Deformation of Concrete

Fall 2010

Dept of Architecture
Seoul National University

12th week

Stress and strain of cracked sections

7. Stress and Strain of Cracked Section

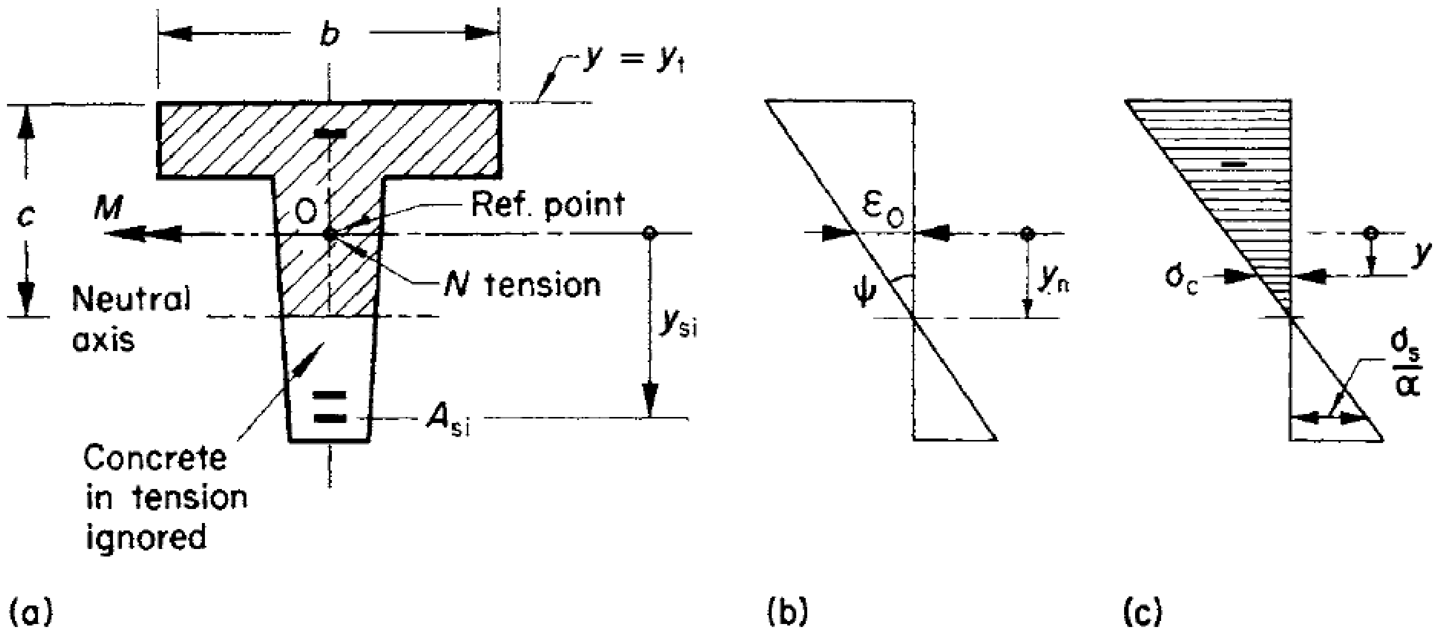
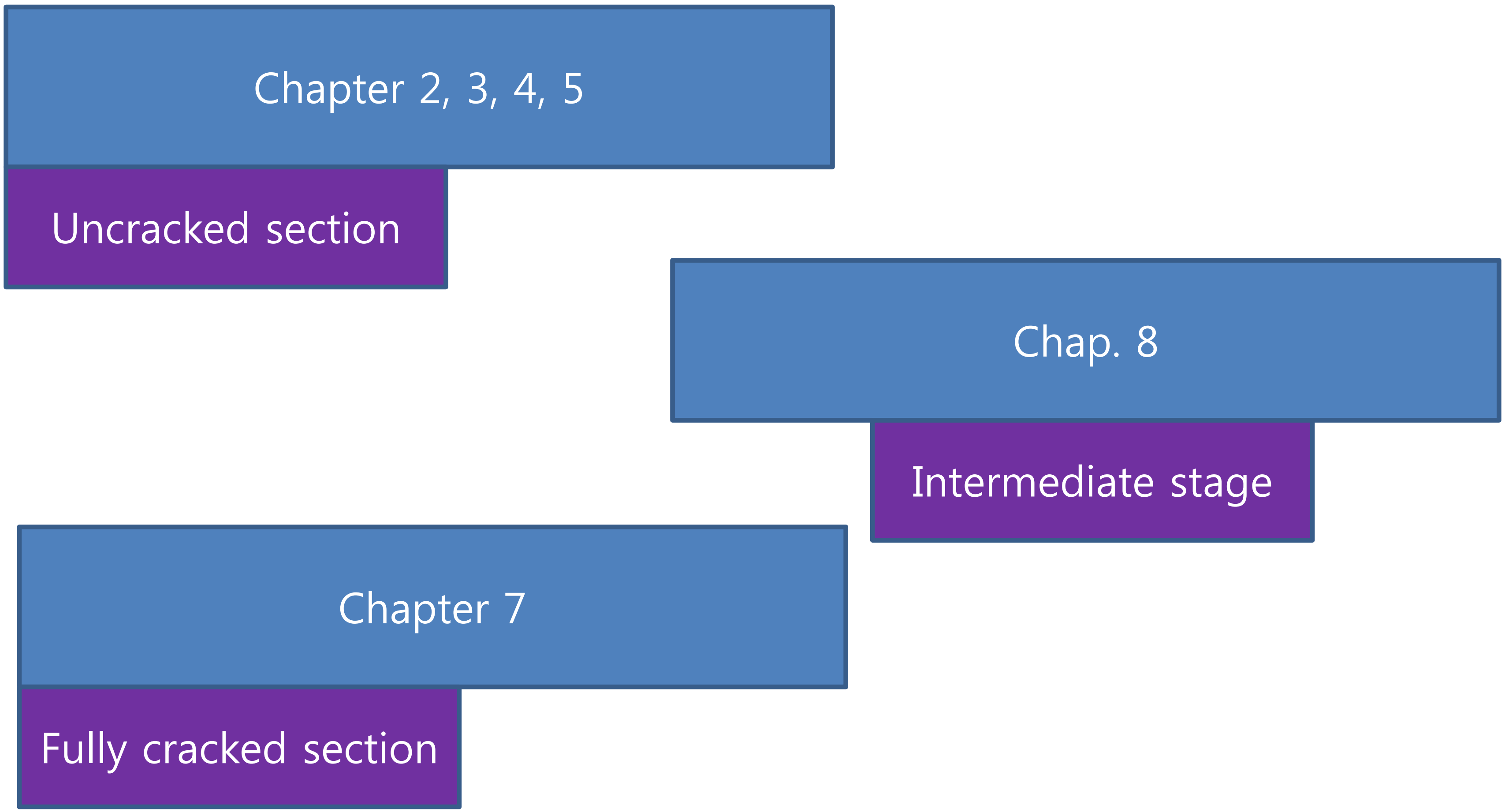


Figure 7.1 Stress (c) and strain (b) distributions in a fully cracked reinforced concrete section (a) (state 2) subjected to M and N. Convention for positive M, N, y, y_n and y_s.

Bounds between uncracked and fully cracked stages



The strain at any fibre (Fig. 7.1(b)) is

$$\varepsilon = \varepsilon_O + y\psi \tag{7.2}$$

The y-coordinate of the neutral axis is:

$$y_n = -\varepsilon_O/\psi \tag{7.3}$$

The stress in concrete at any fibre is

$$\sigma_c = \begin{cases} E_c\left(1 - \dfrac{y}{y_n}\right)\varepsilon_O & y < y_n \quad (7.4) \\ 0 & y \geqslant y_n \quad (7.5) \end{cases}$$

It may be noted that in Fig. 7.1(b), ε_O is a negative quantity since O is chosen in the compression zone. The stress in any steel layer at coordinate y_s is:

$$\sigma_s = E_s\left(1 - \frac{y_s}{y_n}\right)\varepsilon_O \tag{7.6}$$

Integrating the stresses over the area and taking moment about an axis through O gives:

$$\varepsilon_O\left\{E_c\int_{y_t}^{y_n}\left(1 - \frac{y}{y_n}\right)\,\mathrm{d}A + E_s\,\Sigma\left[A_s\left(1 - \frac{y_s}{y_n}\right)\right]\right\} = N \tag{7.7}$$

$$\varepsilon_O\left\{E_c\int_{y_t}^{y_n}y\left(1 - \frac{y}{y_n}\right)\,\mathrm{d}A + E_s\,\Sigma\left[A_s y_s\left(1 - \frac{y_s}{y_n}\right)\right]\right\} = M \tag{7.8}$$

When the section is subjected to bending moment only, N can be set equal to zero in Equation (7.7), giving the following equation which can be solved for the coordinate y_n, defining the position of the neutral axis:

$$\int_{y_t}^{y_n} (y_n - y)\mathrm{d}A + a\,\Sigma\,[A_s(y_n - y_s)] = 0 \tag{7.9}$$

where $a = E_s/E_c$.

When $N \neq 0$, the neutral axis does not coincide with the centroid of the transformed area. The equation to be solved for y_n is obtained by division of Equation (7.8) by (7.7):

$$\frac{\int_{y_t}^{y_n} y(y_n - y)\mathrm{d}A + a\,\Sigma\,[A_s y_s(y_n - y_s)]}{\int_{y_t}^{y_n} (y_n - y)\mathrm{d}A + a\,\Sigma\,[A_s(y_n - y_s)]} - e = 0 \tag{7.10}$$

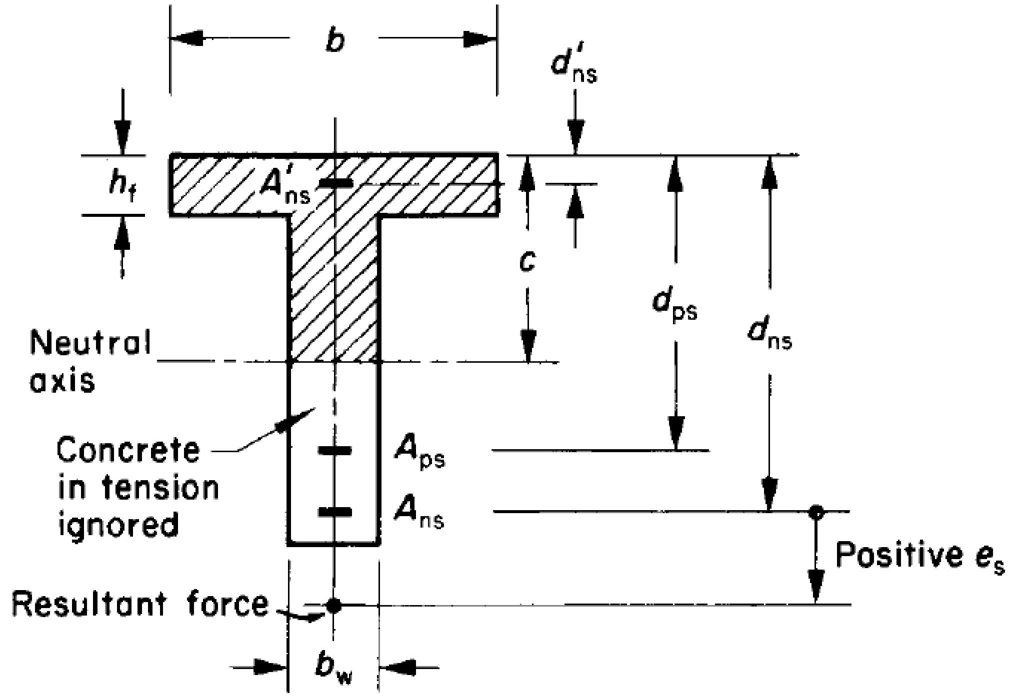


Figure 7.2 Definition of symbols employed in Section 7.4.2.

