix} 0.3564 & -0.779*10^{-3} \\ -0.779*10^{-3} & 42.22*10^3 \end{bmatrix} \begin{Bmatrix} -120.82*10^{-6} \\ -152.96*10^{-6} \end{Bmatrix}$$

$$= 30*10^9 \begin{Bmatrix} -42.94*10^{-6} \\ -6.638*10^{-9} \end{Bmatrix} = \begin{Bmatrix} -1288.2 \text{ kN} \\ -190.9 \text{ kN-m} \end{Bmatrix}$$

$$\begin{Bmatrix} \Delta N \\ \Delta M \end{Bmatrix}_{creep} = -\sum_{i=1}^{m} \left\{ \bar{E}_c \begin{bmatrix} A_c & B_c \\ B_c & I_c \end{bmatrix} \varphi \begin{Bmatrix} \varepsilon_o(t_o) \\ \psi_o(t_o) \end{Bmatrix} \right\}_i \qquad \begin{Bmatrix} \varepsilon_o(t_o) \\ \psi(t_o) \end{Bmatrix} = \begin{Bmatrix} -120.82*10^{-6} \\ -152.96*10^{-6} \end{Bmatrix}$$

$$\begin{Bmatrix} \Delta N \\ \Delta M \end{Bmatrix}_{creep} = -8.82*10^9*3 \begin{bmatrix} 0.3806 & 4.464*10^{-3} \\ 4.464*10^{-3} & 48.773*10^{-3} \end{bmatrix} \begin{Bmatrix} \varepsilon_o(t_o) \\ \psi_o(t_o) \end{Bmatrix}$$

$$\begin{Bmatrix} \Delta N \\ \Delta M \end{Bmatrix}_{shrinkage} = -\sum_{i=1}^{m} \left\{ \bar{E}_c \begin{bmatrix} A_c & B_c \\ B_c & I_c \end{bmatrix} \begin{Bmatrix} \varepsilon_{cs} \\ 0 \end{Bmatrix} \right\}_i$$

$$\begin{Bmatrix} \Delta N \\ \Delta M \end{Bmatrix}_{shrinkage} = -8.82*10^9 \left(-240*10^{-6}\right) \begin{Bmatrix} 0.3806 \\ 4.464*10^{-3} \end{Bmatrix}$$

$$\begin{Bmatrix} \Delta N \\ \Delta M \end{Bmatrix}_{\text{prestressing}} = \sum \begin{Bmatrix} A_{ps} \Delta \bar{\sigma}_{pr} \\ A_{ps} y_{ps} \Delta \bar{\sigma}_{pr} \end{Bmatrix}_i$$

$$\begin{Bmatrix} \Delta N \\ \Delta M \end{Bmatrix}_{\text{relaxation}} = \begin{Bmatrix} 1120 \times 10^{-6}(-80 \times 10^6) \\ 1120 \times 10^{-6} \times 0.45\,(-80 \times 10^6) \end{Bmatrix}$$

$$= 10^6 \begin{Bmatrix} -0.090\ \text{N} \\ -0.0403\ \text{N-m} \end{Bmatrix}$$

Table 2.2 Calculation of $\bar{A}$, $\bar{B}$ and $\bar{I}$ of the age-adjusted transformed section

	Properties of area			Properties of transformed area		
	A (m^2)	B (m^3)	I (m^4)	AE/E_{ref} (m^2)	BE/E_{ref} (m^3)	IE/E_{ref} (m^4)
Concrete	0.3545	-1.625×10^{-3}	41.84×10^{-3}	0.3545	-1.625×10^{-3}	41.84×10^{-3}
Non-prestressed steel	2500×10^{-6}	0.275×10^{-3}	0.756×10^{-3}	0.0567	6.236×10^{-3}	17.24×10^{-3}
Prestressed steel	1120×10^{-6}	0.504×10^{-3}	0.227×10^{-3}	0.0254	11.429×10^{-3}	5.15×10^{-3}
Properties of age-adjusted transformed section				0.4366 $\bar{A}$	16.040×10^{-3} $\bar{B}$	64.12×10^{-3} $\bar{I}$

Change in strain and curvature

$$\begin{Bmatrix} \Delta\varepsilon_o \\ \Delta\psi \end{Bmatrix} = \frac{1}{\bar{E}_c\left(\bar{A}\bar{I} - \bar{B}^2\right)} \begin{bmatrix} \bar{I} & -\bar{B} \\ -\bar{B} & \bar{A} \end{bmatrix} \begin{Bmatrix} -\Delta N \\ -\Delta M \end{Bmatrix}$$

Total effect again

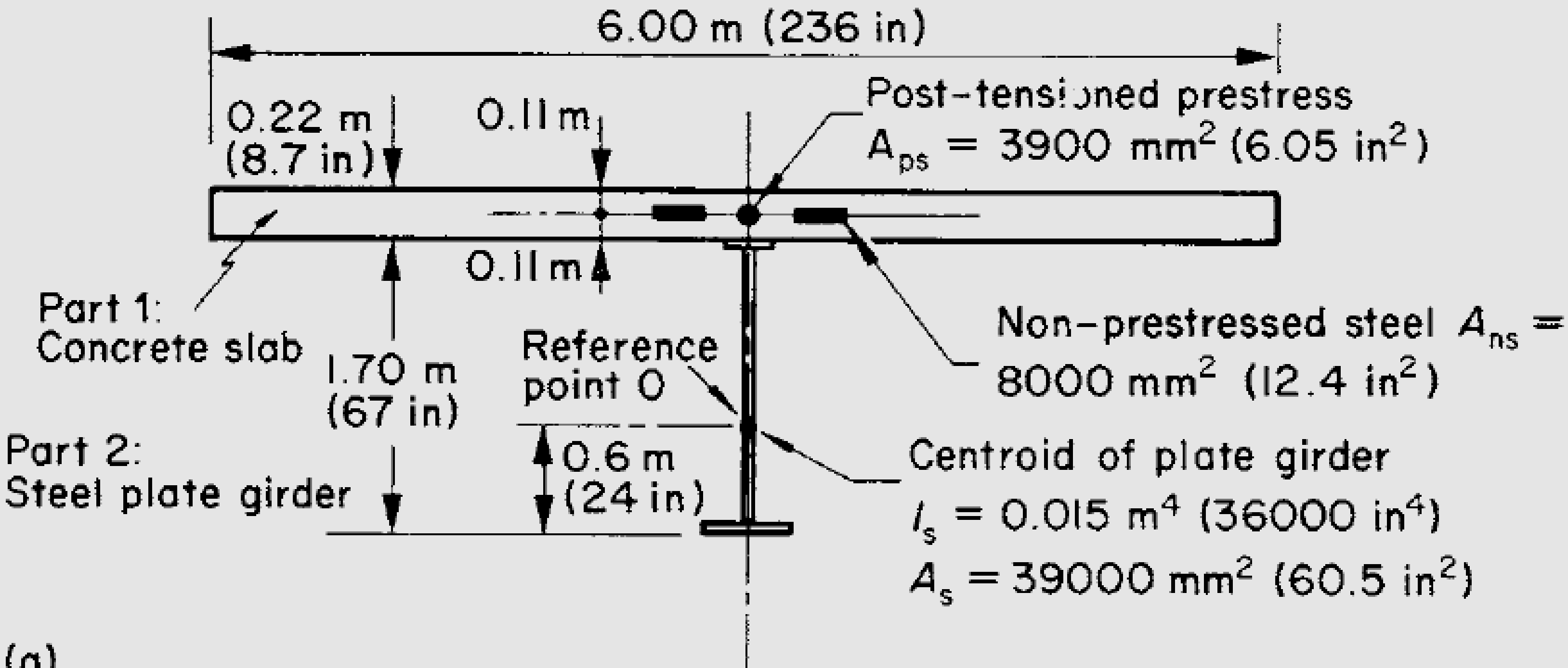
$$\Delta\sigma_c = \sigma_{restrained} + \bar{E}_c\left(t, t_o\right)\left[\Delta\varepsilon_o + y\Delta\psi\right]$$