Consider the case when the section in Fig. 7.2 is subjected to a positive bending moment without an axial force. Application of Equation (7.9) gives the following quadratic equation from which the depth c of the compression zone can be determined:

$$\begin{aligned} &\tfrac{1}{2}b_w c^2 + [h_f(b-b_w) + \alpha_{ns}A_{ns} + \alpha_{ps}A_{ps} + (\alpha_{ns}-1)\,A'_{ns}]c \\ &\quad - [\tfrac{1}{2}\,(b-b_w)h_f^2 + \alpha_{ns}A_{ns}d_{ns} + \alpha_{ps}A_{ps}d_{ps} \\ &\quad + (\alpha_{ns}-1)\,A'_{ns}d'_{ns}] = 0 \quad \text{when } c \geqslant h_f \end{aligned} \tag{7.15}$$

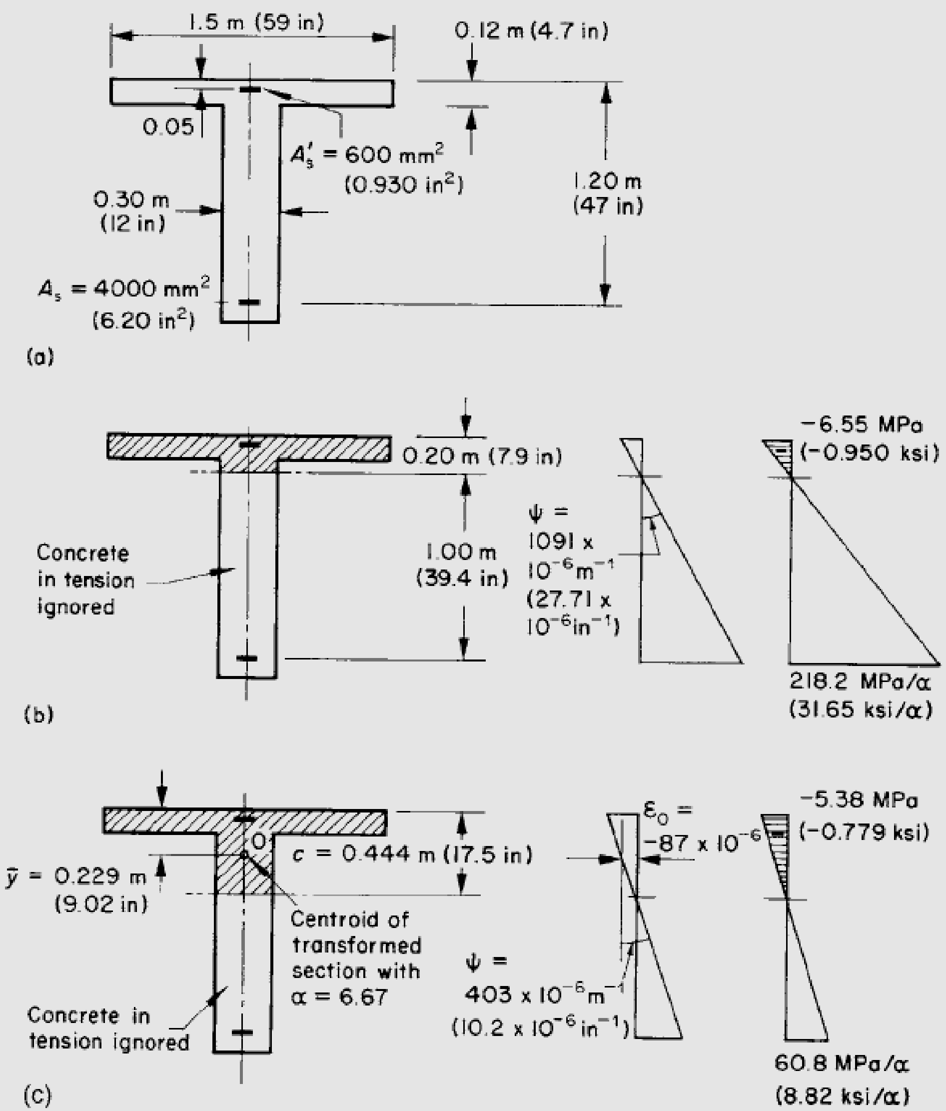
Solution of the quadratic Equation (7.15) gives the depth of the compression zone in a T section subjected to bending moment:

$$c = \frac{-a_2 + \sqrt{(a_2^2 - 4a_1a_3)}}{2a_1} \tag{7.16}$$

where

$$a_1 = b_w/2 \tag{7.17}$$

The T section shown in Fig. 7.7(a) is subjected to a bending moment of 1000 kN-m (8850 kip-in). It is required to find the stress and strain distributions ignoring the concrete in tension. Effects of creep and shrinkage are not considered in this example. The cross-section dimensions are indicated in Fig. 7.7(a); E_c = 30 GPa (4350 ksi); E_s = 200 GPa (29 000 ksi).



$$\alpha = \frac{E_s}{E_c} = \frac{200}{30} = 6.667.$$

In the absence of prestress steel, $A_{ps} = 0$ and the symbols A_{ns} and A'_{ns} have the same meaning as A_s and A'_s.

Substitution in Equations (7.17–19) gives:

$$a_1 = 0.15\,\text{m} \qquad a_2 = 174.07 \times 10^{-3}\,\text{m}^2 \qquad a_3 = -40.812 \times 10^{-3}\,\text{m}^3$$

Equation (7.16) gives the depth of the compression zone

$$c = \frac{-174.07 \times 10^{-3} + \sqrt{[(174.07 \times 10^{-3})^2 + 4(0.15)(40.812 \times 10^{-3})]}}{2 \times 0.15}$$

$$= 0.200\,\text{m} \quad (7.9\text{ in}).$$

The moment of inertia of the transformed section about the centroidal axis (which is the same as the neutral axis):

$$I = 0.3\frac{0.200^3}{3} + (1.5 - 0.3)0.12\left(\frac{0.12^2}{12} + 0.14^2\right)$$

$$+ 6.667(0.004)1.000^2 + 5.667(0.0006)0.15^2$$

$$= 30.54 \times 10^{-3}\,\text{m}^4 \quad (3.53\,\text{ft}^4).$$

Alternatively, if A'_s is ignored, Tables 7.1 and 7.4 can be used giving $c = \bar{y} = 0.202\,\text{m}$ and $I = 30.46 \times 10^{-3}\,\text{m}^4$. The curvature

$$\psi = \frac{1000 \times 10^3}{30 \times 10^9 \times 30.54 \times 10^{-3}} = 1091 \times 10^{-6}\,\text{m}^{-1}\ (28 \times 10^{-6}\,\text{in}^{-1}).$$

$$\text{Stress at the top fibre} = 30 \times 10^9 \times 1091 \times 10^{-6}(-0.200)$$

$$= -6.55\text{ MPa} \quad (-0.950\,\text{ksi}).$$

$$\text{Stress in steel} = 200 \times 10^9 \times 1091 \times 10^{-6}(1.000)$$

$$= 218.2\text{ MPa} \quad (31.65\,\text{ksi}).$$

Strain and stress distributions are shown in Fig. 7.7(b).

$$c = \frac{-a_2 + \sqrt{(a_2^2 - 4a_1a_3)}}{2a_1} \tag{7.16}$$

where

$$a_1 = b_w/2 \tag{7.17}$$

Solve Example 7.1, assuming that the section is subjected to a bending moment of 1000 kN-m (8850 kip-in) and a normal force of –800 kN (–180 kip) at a point 1.0 m (40 in) below the top edge of the section. The cross-section dimensions and moduli of elasticity of steel and concrete are the same as in Example 7.1 (Fig. 7.7(a)).

The resultant force on the section is a normal force of –800 kN at a distance 0.25 m above the top edge. Thus, $e_s = -(0.25 + 1.20) = -1.45$ m. Substituting in Equation (7.20) and solving for c, the height of the compression zone, gives:

$$c = 0.444\,\text{m} \qquad (17.5\,\text{in}).$$

The effective area is shown in Fig. 7.7(c). The transformed section is composed of the area of concrete in compression plus $\alpha(A_s + A'_s)$ with $\alpha = 200/30 = 6.667$. The distance between point O, the centroid of the transformed section, and the top edge is calculated to be $\overline{y} = 0.229$ m (Fig. 7.7(c)). The area and moment of inertia of the transformed section about an axis through its centroid

$$A = 0.3073\,\text{m}^2 \quad I = 31.73 \times 10^{-3}\,\text{m}^4.$$

If A'_s is ignored, Tables 7.1, 7.3 and 7.4 may be used, giving:

$$c = 0.46\,\text{m} \quad \overline{y} = 0.24\,\text{m} \quad I = 30 \times 10^{-3}\,\text{m}^4.$$

Transform the given bending moment and normal force into an equivalent system of a normal force N at the centroid of the transformed section combined with a bending moment M.

$$N = -800\,\text{kN}$$

$$M = 1000 \times 10^3 - 800 \times 10^3(1.000 - 0.229)$$

$$= 383.2\,\text{kN m} \quad (3400\,\text{kip in}).$$

$$\begin{aligned}
&b_w(\tfrac{1}{2}c^2)(d_{ns} - \tfrac{1}{3}c) \\
&+ (b - b_w)h_f[c(d_{ns} - \tfrac{1}{2}h_f) - \tfrac{1}{2}h_f(d_{ns} - \tfrac{2}{3}h_f)] \\
&+ (\alpha_{ns} - 1)A'_{ns}(c - d'_{ns})(d_{ns} - d'_{ns}) - \alpha_{ps}A_{ps}(d_{ps} - c)(d_{ns} - d_{ps}) \\
&+ e_s[b_w(\tfrac{1}{2}c^2) + (b - b_w)h_f(c - \tfrac{1}{2}h_f) + (\alpha_{ns} - 1)A'_{ns}(c - d'_{ns}) \\
&- \alpha_{ps}A_{ps}(d_{ps} - c) - \alpha_{ns}A_{ns}(d_{ns} - c)] = 0 \qquad \text{when } c \geqslant h_f
\end{aligned} \qquad (7.20)$$

The strain at O and the curvature (Equation (2.16))

$$\begin{Bmatrix} \varepsilon_O \\ \psi \end{Bmatrix} = \frac{1}{E_{ref}} \begin{Bmatrix} N/A \\ M/I \end{Bmatrix}$$

$$\varepsilon_O = \frac{1}{30 \times 10^9} \frac{-800 \times 10^3}{0.3073} = -87 \times 10^{-6}$$

$$\psi = \frac{1}{30 \times 10^9} \frac{383.2 \times 10^3}{31.73 \times 10^{-3}}$$

$$= 403 \times 10^{-6} \text{m}^{-1} \quad (10.2 \times 10^{-6} \text{ in}^{-1}).$$

Stress at the top fibre $= 30 \times 10^9[-87 + 403(-0.229)]10^{-6} = -5.38$ MPa.
Stress in bottom steel $= 200 \times 10^9[-87 + 403 \times 0.971]10^{-6} = 60.8$ MPa.

The strain and stress distributions are shown in Fig. 7.7(c).

7.5 RC Section w/o prestress

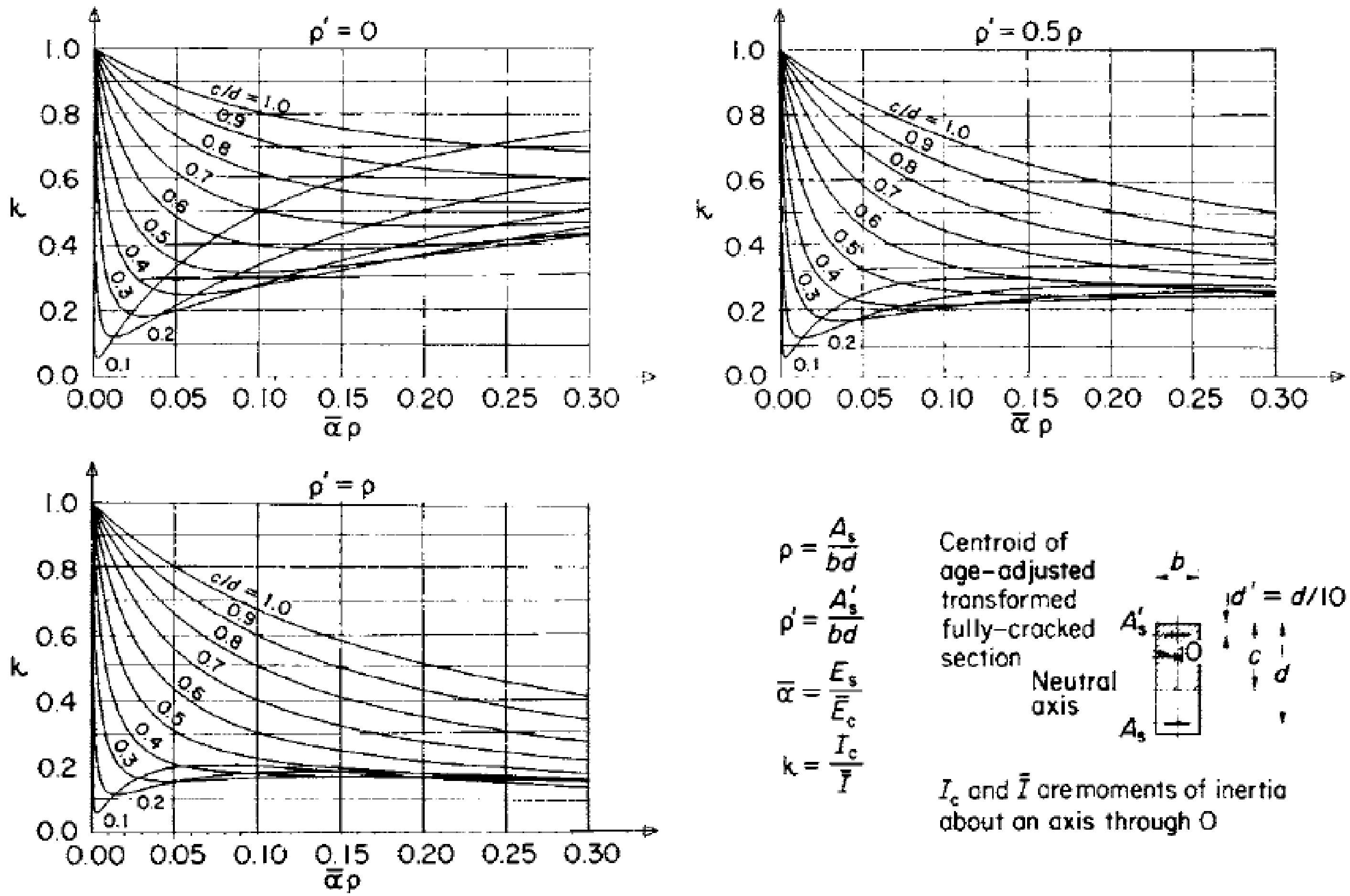


Figure 7.8 Curvature reduction κ for a fully cracked rectangular section.