$$\Delta\sigma_{\mathrm{ns}} = E_{\mathrm{ns}}\left[\Delta\varepsilon_o + y_{\mathrm{ns}}\Delta\psi\right]$$

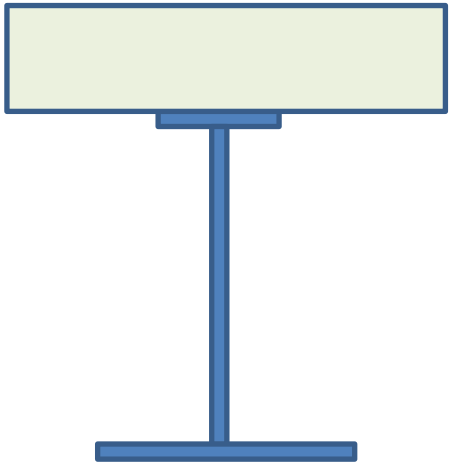
$$\Delta\sigma_{\mathrm{ps}} = \Delta\bar{\sigma}_{\mathrm{pr}} + E_{\mathrm{ps}}\left[\Delta\varepsilon_o + y_{\mathrm{ps}}\Delta\psi\right]$$

Example 2.4 Composite section with post-tension

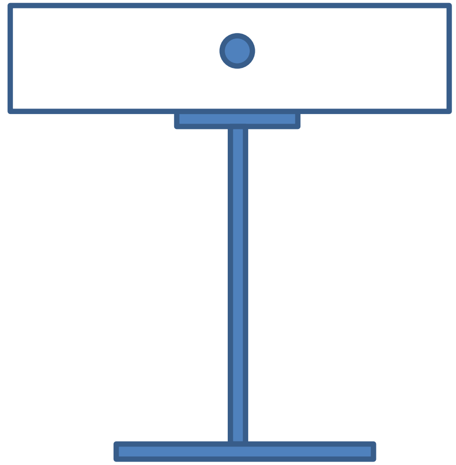
t due to creep, shrinkage and relaxation using the following data: prestressing force $P = 4500 \times 10^3$ N(1010 kip); bending moment introduced at age t_0, $M = 2800 \times 10^3$ N-m. (24 800 kip-in); $\varphi(t, t_0) = 2.5$; $\chi = 0.75$; $\varepsilon_{cs}(t, t_0) = -350 \times 10^{-6}$; reduced relaxation of the prestressed steel $\Delta\bar{\sigma}_{pr} = -90$ MPa (−13 ksi); $E_c(t_0) = 30$ GPa (4350 ksi). The moduli of elasticity of the plate girder, the prestressed and non-prestressed steel are equal; $E_s = E_{ns} = E_{ps} = 200$ GPa (29 000 ksi). The dimensions and properties of the cross-section area of concrete prestressed and non-prestressed steel are given in Fig. 2.9(a). The centroid of the steel girder, its cross-section area and moment of inertia about an axis through its centroid are also given in the same figure.



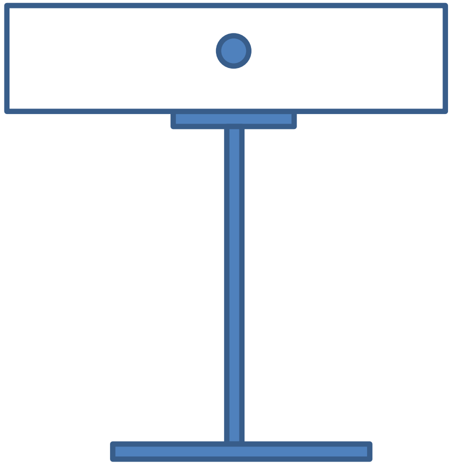
(a)



Stage 1 :
Shored
Cast in situ



Stage 2:
apply prestress
and remove shoring



Stage 3:
connection
and remove shoring

1. Stress and strain in concrete before connection of slab to steel girder

$$A_c = 6*0.22 - 8000*10^{-6} - 3900*10^{-6} = 1,308.1*10^{-3}$$

$$\sum A = A_c + \frac{E_{st}}{E_{ref}} = 1,308.1*10^{-3} + 0.0533 = 1361.4*10^{-3}$$

$$\sigma_c = \frac{P}{\sum A} = 3.305\ \text{MPa}$$

2. After removal of shoring

Table 2.3 Properties of the transformed section used in calculation of stress at time t_0

	Properties of areas			Properties of transformed area		
	A (m^2)	B (m^3)	I (m^4)	AE/E_{ref} (m^2)	BE/E_{ref} (m^3)	IE/E_{ref} (m^4)
Concrete	1.3081	−1.5828	1.9205	1.3081	−1.5828	1.9205
Non-prestressed steel	8000×10^{-6}	−0.0097	0.0117	0.0533	−0.0645	0.0781
Prestressed steel	3900×10^{-6}	−0.0047	0.0057	0.0260	−0.0315	0.0381
Steel girder	39000×10^{-6}	0	0.0150	0.2600	0	0.1000
Properties of transformed section				1.6474 A	−1.6788 B	2.1367 I