A reference point O is chosen at the *centroid of the age-adjusted transformed section*, composed of the area of the compression zone plus $\bar{a}(t, t_0)$ times the area of steel (Figs. 7.8 and 7.9(a)); where $\bar{a}(t, t_0) = E_s/\bar{E}_c(t, t_0)$, with $\bar{E}_c(t, t_0)$ the age-adjusted modulus of elasticity of concrete (see Equation (1.31)). Creep and shrinkage produce the following changes in axial strain at O, in curvature and in stresses:

$$\Delta\varepsilon_O = \eta[\varphi(t, t_0)(\varepsilon_O + \psi y_c) + \varepsilon_{cs}(t, t_0)] \tag{7.26}$$

$$\Delta\psi = \kappa\left[\varphi(t, t_0)\left(\psi + \varepsilon_O\frac{y_c}{r_c^2}\right) + \varepsilon_{cs}(t, t_0)\,\frac{y_c}{r_c^2}\right] \tag{7.27}$$

$$\Delta\sigma_c = \bar{E}_c(t, t_0)[-\varphi(t, t_0)(\varepsilon_O + \psi y) - \varepsilon_{cs}(t, t_0) + \Delta\varepsilon_O + \Delta\psi y] \tag{7.28}$$

$$\Delta\sigma_s = E_s(\Delta\varepsilon_O + \Delta\psi y_s) \tag{7.29}$$

where

- ε_O, ψ = the axial strain at O and the curvature at time t_0 immediately after application of M and N (Fig. 7.9(b))
- $\varphi(t, t_0)$ = coefficient for creep at time t for age at loading t_0
- $\varepsilon_{cs}(t, t_0)$ = the shrinkage that would occur in concrete if it were free, during the period $(t - t_0)$
- y_c = the y-coordinate of the centroid of the concrete area in compression (based on the stress distribution at age t_0). y_c is measured downwards from O

$$r_c^2 = I_c/A_c \tag{7.30}$$

$$\eta = A_c/\overline{A} \tag{7.31}$$

$$\kappa = I_c/\bar{I} \tag{7.32}$$

Example 7.3 Cracked T section: creep and shrinkage effects

Find the changes in strain and stress distributions due to creep and shrinkage in the cross-section of Example 7.2 (Fig. 7.7(a)). Consider that the result of Example 7.2 represents the stress and strain at age t_0 and use the following data:

$$\varphi(t, t_0) = 2.5 \quad \chi(t, t_0) = 0.75 \quad \varepsilon_{cs}(t, t_0) = -300 \times 10^{-6}.$$

The effective area of the section is considered unchangeable with time. Thus, using the result of Example 7.2, the depth of the effective part of the section $c = 0.444$ and the stress distribution at time t_0 is as shown in Fig. 7.7(c).

The area of the effective part of concrete, $A_c = 0.2766\,\mathrm{m}^2$. The distance of the centroid of A_c from top, $\bar{y}_c = 0.138\,\mathrm{m}$ (Fig. 7.9(a)).

The age-adjusted modulus of elasticity of concrete (Equation (1.31))

$$\bar{E}_c(t, t_0) = \frac{30 \times 10^9}{1 + 0.75 \times 2.5} = 10.43\,\text{GPa} \quad (1500\,\text{ksi})$$

$$\bar{\alpha}(t, t_0) = \frac{200}{10.43} = 19.17.$$

The area of a transformed section composed of A_c plus $\bar{\alpha}(A_s + A'_s)$ is

$$\bar{A} = 0.3648\,\text{m}^2 \quad (560\,\text{in}^2).$$

For use of Equations (7.26–31), a reference point O must be chosen at the centroid of the transformed effective area. This centroid is calculated and is found to be at $\bar{y} = 0.358\,\text{m}$ below the top edge.

The moment of inertia of A_c about an axis through O is

$$I_c = 17.56 \times 10^{-3}\,\text{m}^4 \quad r_c^2 = I_c/A_c = 0.0635\,\text{m}^2.$$

The moment of inertia of the transformed section is

$$\bar{I} = 73.01 \times 10^{-3}\,\text{m}^4.$$

$$\Delta\varepsilon_O = \eta[\varphi(t, t_0)(\varepsilon_O + \psi y_c) + \varepsilon_{cs}(t, t_0)] \tag{7.26}$$

$$\Delta\psi = \kappa\left[\varphi(t, t_0)\left(\psi + \varepsilon_O \frac{y_c}{r_c^2}\right) + \varepsilon_{cs}(t, t_0)\frac{y_c}{r_c^2}\right] \tag{7.27}$$

$$\Delta\sigma_c = \bar{E}_c(t, t_0)[-\varphi(t, t_0)(\varepsilon_O + \psi y) - \varepsilon_{cs}(t, t_0) + \Delta\varepsilon_O + \Delta\psi y] \tag{7.28}$$

$$\Delta\sigma_s = E_s(\Delta\varepsilon_O + \Delta\psi y_s) \tag{7.29}$$

$$r_c^2 = I_c/A_c \tag{7.30}$$

$$\eta = A_c/\bar{A} \tag{7.31}$$

$$\kappa = I_c/\bar{I} \tag{7.32}$$

The axial strain and curvature reduction factors (Equations (7.30) and (7.31)) are

$$\eta = \frac{0.2766}{0.3648} = 0.7582 \qquad \kappa = \frac{17.56}{73.01} = 0.2404.$$

If the area A'_s is ignored, Tables 7.3 to 7.5 can be used to calculate $\overline{y}$, $\bar{I}$ and κ.

The y-coordinate of the centroid of A_c (see Fig. 7.9(a)) is

$$y_c = -(0.358 - 0.138) = -0.220\,\text{m}.$$

The strain and stress distributions at time t_0 are shown in Fig. 7.9(b) (copied from the result of Example 7.2, Fig. 7.7(c)):

$$\varepsilon_O = -35 \times 10^{-6} \qquad \psi = 403 \times 10^{-6}\,\text{m}^{-1}.$$

(Note that the reference point O is lower in Fig. 7.9(b) compared to Fig. 7.7(c).)

Changes in strain at O and in curvature due to creep and shrinkage (Equations (7.26) and (7.27)) are

$$\Delta\varepsilon_O = 0.7582\{2.5[-35 + 403(-0.22)]10^{-6} - 300 \times 10^{-6}\} = -462 \times 10^{-6}$$

$$\Delta\psi = 0.2404\left[2.5\left(403 - 35\,\frac{(-0.22)}{0.0635}\right)10^{-6} - 300 \times 10^{-6}\,\frac{(-0.22)}{0.0635}\right]$$

$$= 565 \times 10^{-6}\,\mathrm{m}^{-1}$$

Changes in concrete stresses due to creep and shrinkage (Equation (7.28)) are at the top edge;

$$(\Delta\sigma_c)_{top} = 10.43 \times 10^9\ \{-2.5[-35 + 403(-0.358)]$$

$$+ 300 - 462 + 565(-0.358)\}\ 10^{-6}$$

$$= 0.876\,\mathrm{MPa} \quad (0.127\,\mathrm{ksi})$$

at the lower edge of the effective area;

$$(\Delta\sigma_c)_{\text{at 0.444m below top edge}} = 10.43 \times 10^9(300 - 462 + 565 \times 0.086)10^{-6}$$

$$= -1.182\,\mathrm{MPa} \quad (-0.171\,\mathrm{ksi}).$$

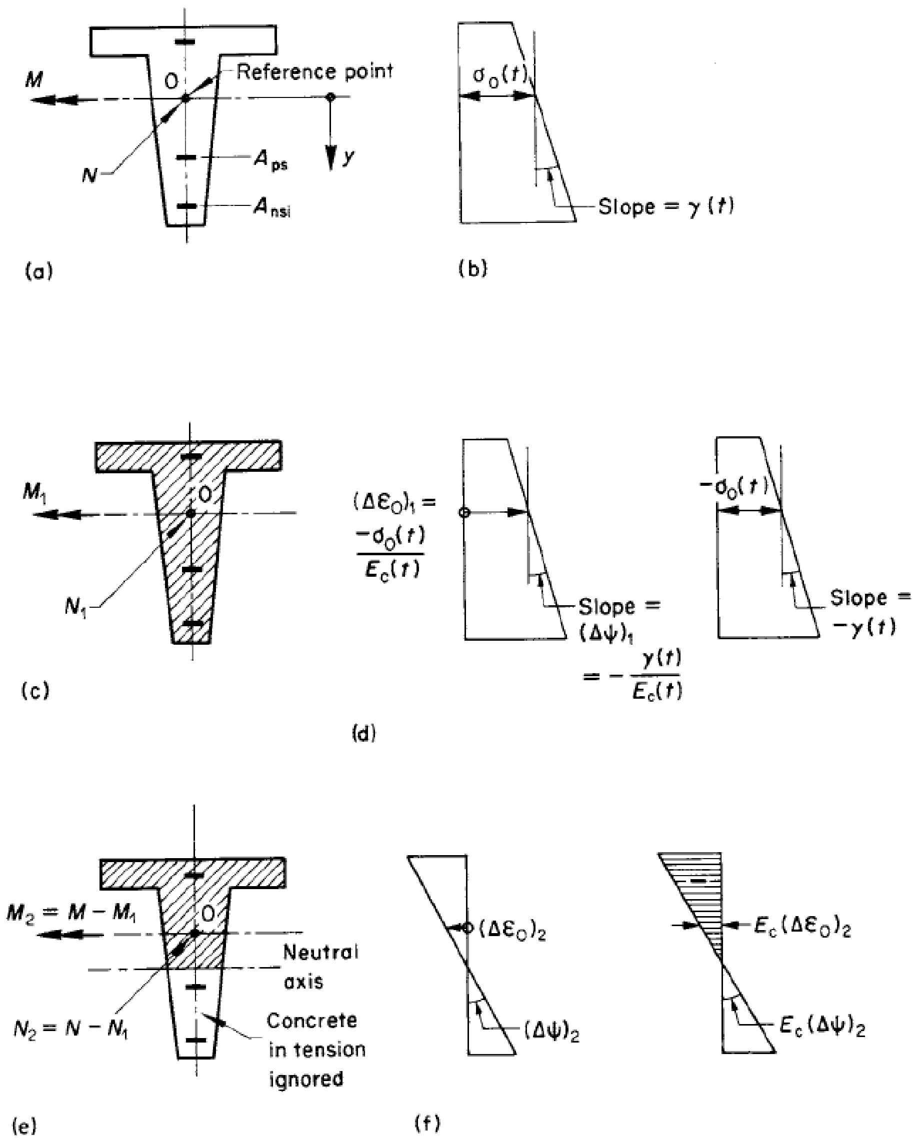
Changes in stress in steel due to creep and shrinkage (Equation (7.29)) are:

$$(\Delta\sigma_s)_{bot} = 200 \times 10^9\ (-462 + 565 \times 0.842)10^{-6} = 2.8\,\mathrm{MPa}\ (0.41\,\mathrm{ksi})$$

$$(\Delta\sigma_s)_{top} = 200 \times 10^9\ (-462 - 565 \times 0.308)10^{-6}$$

$$= -127.2\,\mathrm{MPa} \quad (-18.45\,\mathrm{ksi}).$$

7.6 Partial prestressed sections



Uncracked section

?

Fully cracked section

(1) M_1 and N_1 applied on uncracked section;
(2) M_2 and N_2 applied on a fully cracked section.