Axial force at O and bending moment introduced at removal of shores is:

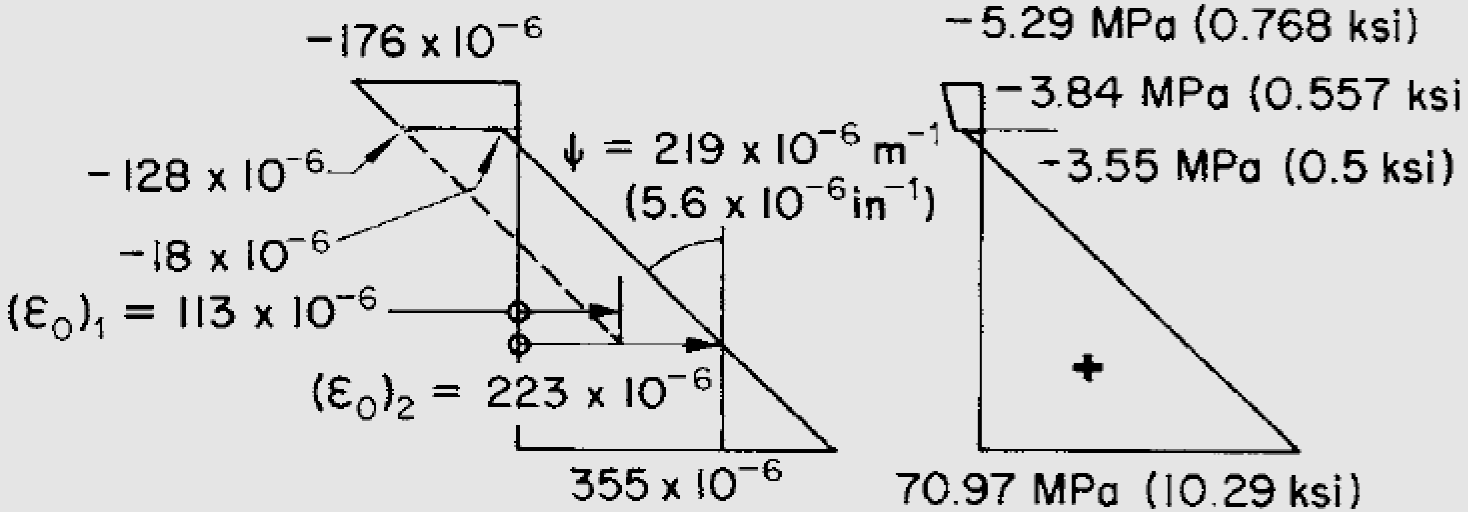
$$\begin{Bmatrix} N \\ M \end{Bmatrix} = \begin{Bmatrix} 0 \\ 2800 \times 10^3 \text{ N-m} \end{Bmatrix}$$

The axial strain at O and the curvature caused by these forces (Equation (2.32)) is

$$\begin{Bmatrix} \varepsilon_O(t_0) \\ \psi(t_0) \end{Bmatrix} = \frac{1}{30 \times 10^9(1.6474 \times 2.1367 - 1.6788^2)}$$

$$\times \begin{bmatrix} 2.1367 & 1.6788 \\ 1.6788 & 1.6474 \end{bmatrix} \begin{Bmatrix} 0 \\ 2800 \times 10^3 \end{Bmatrix}$$

$$= 10^{-6} \begin{Bmatrix} 223 \\ 219 \text{ m}^{-1} \end{Bmatrix}$$



−176 x 10⁻⁶
−128 x 10⁻⁶
−18 x 10⁻⁶
(ε₀)₁ = 113 x 10⁻⁶
(ε₀)₂ = 223 x 10⁻⁶
ψ = 219 x 10⁻⁶ m⁻¹
(5.6 x 10⁻⁶ in⁻¹)
355 x 10⁻⁶
−5.29 MPa (0.768 ksi)
−3.84 MPa (0.557 ksi)
−3.55 MPa (0.5 ksi)
+
70.97 MPa (10.29 ksi)

(b)

(c) Changes in stress and strain due to creep, shrinkage and relaxation

Age-adjusted elasticity modulus is

$$\bar{E}_c(t, t_0) = \frac{30 \times 10^9}{1 + 0.75 \times 2.5} = 10.435\,\text{GPa}.$$

In the restrained condition, stress in concrete is (Equation (2.45)):

$$(\sigma_{c\ \text{restrained}})_{\text{top}} = -10.435 \times 10^9[2.5(-176 \times 10^{-6}) - 350 \times 10^{-6}]$$

$$= 8.24\,\text{MPa}$$

$$(\sigma_{c\ \text{restrained}})_{\text{bot}} = -10.435 \times 10^9[2.5(-128 \times 10^{6}) - 350 \times 10^{-6}]$$

$$= 6.99\,\text{MPa}$$

$$\sigma_{\text{restrained}} = -\bar{E}_c(t, t_0)[\varphi(t, t_0)\varepsilon_c(t_0) + \varepsilon_{cs}] \qquad (2.45)$$

$$\begin{Bmatrix}\Delta N\\ \Delta M\end{Bmatrix}_{\text{creep}} = -10.435\times 10^{9}\times 2.5$$

$$\times\begin{bmatrix}1.3081 & -1.5828\\ -1.5828 & 1.9205\end{bmatrix}\begin{Bmatrix}113\\ 219\end{Bmatrix}10^{-6}$$

$$= 10^{6}\begin{Bmatrix}5.187\text{ N}\\ -6.306\text{ N-m}\end{Bmatrix}$$

Table 2.3 Properties of the transformed section used in calculation of stress at time t_0

	Properties of areas			*Properties of transformed area*		
	A (*m*²)	*B* (*m*³)	*I* (*m*⁴)	AE/E_{ref} (*m*²)	BE/E_{ref} (*m*³)	IE/E_{ref} (*m*⁴)
Concrete	1.3081	−1.5828	1.9205	1.3081	−1.5828	1.9205
Non-prestressed steel	8000 × 10⁻⁶	−0.0097	0.0117	0.0533	−0.0645	0.0781
Prestressed steel	3900 × 10⁻⁶	−0.0047	0.0057	0.0260	−0.0315	0.0381
Steel girder	39000 × 10⁻⁶	0	0.0150	0.2600	0	0.1000
Properties of transformed section				1.6474 *A*	−1.6788 *B*	2.1367 *I*

$$\begin{Bmatrix}\Delta N\\ \Delta M\end{Bmatrix}_{\text{shrinkage}} = -10.435\times 10^{9}(-350\times 10^{-6})\begin{Bmatrix}1.3081\\ -1.5828\end{Bmatrix}$$

$$= 10^{6}\begin{Bmatrix}4.777\text{ N}\\ -5.781\text{ N-m}\end{Bmatrix}$$

$$\begin{Bmatrix}\Delta N\\ \Delta M\end{Bmatrix}_{\text{relaxation}} = \begin{Bmatrix}3900\times 10^{-6}(-90\times 10^{6})\\ 3900\times 10^{-6})(-1.21)(-90\times 10^{6})\end{Bmatrix}$$

$$= 10^{6}\begin{Bmatrix}-0.351\text{ N}\\ 0.425\text{ N-m}\end{Bmatrix}$$

Age adjusted Elasticity
10.435 GPa

$$\begin{Bmatrix}\Delta N\\ \Delta M\end{Bmatrix} = 10^{6}\begin{Bmatrix}5.187+4.777-0.351\\ -6.306-5.781+0.425\end{Bmatrix} = 10^{6}\begin{Bmatrix}9.613\text{ N}\\ -11.662\text{ N-m}\end{Bmatrix}$$

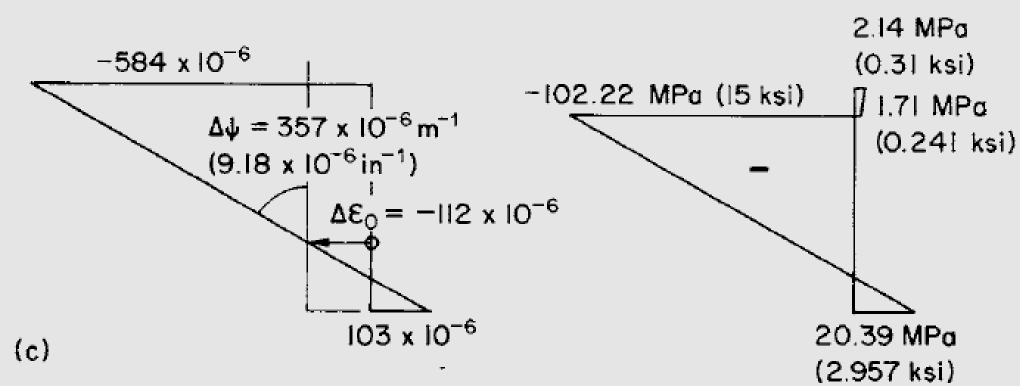
$$\bar{A} = 2.284\,\text{m}^2; \quad \bar{B} = -1.859\,\text{m}^3; \quad \bar{I} = 2.542\,\text{m}^4.$$

E_{ref} used in the calculation of the above values is:

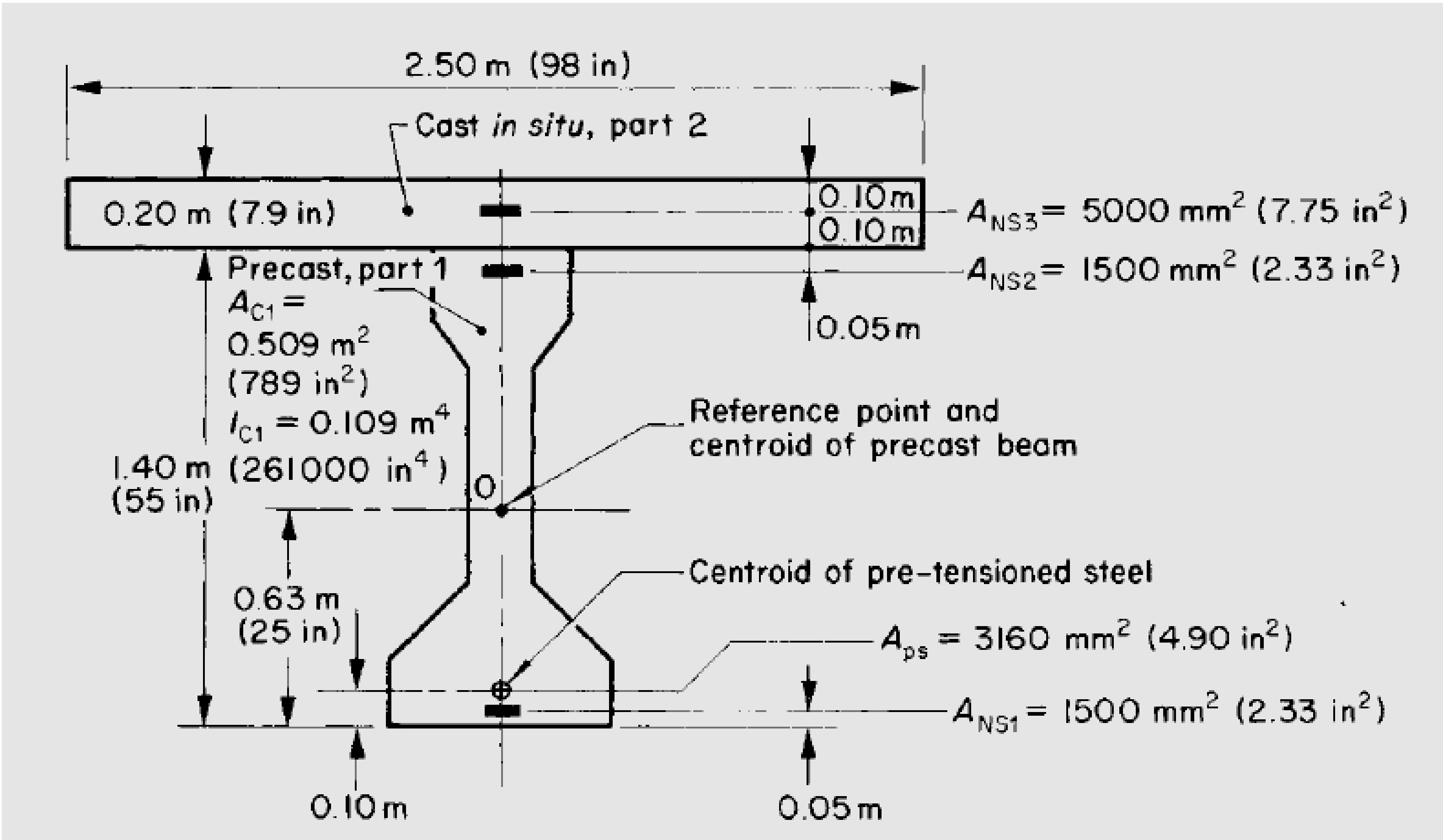
$$E_{ref} = \bar{E}_c(t, t_0) = 10.435\,\text{GPa}.$$