The strain changes in the two stages are given by Fig. 7.10(d) and (f):

$$(\Delta\varepsilon)_1 = (\Delta\varepsilon_O)_1 + (\Delta\psi)_1 y \tag{7.35}$$

$$(\Delta\varepsilon)_2 = (\Delta\varepsilon_O)_2 + (\Delta\psi)_2 y \tag{7.36}$$

The total instantaneous change in strain due to M and N is

$$\Delta\varepsilon = (\Delta\varepsilon)_1 + (\Delta\varepsilon)_2 \tag{7.37}$$

The stress produced in stage 1 is simply equal to the stress in Fig. 7.10(b) reversed in sign, as shown in Fig. 7.10(d). The corresponding strain in stage 1 is obtained by division of stress values by $E_c(t)$; the strain distribution in stage 1 is also shown in Fig. 7.10(d). Thus, the stress in concrete is zero after application of M_1 and N_1. The final stress in concrete is given by the analysis of the effects of M_2 and N_2 only (Fig. 7.10(f)). It should, however, be noted that M_1 and N_1 bring to zero the stress in concrete but not in steel.

The values of M_1 and N_1 are equal and opposite to the resultants of stresses $\sigma_c(t)$ on the concrete and α times this stress on steel, with $\sigma_c(t)$ being the stress existing before application of M and N (Fig. 7.10(b)). M_1 and N_1 are sometimes referred[1] to as decompression forces, because $\sigma_c(t)$ is generally compressive. (In all the stress and strain diagrams in Fig. 7.10, the variables ε_O, ψ, σ_O and γ are plotted as positive quantities.) The decompression forces are given by:

$$N_1 = -\int \sigma \mathrm{d}A \tag{7.38}$$

$$M_1 = -\int \sigma y \mathrm{d}A \tag{7.39}$$

7.6 Partial prestressed sections

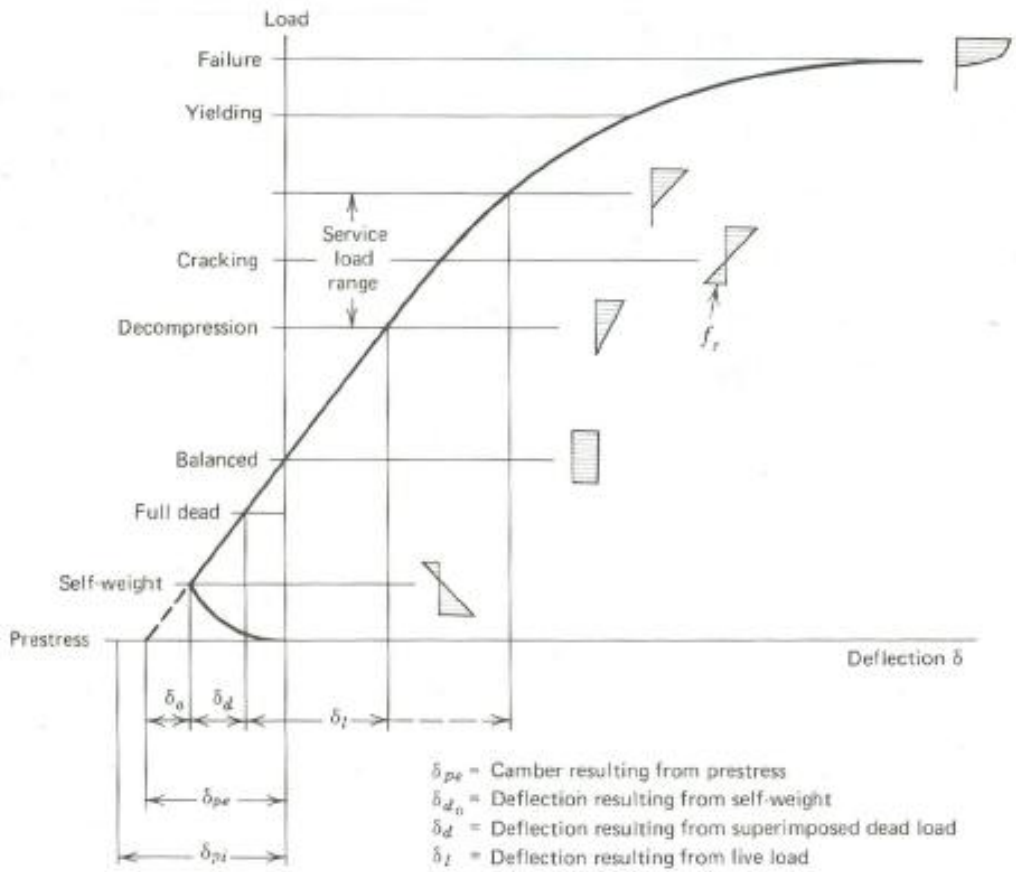


FIGURE 4.1 Load-deflection curve for typical beam.

When the stress varies over the full height of the section as *one* straight line, the integrals in Equations (7.38) and (7.39) may be eliminated (see Equations (2.2–8)):

$$N_1 = -(A\sigma_O + B\gamma) \tag{7.40}$$

$$M_1 = -(B\sigma_O + I\gamma) \tag{7.41}$$

where A is the area of a transformed section composed of the full concrete area plus a times the area of steel, prestressed and non-prestressed;

If O is chosen at the centroid of the above-mentioned transformed area, $B = 0$ and Equations (7.40) and (7.41) become:

$$N_1 = -A\sigma_O \tag{7.43}$$

$$M_1 = -I\gamma \tag{7.44}$$

The changes in axial strain and curvature due to M_1 and N_1 simply are:

$$(\Delta\varepsilon_O)_1 = -\frac{1}{E_c}\sigma_O \tag{7.45}$$

$$(\Delta\psi)_1 = -\frac{1}{E_c}\gamma \tag{7.46}$$

Example 7.4 Pre-tensioned tie before and after cracking

Fig. 7.11 shows a square cross-section of a precast pretensioned tie. Immediately before transfer, the force in the tendon is 1100 kN (247 kip), the age of concrete t_0 and no dead load is simultaneously applied with the prestress. At a much older age t, a normal tensile force 1200 kN (270 kip) is applied at the centre of the section. It is required to find the axial strain and stress in the concrete and steel immediately after prestressing, and just before and after application of the 1200 kN force. The following data are given: the moduli of elasticity of concrete and steel, $E_c(t_0) = 24$ GPa (3480 ksi); $E_c(t) = 35$ GPa (5076 ksi); $E_s =$ 200 GPa (29 000 ksi) (for prestressed and non-prestressed reinforcements); creep coefficient $\varphi(t, t_0) = 2.4$; aging coefficient $\chi(t, t_0) = 0.80$; during the period $(t - t_0)$, the reduced relaxation $\Delta\bar{\sigma}_{pr} = -90$ MPa (−13 ksi) and the free shrinkage $\varepsilon_{cs}(t, t_0) = -270 \times 10^{-6}$.

(a) Strain and stress immediately after transfer

The area of the transformed section is composed of $A_c + \alpha(A_{ps} + A_{ns})$, where $\alpha = E_s/E_c(t_0)$.

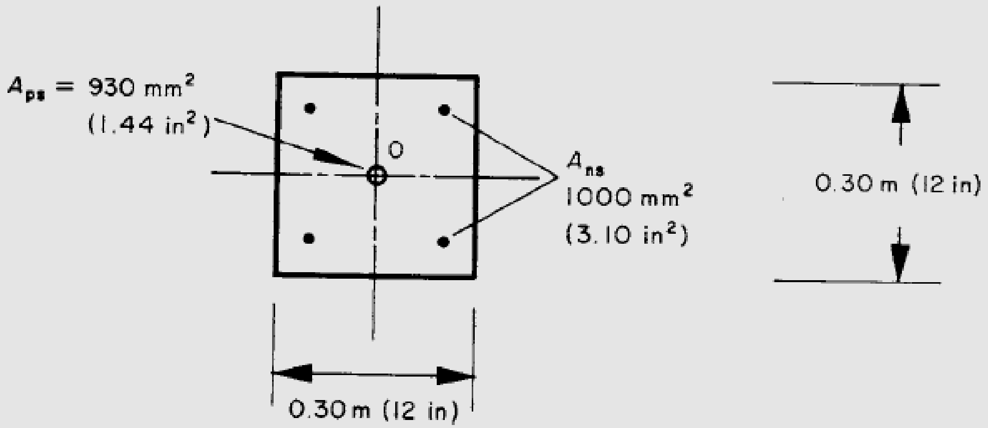


Figure 7.11 Cross-section of a partially prestressed tie analysed for strain and stress in Example 7.4.

$$A_c = 0.30 \times 0.30 - (930 + 1000)10^{-6} = 0.0881\,\mathrm{m}^2$$

$$\alpha = 200/24 = 8.33$$

$$A = 0.0881 + 8.33(930 + 1000)10^{-6} = 0.1042\,\mathrm{m}^2.$$

The axial strain at transfer (Equation (2.33)) is

$$\varepsilon(t_0) = -\frac{1100 \times 10^3}{24 \times 10^9 \times 0.1042} = -440 \times 10^{-6}.$$

The stress in concrete (Equation (2.35)) is

$$\sigma(t_0) = 24 \times 10^9(-440 \times 10^{-6}) = -10.559\,\mathrm{MPa} \quad (-1.532\,\mathrm{ksi}).$$

The stress in non-prestressed and in prestressed steel is

$$\sigma_{ns} = 200 \times 10^9(-440 \times 10^{-6}) = -88.0\,\mathrm{MPa} \quad (-12.8\,\mathrm{ksi})$$

$$\sigma_{ps} = \frac{1100 \times 10^3}{930 \times 10^{-6}} + 200 \times 10^9(-440 \times 10^{-6})$$

$$= 1094.8\,\mathrm{MPa} \quad (158.8\,\mathrm{ksi}).$$

(b) Changes in strain and in stress due to creep, shrinkage and relaxation

The transformed section to be used here is composed of $A_c + \overline{\alpha}(A_{ps} + A_{ns})$; where $\overline{\alpha} = E_s/\overline{E}_c(t, t_0)$

Using Equation (1.31)

$$\overline{E}_c = \frac{24 \times 10^9}{1 + 2.4 \times 0.8} = 8.215\,\mathrm{GPa} \quad (1192\,\mathrm{ksi})$$

$$\overline{\alpha} = \frac{200}{8.215} = 24.33.$$

The transformed area

$$\overline{A} = 0.0881 + 24.33(930 + 1000)10^{-6} = 0.1351\,\text{m}^2.$$

The artificial force that would be necessary to prevent strain due to creep, shrinkage and relaxation (Equations (2.41–44)) is

$$\Delta N = -8.215 \times 10^9 \times 2.4 \times 0.0881 \quad (-440 \times 10^{-6})$$
$$-8.215 \times 10^9(-270 \times 10^{-6})0.0881 + 930 \times 10^{-6}(-90 \times 10^6)$$
$$= 0.8759 \times 10^6\ \text{N} \quad (196.9\,\text{kip}).$$

The change in axial strain in concrete when the restraint is removed (Equation (2.40)) is

$$\Delta\varepsilon = -\frac{0.8758 \times 10^6}{8.215 \times 10^9 \times 0.1351} = -789 \times 10^{-6}.$$

The change in concrete stress (Equations (2.45) and (2.46)) is

$$\Delta\sigma = -8.215 \times 10^9[2.4(-440 \times 10^{-6}) - 270 \times 10^{-6}]$$
$$+ 8.215 \times 10^9(-789 \times 10^{-6})$$
$$= 4.407\,\text{MPa} \quad (0.6392\,\text{ksi}).$$

Changes in stress in non-prestressed and prestressed steels (Equations (2.47) and (2.48)) are

$$\Delta\sigma_{\text{ns}} = 200 \times 10^9(-789 \times 10^{-6}) = -157.9\,\text{MPa} \quad (-22.90\,\text{ksi})$$
$$\Delta\sigma_{\text{ps}} = -90 \times 10^6 + 200 \times 10^9(-789 \times 10^{-6})$$
$$= -247.9\,\text{MPa} \quad (-35.95\,\text{ksi}).$$

The stress in concrete after creep, shrinkage and relaxation is

$$\sigma(t) = -10.559 + 4.407 = -6.152\,\text{MPa} \quad (-0.8923\,\text{ksi}).$$

(c) Changes in strain and stress in the decompression stage

The transformed area to be used here is composed of $A_c + \alpha(A_{ps} + A_{ns})$; where $\alpha = E_s/E_c(t)$

$$\alpha = 200/35 = 5.71.$$

The transformed area is

$$A = 0.0881 + 5.71(930 + 1000)10^{-6} = 0.0991\,\text{m}^2.$$

The decompression force (Equation (7.43)) is

$$N_1 = -0.0991(-6.152 \times 10^6) = 609.8\,\text{kN} \quad (137.1\,\text{kip}).$$

The change in strain due to N_1 (Equation (7.45)) is

$$(\Delta\varepsilon)_1 = \frac{6.152 \times 10^6}{35 \times 10^9} = 176 \times 10^{-6}.$$

The change in stress in the two types of reinforcement is

$$(\Delta\sigma_{ns})_1 = (\Delta\sigma_{ps})_1 = 200 \times 10^9 \times 176 \times 10^{-6} = 35.2\,\text{MPa} \quad (5.11\,\text{ksi}).$$

(d) Changes in strain and stress in the cracking stage

All the concrete area will be in tension; thus, the transformed area is composed of $\alpha(A_{ps} + A_{ns})$, with α the same as in (c) above.