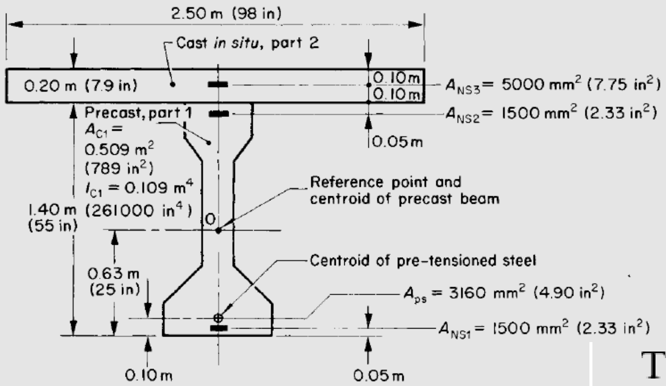
$$\begin{Bmatrix}\Delta\varepsilon_O \\ \Delta\psi\end{Bmatrix} = \frac{10^6}{10.435 \times 10^9(2.284 \times 2.542 - 1.859^2)}\begin{bmatrix}2.542 & 1.859 \\ 1.859 & 2.284\end{bmatrix}\begin{Bmatrix}-9.613 \\ 11.662\end{Bmatrix}$$

$$= 10^{-6}\begin{Bmatrix}-112 \\ 357\,\text{m}^{-1}\end{Bmatrix}$$



Example 2.5 composite section with pre-tensioned





The cross-section shown in Fig. 2.10 is composed of a precast pre-tensioned beam (part 1) and a slab cast *in situ* (part 2). It is required to find the stress and strain distribution in the section immediately after prestressing, and the changes in these values occurring between prestressing and casting of the deck slab and after a long period using the following data.

Ages of precast beam at the time of prestress, $t_1 = 3$ days and at the time of casting of the deck slab, $t_2 = 60$ days; the final stress and strain are required at age $t_3 = \infty$. The prestress force, $P = 4100 \times 10^3$ N; (920 kip); the bending moment due to self-weight of the prestress beam (which is introduced at the same time as the prestress), $M_1 = 1400 \times 10^3$ N-m (12400 kip-in); additional bending moment introduced at age t_2 (representing the effect of the weight of the slab plus superimposed dead load), $M_2 = 1850 \times 10^3$ N-m (16400 kip-in). The modulus of elasticity of concrete of the precast beam $E_{c1}(3) = 25$ GPa (3600 ksi) and $E_{c1}(60) = 37$ GPa (5400 ksi).

Soon after hardening of the concrete, the composite action starts to develop gradually. Here we will ignore the small composite action occurring during the first three days. Consider that age $t_2 = 60$ days for the precast beam corresponds to age = 3 days of the deck at which time the modulus of elasticity of the deck $E_{c2}(3) = 23$ GPa (3300 ksi).

Concrete part 1:

$[\varphi(60, 3)]_1 = 1.20$ $[\varphi(\infty, 3)]_1 = 2.30$ $[\varphi(\infty, 60)]_1 = 2.27$;

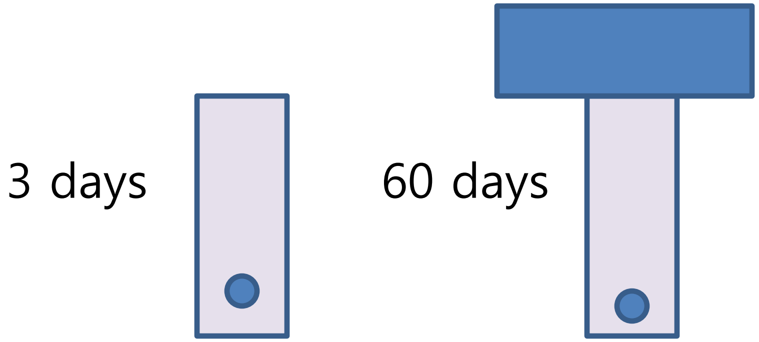
$[\chi(60, 3)]_1 = 0.86$ $[\chi(\infty, 60)]_1 = 0.80$

$[\varepsilon_{cs}(60, 3)]_1 = -57 \times 10^{-6}$ $[\varepsilon_{cs}(\infty, 60)]_1 = -205 \times 10^{-6}$

Concrete part 2:

$[\varphi(\infty, 3)]_2 = 2.40$ $[\chi(\infty, 3)]_2 = 0.78$

$[\varepsilon_{cs}(\infty, 3)]_2 = -269 \times 10^{-6}$



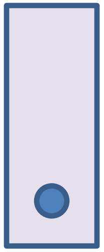
Reduced relaxation $\Delta\overline{\sigma}_{pr} = -85$ MPa (12 ksi) of which -15 MPa (2.2 ksi) in the first 57 days. Modulus of elasticity of the prestressed and non-prestressed steels = 200 GPa.

The dimensions and properties of areas of the concrete and steel in the two parts are given in Fig. 2.10.

Table 2.4 Properties of the precast section employed in calculation of stress and strain at time $t_1 = 3$ days

	Properties of area			Properties of transformed area		
	A (m^2)	B (m^3)	I (m^4)	AE/E_{ref} (m^2)	BE/E_{ref} (m^3)	IE/E_{ref} (m^4)
Concrete	0.5090	0.0	0.1090	0.5090	0.0	0.1090
Non-prestressed steel	3000×10^{-6}	-210×10^{-6}	1282×10^{-6}	0.0240	−0.0017	0.0103
Prestressed steel	3160×10^{-6}	1675×10^{-6}	888×10^{-6}	0.0253	0.0134	0.0071
Properties of transformed section				0.5583 A	0.0117 B	0.1264 I

3 days



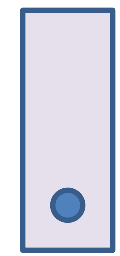
$$\begin{Bmatrix} N \\ M \end{Bmatrix}_{equivalent} = \begin{Bmatrix} -4100 \times 10^3 \\ 1400 \times 10^3 - 4100 \times 10^3 \times 0.53 \end{Bmatrix} = \begin{Bmatrix} -4100 \times 10^3 \text{ N} \\ -773 \times 10^3 \text{ N-m} \end{Bmatrix}$$

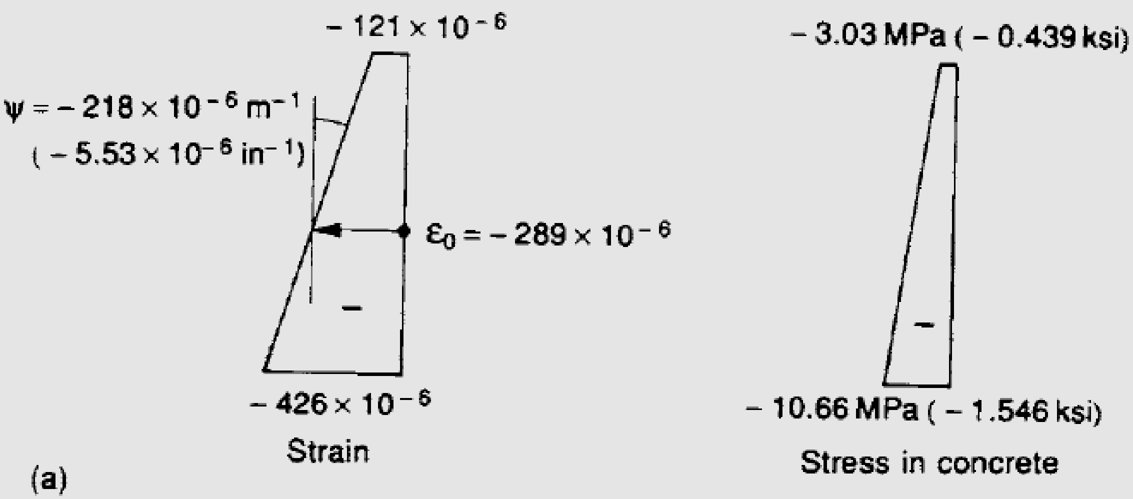
Instantaneous axial strain and curvature at $t_1 = 3$ days (Equation (2.32)) are,

$$\begin{Bmatrix} \varepsilon_O(t_1) \\ \psi(t_1) \end{Bmatrix} = \frac{1}{25 \times 10^9 (0.5583 \times 0.1264 - 0.0117^2)} \begin{bmatrix} 0.1264 & -0.0117 \\ -0.0117 & 0.5583 \end{bmatrix}$$

$$\times \begin{Bmatrix} -4100 \times 10^3 \\ -773 \times 10^3 \end{Bmatrix} = 10^{-6} \begin{Bmatrix} -289 \\ -218 \text{ m}^{-1} \end{Bmatrix}$$

3 days





$$\sigma = E\left(\varepsilon_o + \psi y\right)$$

(b) Change in stress and strain occurring between t = 3 days and t = 60 days

The age-adjusted elasticity modulus of concrete (Equation (1.31)) is:

t=60 days

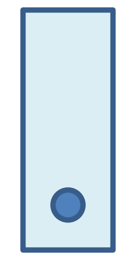
$$\bar{E}_c(60, 3) = \frac{25 \times 10^9}{1 + 0.86 \times 1.20} = 12.30\,\text{GPa (1780 ksi)}.$$

$$\begin{Bmatrix} \Delta N \\ \Delta M \end{Bmatrix}_{\text{creep}} = -12.30 \times 10^9 \times 1.2 \begin{bmatrix} 0.5090 & 0 \\ 0 & 0.1090 \end{bmatrix} \begin{Bmatrix} -289 \\ -218 \end{Bmatrix} 10^{-6}$$

Creep fn

Ac, Ic at t=3days

60 days



$$= 10^6 \begin{Bmatrix} 2.171\ \text{N} \\ 0.351\ \text{N-m} \end{Bmatrix}$$

$$\begin{Bmatrix} \Delta N \\ \Delta M \end{Bmatrix}_{\text{shrinkage}} = -12.3 \times 10^9(-57 \times 10^{-6}) \begin{Bmatrix} 0.5090 \\ 0 \end{Bmatrix} = 10^6 \begin{Bmatrix} 0.357\ \text{N} \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} \Delta N \\ \Delta M \end{Bmatrix}_{\text{relaxation}} = \begin{Bmatrix} 3160 \times 10^{-6}(-15 \times 10^6 \\ 3160 \times 10^{-6} \times 0.53(-15 \times 10^6) \end{Bmatrix} = 10^6 \begin{Bmatrix} -0.047\ \text{N} \\ -0.025\ \text{N-m} \end{Bmatrix}$$

The total restraining forces are

$$\begin{Bmatrix} \Delta N \\ \Delta M \end{Bmatrix} = 10^6 \begin{Bmatrix} 2.171 + 0.357 - 0.047 \\ 0.0351 + 0 - 0.025 \end{Bmatrix} = 10^6 \begin{Bmatrix} 2.481\ \text{N} \\ 0.326\ \text{N-m} \end{Bmatrix}$$

Change in strain and curvature

$$\begin{Bmatrix} \Delta\varepsilon_o \\ \Delta\psi \end{Bmatrix} = \frac{1}{\bar{E}_c\left(\bar{A}\bar{I}-\bar{B}^2\right)} \begin{bmatrix} \bar{I} & -\bar{B} \\ -\bar{B} & \bar{A} \end{bmatrix} \begin{Bmatrix} -\Delta N \\ -\Delta M \end{Bmatrix}$$