Transformed area is

$$A = 5.71(930 + 1000)10^{-6} = 0.0110\,\text{m}^2.$$

Force producing cracking (Equation (7.34)) is

$$N_2 = 1200 - 609.8 = 590.2\,\text{kN} \quad (113\,\text{kip}).$$

The change in strain due to N_2 (Equation (2.16)) is

$$(\Delta\varepsilon)_2 = \frac{590.2 \times 10^3}{35 \times 10^9 \times 0.0110} = 1530 \times 10^{-6}.$$

The change in stress in any of the two types of reinforcement is

$$(\Delta\sigma_{ns})_2 = (\Delta\sigma_{ps})_2 = 200 \times 10^9 \times 1530 \times 10^{-6}.$$

$$= 306.0\,\text{MPa} \quad (44.4\,\text{ksi}).$$

$$(\Delta\varepsilon)_2 = \frac{590.2 \times 10^3}{200 \times 10^9 \times 1930 \times 10^{-6}} = 1530 \times 10^{-6}.$$

The stress in non-prestressed steel is

$$-88.0 - 157.9 + 35.2 + 306.0 = 95.3\,\text{MPa} \quad (13.8\,\text{ksi}$$

The stress in prestressed steel is

$$1094.8 - 247.9 + 35.2 + 306.0 = 1188.1\,\text{MPa} \quad (172.3\,\text{k}$$

The strain in the non-prestressed steel immediately before cracking the sum of strain values calculated in steps (a), (b) and (c) = $(-440 - 7$ $+ 176)10^{-6} = -1053 \times 10^{-6}$. At cracking, the change in strain in the prestressed or non-prestressed steel is $(\Delta\varepsilon)_2 = 1530 \times 10^{-6}$. Thus, the strain in the non-prestressed steel after cracking is 477×10^{-6}. At this stage the concrete is not participating in resisting any force. The strains in concrete and steel are no more compatible and slip must occur in the vicinity of cracks. This will be discussed further in Chapter 8.

$$\sigma_{ns} = 200 \times 10^9(-440 \times 10^{-6}) = -88.0\,\text{MPa} \quad (-12.8\,\text{ksi})$$

$$\sigma_{ps} = \frac{1100 \times 10^3}{930 \times 10^{-6}} + 200 \times 10^9(-440 \times 10^{-6})$$

$$= 1094.8\,\text{MPa} \quad (158.8\,\text{ksi}).$$

$$\Delta\sigma_{ns} = 200 \times 10^9(-789 \times 10^{-6}) = -157.9\,\text{MPa} \quad (-22.90\,\text{ksi})$$

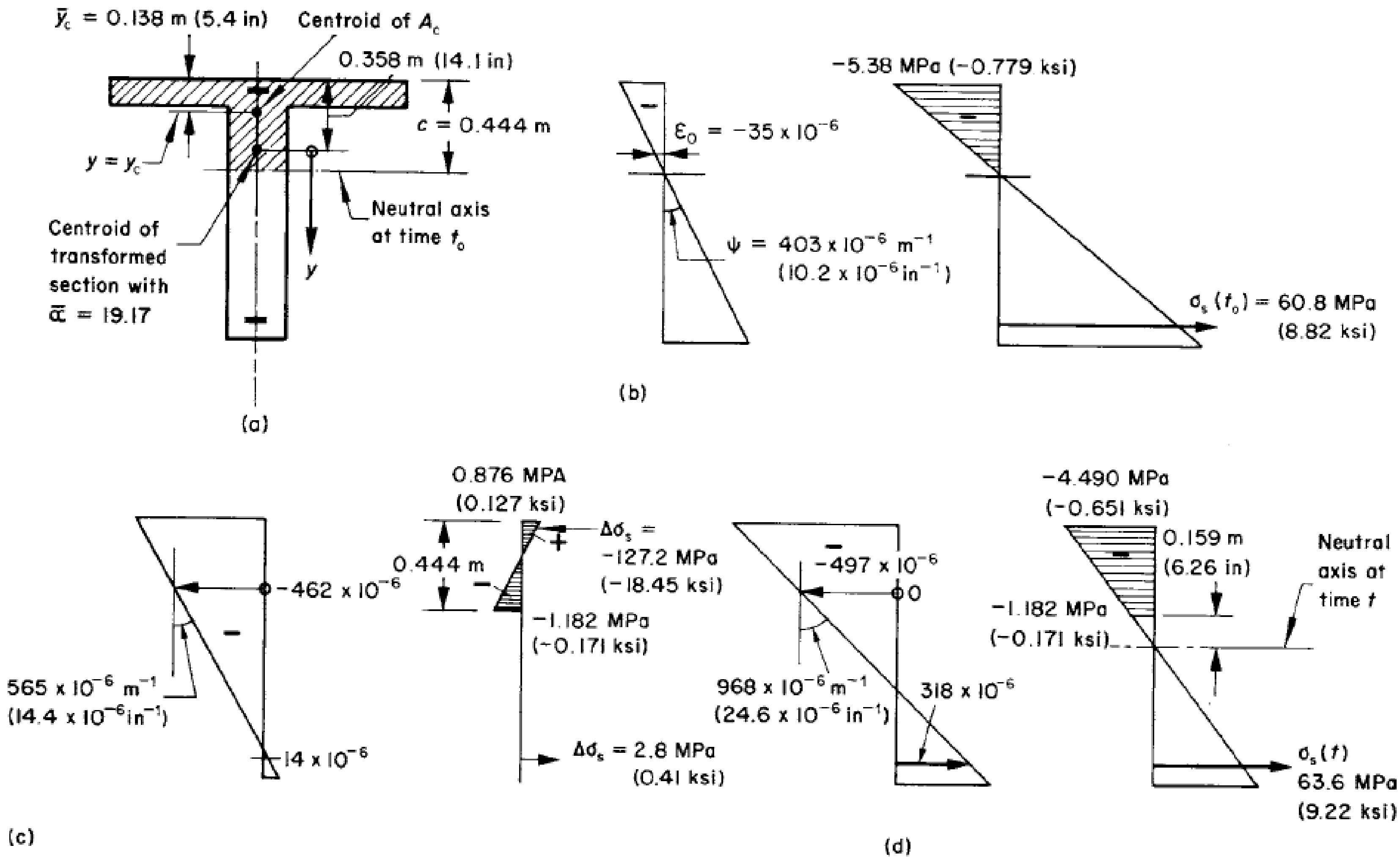
$$\Delta\sigma_{ps} = -90 \times 10^6 + 200 \times 10^9(-789 \times 10^{-6})$$

$$= -247.9\,\text{MPa} \quad (-35.95\,\text{ksi}).$$

$$(\Delta\sigma_{ns})_1 = (\Delta\sigma_{ps})_1 = 200 \times 10^9 \times 176 \times 10^{-6} = 35.2\,\text{MPa} \quad (5.11\,\text{ksi}).$$

$$(\Delta\sigma_{ns})_2 = (\Delta\sigma_{ps})_2 = 200 \times 10^9 \times 1530 \times 10^{-6}.$$

$$= 306.0\,\text{MPa} \quad (44.4\,\text{ksi}).$$



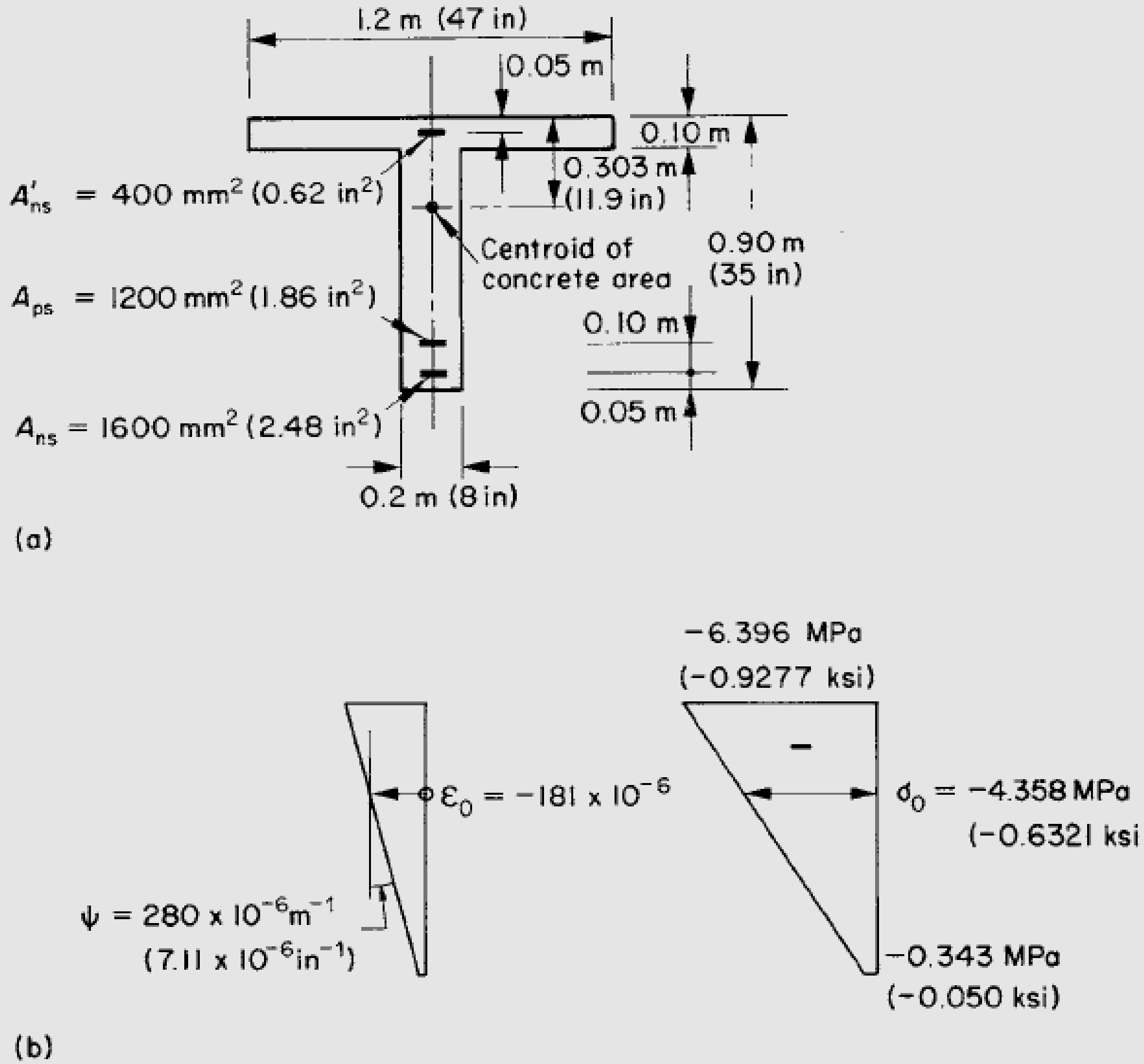
$\bar{y}_c$ = 0.138 m (5.4 in)
Centroid of A_c
0.358 m (14.1 in)
c = 0.444 m
y = y_c
Neutral axis at time t_0
Centroid of transformed section with $\bar{\alpha}$ = 19.17
y
(a)
ε_0 = −35 x 10^{-6}
ψ = 403 x 10^{-6} m^{-1} (10.2 x 10^{-6} in^{-1})
−5.38 MPa (−0.779 ksi)
$\sigma_s(t_0)$ = 60.8 MPa (8.82 ksi)
(b)
−462 x 10^{-6}
565 x 10^{-6} m^{-1} (14.4 x 10^{-6} in^{-1})
14 x 10^{-6}
0.876 MPA (0.127 ksi)
0.444 m
$\Delta\sigma_s$ = −127.2 MPa (−18.45 ksi)
−1.182 MPa (−0.171 ksi)
$\Delta\sigma_s$ = 2.8 MPa (0.41 ksi)
(c)
−497 x 10^{-6}
O
968 x 10^{-6} m^{-1} (24.6 x 10^{-6} in^{-1})
318 x 10^{-6}
−4.490 MPa (−0.651 ksi)
0.159 m (6.26 in)
−1.182 MPa (−0.171 ksi)
Neutral axis at time t
$\sigma_s(t)$ 63.6 MPa (9.22 ksi)
(d)