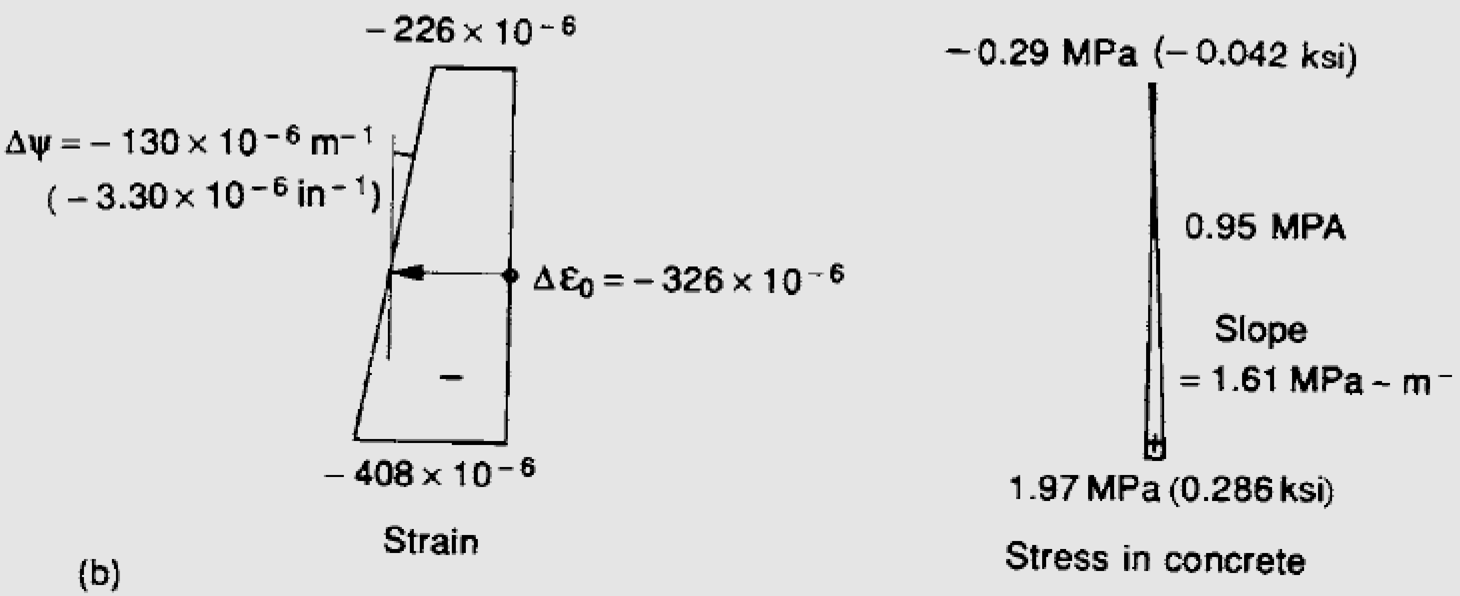
Total effect again

Section properties at t=60days

Removal of the restraining forces results in the following increments of axial strain and curvature during the period t_1 to t_2 (Equation (2.40)):

$$\begin{Bmatrix} \Delta\varepsilon_O(t_2, t_1) \\ \Delta\psi(t_2, t_1) \end{Bmatrix} = 10^{-6} \begin{Bmatrix} -326 \\ -130\,\mathrm{m}^{-1} \end{Bmatrix}$$

$$\Delta\sigma_c = \sigma_{restrained} + \bar{E}_c\left(t, t_o\right)\left[\Delta\varepsilon_o + y\Delta\psi\right]$$

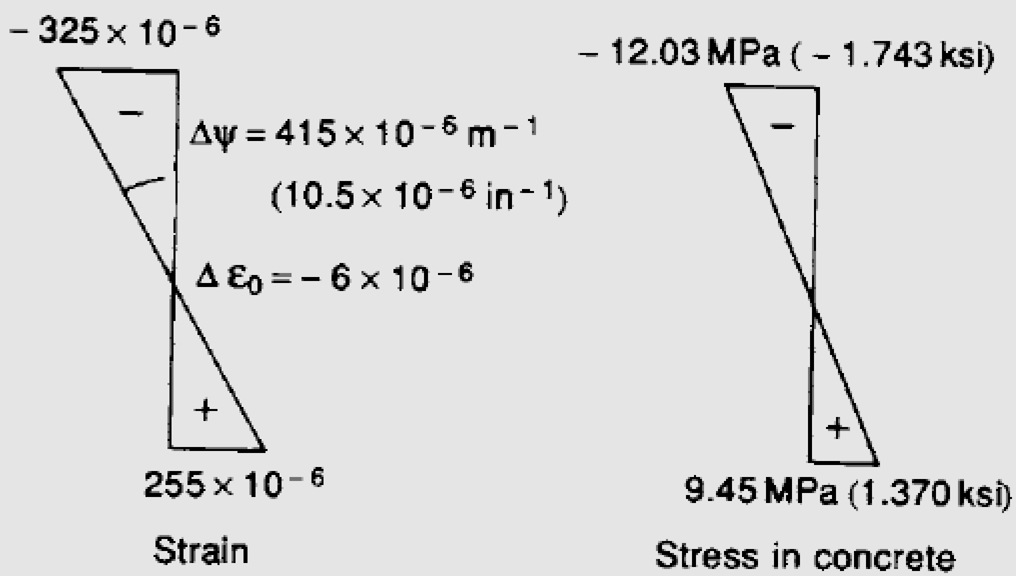


(c) Instantaneous increments of stress and strain at $t_2 = 60$ days

The bending moment $M = 1850 \times 10^3$ N-m is resisted only by the prestressed beam. The properties of the transformed section are calculated in the same way in Table 2.4 using $E_{ref} = E_c(60) = 37$ GPa, giving: $A = 0.5423\,m^2$; $B = 0.0079\,m^3$; $I = 0.1207\,m^4$. Substitution in Equation (2.32) gives the instantaneous increments in axial strain and curvature occurring at t_2:

$$\begin{Bmatrix} \Delta\varepsilon_O(t_2) \\ \Delta\psi(t_2) \end{Bmatrix} = 10^{-6} \begin{Bmatrix} -6 \\ 415\,m^{-1} \end{Bmatrix}$$

$$\begin{Bmatrix} \varepsilon_O(t_0) \\ \psi(t_0) \end{Bmatrix} = \frac{1}{E_{ref}\left(AI - B^2\right)} \begin{bmatrix} I & -B \\ -B & A \end{bmatrix} \begin{Bmatrix} N \\ M \end{Bmatrix}_{equiv}$$



(c)

Concrete part 1:

$[\varphi(60, 3)]_1 = 1.20$ $[\varphi(\infty, 3)]_1 = 2.30$

$[\chi(60, 3)]_1 = 0.86$ $[\chi(\infty, 60)]_1 = 0.80$

$[\varepsilon_{cs}(60, 3)]_1 = -57 \times 10^{-6}$ $[\varepsilon_{cs}(\infty, 60)]_1 = -205$

Concrete part 2:

$[\varphi(\infty, 3)]_2 = 2.40$ $[\chi(\infty, 3)]_2 = 0.78$

$[\varepsilon_{cs}(\infty, 3)]_2 = -269 \times 10^{-6}$

(d) Changes in stress and strain due to creep, shrinkage and relaxation during the period $t_2 = 60$ days to $t_3 = \infty$.

The age-adjusted moduli of elasticity for the precast beam and slab are

Ec(60)

$$\bar{E}_{c1}(\infty, 60) = \frac{37 \times 10^9}{1 + 0.8 \times 2.27} = 13.14\,\text{GPa}\ (1900\,\text{ksi})$$

Ec(3)

$$\bar{E}_{c2}(\infty, 3) = \frac{23 \times 10^9}{1 + 0.78 \times 2.40} = 8.01\,\text{GPa}\ (1160\,\text{ksi})$$

The stresses shown in Figs 2.11(a), (b) and (c) are introduced at various ages and thus have different coefficients for creep occurring during the period considered. In the following, the stresses in Figs 2.11(a) and (b) are combined and treated as if the combined stress were introduced when the age of the precast beam is 3 days; thus the creep coefficient to be used is $\varphi(\infty, 3) - \varphi(60, 3) = 2.30 - 1.20 = 1.10$. The stress in Fig. 2.11(c) is introduced when the precast beam is 60 days old; thus the coefficient of creep for the period considered is $\varphi(\infty, 60) = 2.27$.

For more accuracy, the stress in Fig. 2.11(b) which is gradually introduced between the age 3 days and 60 days may be treated as if it were introduced at some intermediate time $\bar{t}$, such that:

$$\frac{1}{E_c(\bar{t})}[1+\varphi(60,\bar{t})]=\frac{1}{E_c(3)}[1+\chi(60,3)\,\varphi(60,3)].$$

The stresses in the precast beam necessary to artificially restrain creep and shrinkage (Equation (2.45)) are:

$$(\Delta\sigma_{c\ \mathrm{restrained}})_{\mathrm{top}} = -13.14\times10^{9}\Big[1.10\left(-121\times10^{-6}-\frac{0.29\times10^{6}}{25\times10^{9}}\right)$$

$$+\,2.27\,(-325\times10^{-6})+(-205\times10^{-6})\Big]$$

$$=14.304\,\mathrm{MPa}$$

$$(\Delta\sigma_{c\ \mathrm{restrained}})_{\mathrm{bot}} = -13.14\times10^{9}\Big[1.10\left(-426\times10^{-6}+\frac{1.97\times10^{6}}{25\times10^{9}}\right)$$

$$+\,2.27(255\times10^{-6})+(-205\times10^{-6})\Big]$$

$$=0.106\,\mathrm{MPa}.$$

Table 2.5 Properties of the composite age-adjusted transformed section used in calculation of the changes of stress and strain between $t_2 = 60$ days and $t_3 = \infty$.

	Properties of area			*Properties of transformed area*		
	A (m^2)	B (m^3)	I (m^4)	AE/E_{ref} (m^2)	BE/E_{ref} (m^3)	IE/E_{ref} (m^4)
Concrete of deck slab	0.495	−0.4307	0.3763	0.3017	−0.2625	0.2294
Non- prestressed *steel in deck* slab	5000×10^{-6}	-4350×10^{-6}	3785×10^{-6}	0.0761	−0.0662	0.05763
Concrete in beam	0.5090	0.0	0.1090	0.5090	0.0	0.1090
Non- prestressed *steel in beam*	3000×10^{-6}	-210×10^{-6}	1282×10^{-6}	0.0457	−0.0032	0.0195
Prestressed steel	3160×10^{-6}	1675×10^{-6}	887.6×10^{-6}	0.0481	0.0255	0.0135
Properties of the age-adjusted transformed section				0.9806 $\bar{A}$	−0.3064 $\bar{B}$	0.4290 $\bar{I}$

The properties of the age-adjusted transformed section for the period t_2 to t_3 are calculated in Table 2.5, using $E_{ref} = \bar{E}_{c1}(\infty, 60) = 13.14$ GPa.

The forces necessary to restrain creep, shrinkage and relaxation during the period t_2 to t_3 are (Equations (2.41) to (2.44)):

$$\begin{Bmatrix} \Delta N \\ \Delta M \end{Bmatrix}_{creep} = -13.14 \times 10^9 \begin{bmatrix} 0.509 & 0 \\ 0 & 0.109 \end{bmatrix} \times \begin{Bmatrix} 1.10\left(-289 + \dfrac{0.95 \times 10^{12}}{25 \times 10^9}\right) + 2.27(-6) \\ 1.10\left(-218 + \dfrac{1.61 \times 10^{12}}{25 \times 10^9}\right) + 2.27(415) \end{Bmatrix} 10^{-6}$$

$$= 10^6 \begin{Bmatrix} 1.937 \text{ N} \\ -1.108 \text{ N-m} \end{Bmatrix}$$

$E_c(3) = 25$ GPa

$$\varepsilon_O(60,3) = \varepsilon_O(3) + \left\{\sigma_{restrained} + \bar{E}_c(60)\Delta\varepsilon_O(60,3)\right\} / E_c(3)$$

$$\sigma_{restrained} = -\bar{E}_c(t, t_0)[\varphi(t, t_0)\varepsilon_c(t_0) + \varepsilon_{cs}] \qquad (2.45)$$

$$\begin{Bmatrix} \Delta N \\ \Delta M \end{Bmatrix}_{\text{shrinkage}} = -13.14 \times 10^9 (-205 \times 10^{-6}) \begin{Bmatrix} 0.509 \\ 0 \end{Bmatrix}$$

$$-8.01 \times 10^9 (-269 \times 10^{-6}) \begin{Bmatrix} 0.495 \\ -0.4307 \end{Bmatrix}$$

$$= 10^6 \begin{Bmatrix} 2.437 \text{ N} \\ -0.928 \text{ N-m} \end{Bmatrix}$$

$$\begin{Bmatrix} \Delta N \\ \Delta M \end{Bmatrix}_{\text{relaxation}} = \begin{Bmatrix} 3160 \times 10^{-6} (-70 \times 10^6) \\ 3160 \times 10^{-6} \times 0.53 (-70 \times 10^6) \end{Bmatrix} = 10^6 \begin{Bmatrix} -0.221 \text{ N} \\ -0.117 \text{N-m} \end{Bmatrix}$$

The total restraining forces

$$\begin{Bmatrix} \Delta N \\ \Delta M \end{Bmatrix} = 10^6 \begin{Bmatrix} 1.937 + 2.437 - 0.221 \\ -1.108 - 0.928 - 0.117 \end{Bmatrix} = 10^6 \begin{Bmatrix} 4.153 \text{ N} \\ -2.153 \text{ N-m} \end{Bmatrix}$$

The increments of axial strain and curvature during the period t_2 to t_3 are obtained by substitution in Equation (2.40) and are plotted in Fig. 2.12:

$$\begin{Bmatrix} \Delta \varepsilon_O(t_3, t_2) \\ \Delta \psi(t_3, t_2) \end{Bmatrix} = 10^{-6} \begin{Bmatrix} -261 \\ 195 \text{m}^{-1} \end{Bmatrix}$$