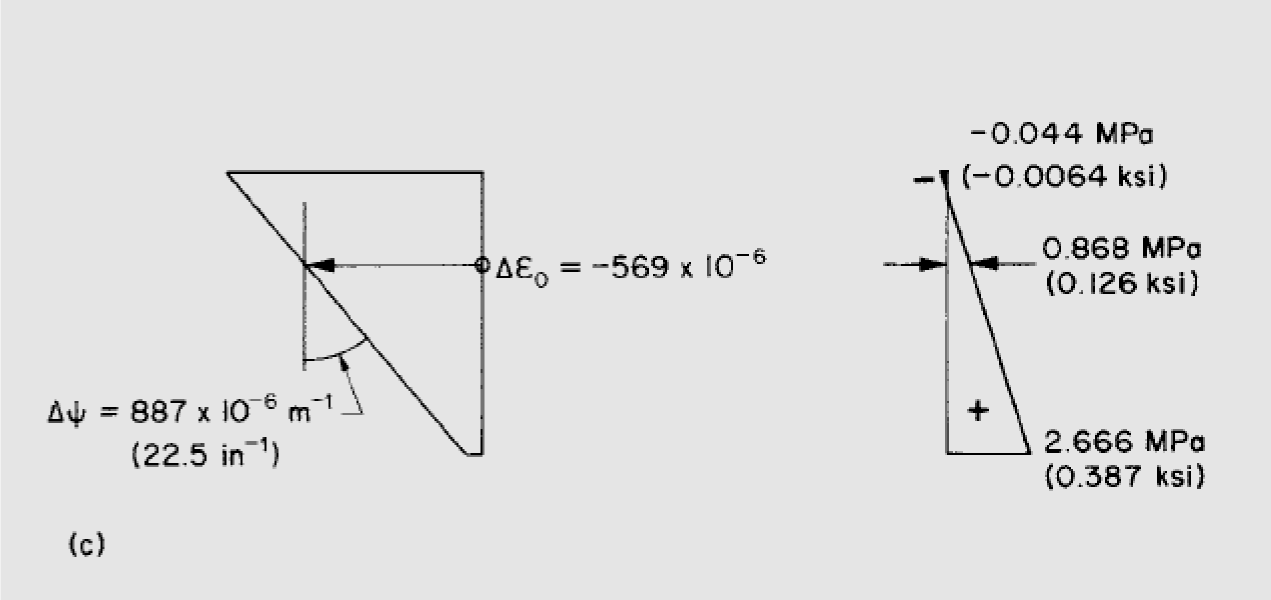
Example 7.5 Pre-tensioned section in flexure: live-load cracking

Fig. 7.12(a) shows the cross-section of a pre-tensioned partially prestressed beam. A 700 kN-m (6200 kip-in) bending moment due to a dead load is introduced at age t_0 at the same time as the prestress transfer. This bending moment includes the effect of the superimposed dead load introduced shortly after transfer, but is considered here as if it were applied simultaneously with the prestress transfer. At time t, long after t_0, a live load is applied, producing a bending moment of 400 kN-m (3540 kip-in). Find the strain and stress distributions immediately after application of the live load bending moment, given the following data.

Tension in prestressed tendon before transfer = 1250 kN (281 kip); moduli of elasticity of concrete at ages t_0 and t, $E_c(t_0) = 24$ GPa (3480 ksi) and $E_c(t) = 30$ GPa (4350 ksi); $E_s = 200$ GPa (29 000 ksi) for all reinforcements; $\varphi(t, t_0) = 2.0$; $\chi(t, t_0) = 0.8$; reduced relaxation for the period $(t - t_0) = -90$ MPa (−13 ksi); shrinkage during the same period, $\varepsilon_{cs}(t, t_0) = -300 \times 10^{-6}$.

As in Example 7.4, the analysis may be done in five parts:


1.2 m (47 in)
0.05 m
0.10 m
0.303 m
(11.9 in)
A'_{ns} = 400 mm² (0.62 in²)
Centroid of concrete area
0.90 m
(35 in)
A_{ps} = 1200 mm² (1.86 in²)
0.10 m
A_{ns} = 1600 mm² (2.48 in²)
0.05 m
0.2 m (8 in)
(a)
−6.396 MPa
(−0.9277 ksi)
$\varepsilon_0 = -181 \times 10^{-6}$
$\sigma_0 = -4.358$ MPa
(−0.6321 ksi)
$\psi = 280 \times 10^{-6}\text{m}^{-1}$
$(7.11 \times 10^{-6}\text{in}^{-1})$
−0.343 MPa
(−0.050 ksi)
(b)



$\Delta\varepsilon_0 = -569 \times 10^{-6}$
$\Delta\psi = 887 \times 10^{-6}\ m^{-1}$
$(22.5\ in^{-1})$
(c)
−0.044 MPa
(−0.0064 ksi)
0.868 MPa
(0.126 ksi)
2.666 MPa
(0.387 ksi)
−
+

(a) Strain and stress immediately after transfer

The calculations in this part follow the procedure presented in Section 2.3 and applied in Example 2.2. Thus, here only the results of the calculations are presented (Fig. 7.12(b)). The stress in the bottom non-prestressed reinforcement, $\sigma_{ns} = -5.6\,\text{MPa}$ and in the prestressed steel, $\sigma_{ps} = 1030.5\,\text{MPa}$.

(b) Changes in strain and in stress due to creep, shrinkage and relaxation

The analysis for this part follows the method discussed in Section 2.5 and applied in Example 2.2. The results are shown in Fig. 7.12(c). The changes in stress in the bottom non-prestressed steel, $\Delta\sigma_{ns} = -16.8\,\text{MPa}$ and in the prestress steel, $\Delta\sigma_{ps} = -124.5\,\text{MPa}$.

After occurrence of the time-dependent changes, the distribution of stress $\sigma(t)$ becomes as shown in Fig. 7.13(b).

(c) Changes in strain and stress in the decompression stage

The transformed area to be used here is composed of A_c plus α times the area of all reinforcements; where $\alpha = E_s/E_c(t)$; A_c = area of concrete section = 0.2768 m^2:

$$\alpha = 200/30 = 6.667.$$

Choose reference point O at the centroid of A_c, at 0.303 m below the top edge (Fig. 7.13(a)). The moment of inertia of A_c about an axis through O = 21.78×10^{-3} m^4; $A_c = 0.2768$ m^2.

The area of the transformed section, its first and second moments about an axis through O are:

$$A = 0.2768 + 6.667(1600 + 1200 + 400)10^{-6} = 0.2981\,\text{m}^2$$

$$B = 6.667(1600 \times 0.547 + 1200 \times 0.447 - 400 \times 0.253)10^{-6}$$

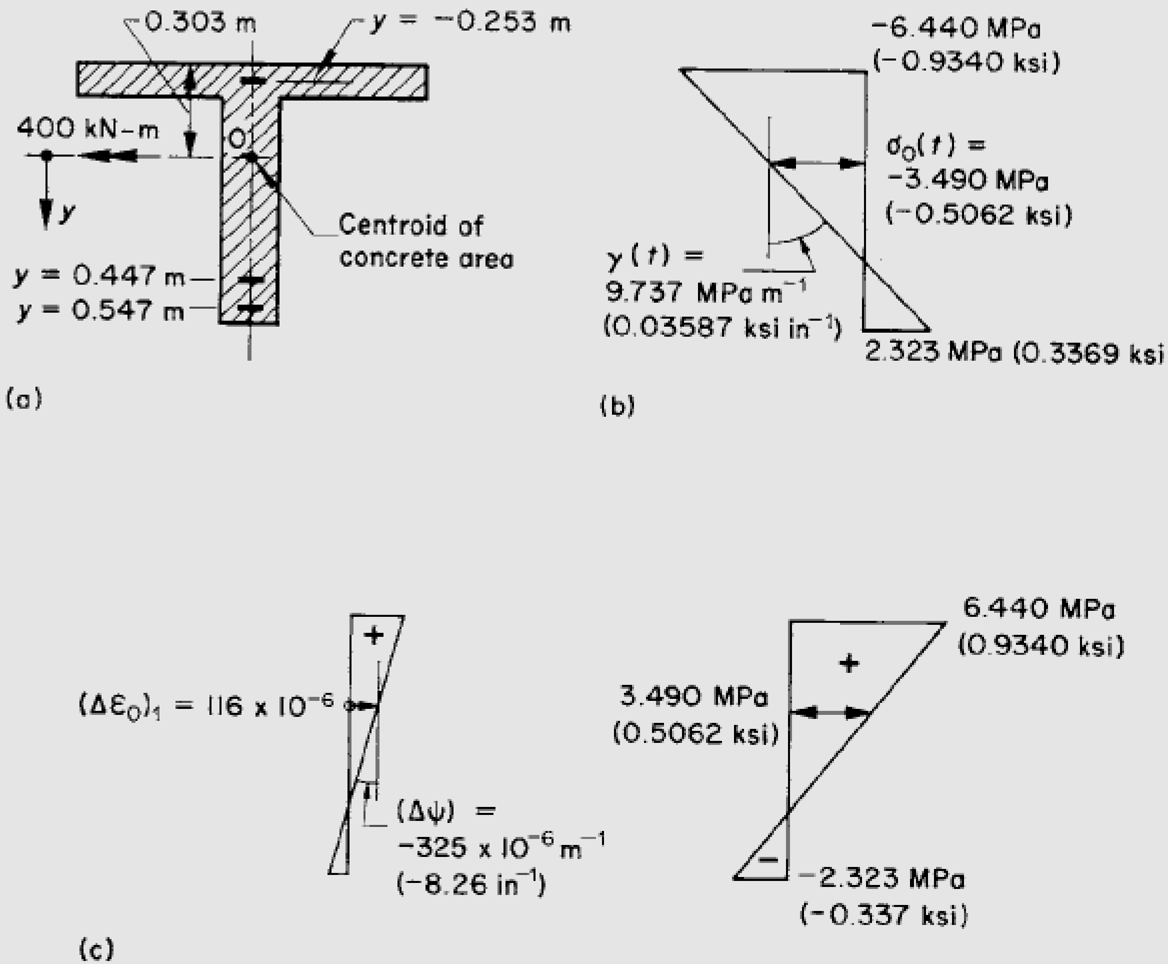
$$= 8.734 \times 10^{-3}\,\text{m}^3$$

$$I = 21.78 \times 10^{-3} + 6.667(1600 \times 0.547^2$$

$$+ 1200 \times 0.447^2 + 400 \times 0.253^2)10^{-6}$$

$$= 26.74 \times 10^{-3}\,\text{m}^4.$$

The stress distribution in Fig. 7.13(b) may be defined by the value of stress at O and the slope of diagram:



0.303 m
y = −0.253 m
400 kN-m
O
y
Centroid of
concrete area
y = 0.447 m
y = 0.547 m
(a)
−6.440 MPa
(−0.9340 ksi)
σ0(t) =
−3.490 MPa
(−0.5062 ksi)
γ(t) =
9.737 MPa m−1
(0.03587 ksi in−1)
2.323 MPa (0.3369 ksi)
(b)
+
(Δε0)1 = 116 x 10−6
(Δψ) =
−325 x 10−6 m−1
(−8.26 in−1)
6.440 MPa
(0.9340 ksi)
+
3.490 MPa
(0.5062 ksi)
−
−2.323 MPa
(−0.337 ksi)
(c)

$$\sigma_O(t) = -3.490\,\text{MPa} \quad \gamma(t) = 9.737\,\text{MPa/m}.$$

The decompression forces (Equations (7.40) and (7.41)) are

$$N_1 = -[0.2981(-3.490 \times 10^6) + 8.734 \times 10^{-3} \times 9.737 \times 10^6]$$
$$= 0.955 \times 10^6\,\text{N}$$
$$M_1 = -[8.734 \times 10^{-3}(-3.490 \times 10^6) + 26.74 \times 10^{-3} \times 9.737 \times 10^6]$$
$$= -229.9 \times 10^3\,\text{N-m}.$$

The changes in strain at O and in curvature (Equations (7.45) and (7.46)) are

$$(\Delta\varepsilon_O)_1 = \frac{3.490 \times 10^6}{30 \times 10^9} = 116 \times 10^{-6}$$

$$(\Delta\psi)_1 = -\frac{9.737 \times 10^6}{30 \times 10^9} = -325 \times 10^{-6}\,\text{m}^{-1}.$$

The changes in stress in the bottom reinforcement and in the prestressed steel are:

$$(\Delta\sigma_{ns})_1 = 200 \times 10^9(116 - 325 \times 0.547)10^{-6} = -12.3\,\text{MPa}$$
$$(\Delta\sigma_{ps})_1 = 200 \times 10^9(116 - 325 \times 0.447)10^{-6} = -5.8\,\text{MPa}.$$

The changes in strain and in stress distributions in the decompression stage are shown in Fig. 7.13(c).

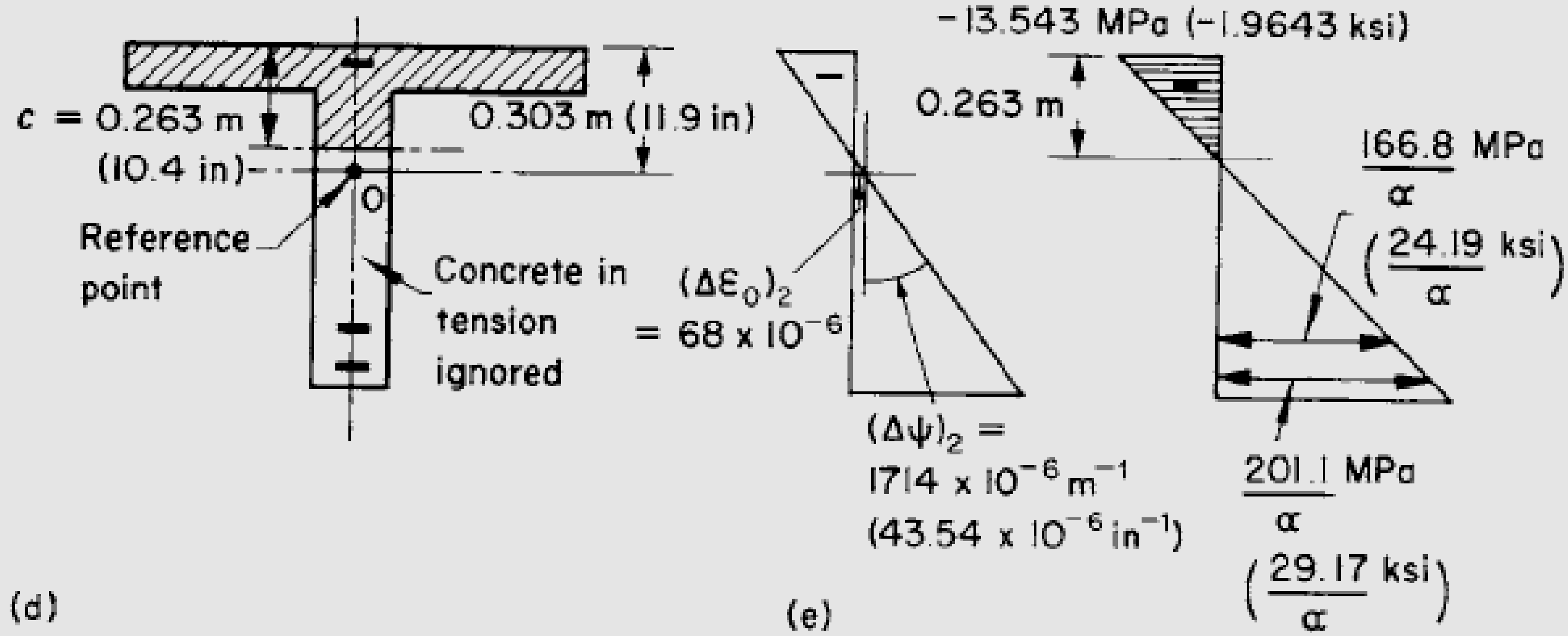


Figure 7.13 Changes in strain and stress in the cross-section of Fig. 7.12 due to a

(d) Changes in strain and stress in the cracking stage

Internal forces producing cracking (Equations (7.33) and (7.34)) are

$$M_2 = 400 \times 10^3 - (-229.9 \times 10^3) = 629.9 \times 10^3 \text{ N-m}$$

$$N_2 = 0 - 0.955 \times 10^6 = -0.955 \times 10^6 \text{ N.}$$

Eccentricity of the resultant of M_2 and N_2 measured from the bottom reinforcement

$$e_s = \frac{629.9 \times 10^3}{-0.955 \times 10^6} - 0.547 = -1.206 \text{ m.}$$

Substitution in Equation (7.20) and solution by trial or use of Table 7.1 gives the depth of the compression zone (Fig. 7.13(d)):

$$c = 0.263\,\text{m}.$$

The transformed section to be used here is composed of the area of concrete in compression plus α times the area of all reinforcements; where $\alpha = E_s/E_c(t) = 200/30 = 6.667$.

The transformed area, its first and second moments about an axis through the reference point O are (Tables 7.3 and 4 may be used for this purpose):

$$A = 0.1736\,\text{m}^2 \quad B = -25.484 \times 10^{-3}\,\text{m}^3 \quad I = 13.270 \times 10^{-3}\,\text{m}^4.$$

Changes in axial strain and in curvature produced by M_2 and N_2 (Equation (2.15) with $E_{ref} = 30\,\text{GPa}$) are

$$(\Delta\varepsilon_O)_2 = 68 \times 10^{-6} \quad (\Delta\psi)_2 = 1714 \times 10^{-6}\,\text{m}^{-1}.$$

The distributions of strain and stress changes are shown in Fig. 7.13(e).

The changes in stress in the bottom reinforcement and in the prestress steel are:

$$(\Delta\sigma_{ns})_2 = 200 \times 10^9(68 + 1714 \times 0.547)10^{-6} = 201.1\,\text{MPa} \quad (29.17\,\text{ksi})$$

$$(\Delta\sigma_{ps})_2 = 200 \times 10^9(68 + 1714 \times 0.447)10^{-6} = 166.8\,\text{MPa} \quad (24.19\,\text{ksi})$$

(e) Strain and stress immediately after cracking

The stress diagram in Fig. 7.13(e), obtained by multiplying the strain diagram in the same figure by the value $E_c(t) = 30\,\text{GPa}$, represents the final stress in concrete after cracking. The final stress in the reinforcement may be obtained by summing up the stress values calculated above in steps (a) to (d). Thus, the stress in the bottom non-prestressed steel is

$$-5.6 - 16.8 - 12.3 + 201.1 = 166.4\,\text{MPa} \quad (24.13\,\text{ksi}).$$

The stress in the prestressed steel is

$$1030.5 - 124.5 - 5.8 + 166.8 = 1067.0\,\text{MPa} \quad (155\,\text{ksi}).$$

Similarly, summing up the strains (Fig. 7.12(b) and (c) and Figs 7.13(c) and (e)) gives the strain at the reference point O:

$$(-181 - 569 + 116 + 68)10^{-6} = -566 \times 10^{-6}$$

and curvature

$$(280 + 887 - 325 + 1714)10^{-6} = 2556 \times 10^{-6}\,\text{m}^{-1}\ (64.92 \times 10^{-6}\,\text{in}^{-1}).$$

8.1 Introduction

1. Reduction in stiffness due to cracking
2. Elongation and curvature
3. Stiffness varies from minimum to maximum (crack to midway between cracks)
4. Two extreme states
5. Interpolation :extent of cracking

8.2 Basic assumptions

1. Free from crack: state 1
2. Fully crack : state 2
3. Bernoulli's hypothesis
4. Contribution of concrete in the tension zone : tension stiffening

8.3 Strain due to axial tension

1. Free from crack

$$N_r = f_{ct}(A_c + \alpha A_s) = f_{ct} A_1 \qquad (8.1)$$

2. Section in state 2

$$\sigma_{sr} = N_r / A_s \qquad (8.2)$$

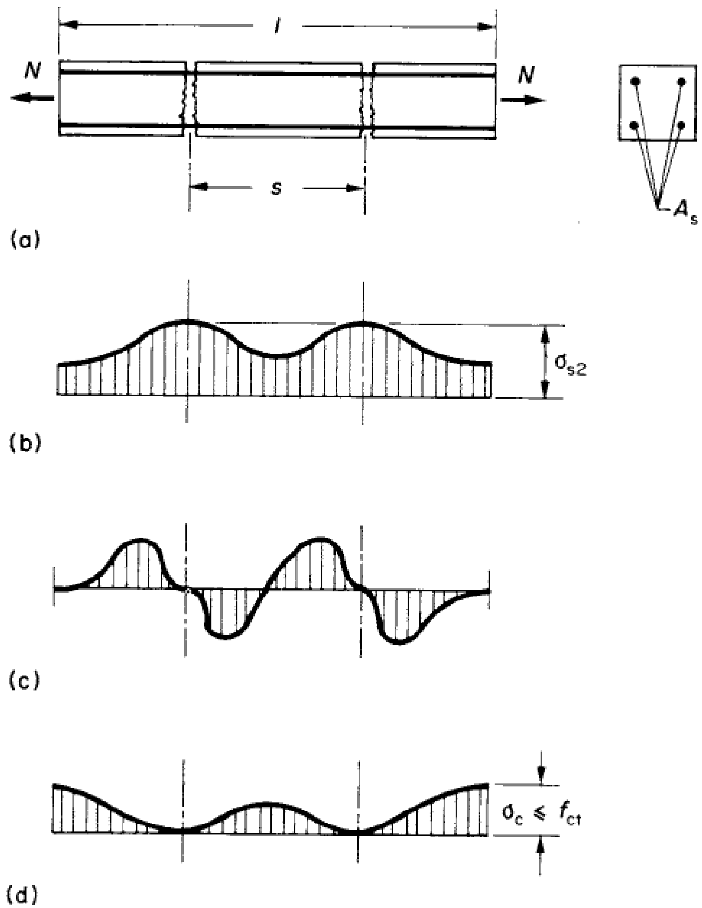


Figure 8.1 Stresses in a reinforced concrete member cracked due to axial force; (a) cracking of a tie; (b) stress in reinforcement; (c) bond stress; (d) stress in concrete ($\sigma_c \leqslant f_{ct}$).

A reinforced concrete member subjected to axial tension N (Fig. 8.1(a)) will be free from cracks when the value of N is lower than

$$N_r = f_{ct}(A_c + \alpha A_s) = f_{ct} A_1 \tag{8.1}$$

where f_{ct} is the strength of concrete in tension; N_r is the value of the axial force that produces first cracking; A_c and A_s are the cross-section areas of concrete and steel and $\alpha = E_s/E_c$, with E_s being the modulus of elasticity of steel and E_c the secant modulus of elasticity of concrete for a loading of short

Just before cracking, the section is in state 1; the stress in concrete is f_{ct} and the stress in steel is αf_{ct}. Immediately after cracking, the section at a crack is in state 2, the stress in steel

$$\sigma_{sr} = N_r/A_s \tag{8.2}$$

At a crack, the section is in state 2, the concrete stress is zero and the steel stress and strain when $N > N_r$

$$\sigma_{s2} = N/A_s \tag{8.3}$$

$$\varepsilon_{s2} = N/E_sA_s \tag{8.4}$$

Midway between consecutive cracks, the tensile stress in concrete has some unknown value smaller than f_{ct} and the steel stress has value smaller than σ_{s2}. Thus, the strain in the reinforcement varies along the length of the member; a mean value of the steel strain is

$$\varepsilon_{sm} = \Delta l/l \tag{8.5}$$

where l is the original length of the member and Δl is the member extension. The symbol ε_{sm} represents an overall mean strain value for the cracked member. Obviously, ε_{sm} is smaller than ε_{s2} which is the steel strain at the cracked section. Let

$$\varepsilon_{sm} = \varepsilon_{s2} - \Delta\varepsilon_s \tag{8.6}$$

where $\Delta\varepsilon_s$ is a reduction in steel strain caused by the participation of concrete in carrying the tensile stress between the cracks. Fig. 8.2 shows the variation

Thus, the strain in the reinforcement varies along the length of the member; a mean value of the steel strain is

$$\varepsilon_{sm} = \Delta l / l \tag{8.5}$$

where l is the original length of the member and Δl is the member extension. The symbol ε_{sm} represents an overall mean strain value for the cracked member. Obviously, ε_{sm} is smaller than ε_{s2} which is the steel strain at the cracked section. Let

$$\varepsilon_{sm} = \varepsilon_{s2} - \Delta\varepsilon_s \tag{8.6}$$

value, $\Delta\varepsilon_{s\ max}$, at the start of cracking, when $N = N_r$. Based on experimental evidence, it is assumed that $\Delta\varepsilon_s$ has hyperbolic variation with σ_{s2} as follows:

$$\Delta\varepsilon_s = \Delta\varepsilon_{s\ max} \frac{\sigma_{sr}}{\sigma_{s2}} \tag{8.8}$$

Substitution of Equations (8.8) and (8.9) into Equation (8.6) gives for a cracked member an overall strain value, which is also the mean strain in steel:

$$\varepsilon_{sm} = (1 - \zeta)\varepsilon_{s1} + \zeta\varepsilon_{s2} \tag{8.10}$$

where ζ is a dimensionless coefficient, between 0 and 1, representing the extent of cracking. $\zeta = 0$ for an uncracked section ($N < N_r$), and $0 < \zeta < 1$ for a cracked section. The value of ζ is given by:

$$\zeta = 1 - \left(\frac{\sigma_{sr}}{\sigma_{s2}}\right)^2 \quad (\text{with } \sigma_{s2} > \sigma_{sr}) \tag{8.11}$$

The second term in Equation (8.10) ($\zeta\varepsilon_{s2}$) represents the supplementary strain of steel compared with the strain of concrete.[3] Thus, the average width of a crack is

$$w_m = s_{rm}\zeta\varepsilon_{s2} \tag{8.14}$$

1) N before cracking lower than the following value

$$N_r = f_{ct}\left(A_c + nA_s\right) = f_{ct}A_1$$

$$n = \frac{E_s}{E_c}$$

$$\sigma_{sr} = \frac{N_r}{A_s}$$

Stress in steel

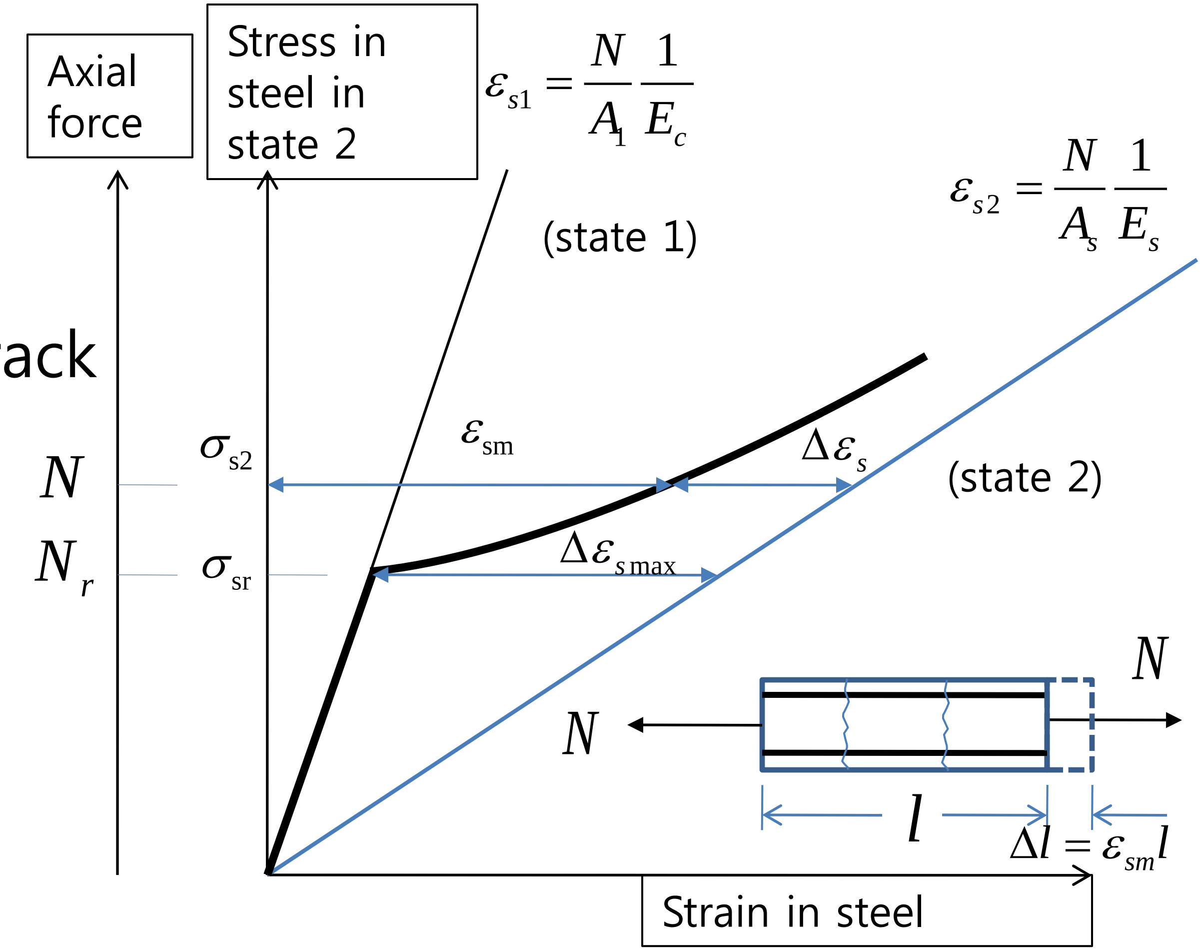
2) After cracking

$$N > N_r$$

Stress in steel at a crack

$$\sigma_{s2} = \frac{N}{A_s}$$

$$\varepsilon_{s2} = \frac{N}{A_s}\frac{1}{E_s}$$



$$\varepsilon_{s1} = \varepsilon_{c1} = \frac{N}{E_c\left(A_c + nA_s\right)} = \frac{N}{E_c A_1}$$

$$\Delta\varepsilon_s = \underline{\varepsilon_{s2}} - \underline{\varepsilon_{sm}}$$

Fully cracked Mean steel strain

Assume strain difference has hyperbolic variation with stress in steel

$$\Delta\varepsilon_s = \Delta\varepsilon_{s\max}\frac{\sigma_{sr}}{\sigma_{s2}}$$

$$\boxed{\Delta\varepsilon_{s\max} = \left(\varepsilon_{s2} - \varepsilon_{s1}\right)\frac{\sigma_{sr}}{\sigma_{s2}}}$$

$$\begin{aligned}
\varepsilon_{sm} &= \varepsilon_{s2} - \Delta\varepsilon_s \\
&= \varepsilon_{s2} - \Delta\varepsilon_{s\max}\frac{\sigma_{sr}}{\sigma_{s2}} \\
&= \varepsilon_{s2} - \left(\varepsilon_{s2} - \varepsilon_{s1}\right)\left(\frac{\sigma_{sr}}{\sigma_{s2}}\right)^2 \\
&= \left(\frac{\sigma_{sr}}{\sigma_{s2}}\right)^2\varepsilon_{s1} + \left[1 - \left(\frac{\sigma_{sr}}{\sigma_{s2}}\right)^2\right]\varepsilon_{s2}
\end{aligned}$$

Let $\zeta = 1 - \left(\frac{\sigma_{sr}}{\sigma_{s2}} \right)^2$

$$\varepsilon_{sm} = \left(1 - \zeta\right) \varepsilon_{s1} + \zeta \varepsilon_{s2}$$

$$\zeta = 1 - \beta_1 \beta_2 \left(\frac{\sigma_{sr}}{\sigma_{s2}} \right)^2$$

β_1 : bond effect

β_2 : loading characteristics

Example 8.1 Mean axial strain in a tie

Find the mean strain, excluding the effect of creep in a reinforced concrete member (Fig. 8.1(a)) having a square cross-section $0.20 \times 0.20\,\mathrm{m}^2$ ($62\,\mathrm{in}^2$) subjected to an axial tensile force $N = 200\,\mathrm{kN}$ (45 kip), given the following data: $A_s = 804\,\mathrm{mm}^2$ ($1.25\,\mathrm{in}^2$); $E_s = 200\,\mathrm{GPa}$ (29 000 ksi); $E_c = 30\,\mathrm{GPa}$ (4350 ksi); $f_{ct} = 2.0\,\mathrm{MPa}$ (290 psi); $\beta_1 = 1$ and $\beta_2 = 0.5$. What is the width of a crack assuming $s_{rm} = 200\,\mathrm{mm}$ (8 in)?

Equation (8.1) gives $N_r = 89.1\,\mathrm{kN}$ (20.0 kip). The stresses in steel, assuming state 2 prevails (Equations (8.2) and (8.3)):

$$\sigma_{sr} = \frac{N_r}{A_s} = 111\,\mathrm{MPa} \qquad \sigma_{s2} = 249\,\mathrm{MPa}.$$

Substitution in Equation (8.13) gives $\zeta = 0.90$. The strains in steel due to N, calculated with the assumption that the section is in states 1 and 2, are (Equations (8.7) and (8.4)):

$$\varepsilon_{s1} = 150 \times 10^{-6} \qquad \varepsilon_{s2} = 1244 \times 10^{-6}.$$

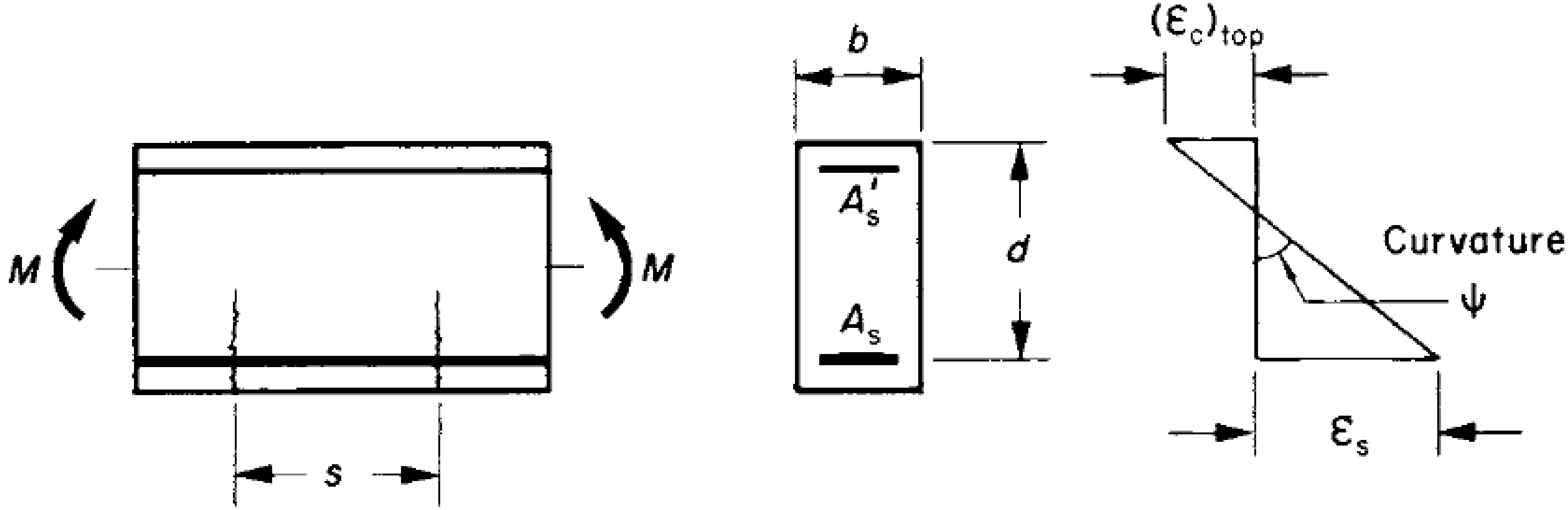
The mean strain for the member (Equation (8.10)) is

$$\varepsilon_{sm} = 150 \times 10^{-6}(1 - 0.90) + 1244 \times 10^{-6} \times 0.90 = 1134 \times 10^{-6}.$$

The width of a crack (Equation (8.14)) is

$$w_m = 200 \times 0.90 \times 1244 \times 10^{-6} = 0.22\,\text{mm}(8.8 \times 10^{-3}\,\text{in}).$$

8.4 Curvature due to bending



For a bending moment $M > M_r$, cracking occurs and the steel stress along the reinforcement varies from a maximum value at the crack location to a minimum value at the middle of the spacing between the cracks. Assuming that the concrete between the cracks has the same effect on the mean strain in steel as in the case of axial force, Equation (8.10) can be adopted. Thus,

$$\varepsilon_{sm} = (1 - \zeta)\varepsilon_{s1} + \zeta\varepsilon_{s2} \tag{8.16}$$

where

$$\zeta = 1 - \beta_1\beta_2\left(\frac{\sigma_{sr}}{\sigma_{s2}}\right)^2 = 1 - \beta_1\beta_2\left(\frac{M_r}{M}\right)^2 \tag{8.17}$$

$$\psi = \frac{\varepsilon_s - (\varepsilon_c)_{top}}{d} \tag{8.20}$$

where ψ is the curvature; E is the modulus of elasticity; I is the moment of inertia of the section; ε_s is the strain in steel reinforcement and $(\varepsilon_c)_{top}$ is the strain at the extreme fibre of the compression zone and d is the distance between steel in tension and the extreme compression fibre (Fig. 8.4). Assume that cracking has an effect on curvature similar to its effect on the strain in axial tension. Thus, the mean curvature is expressed in this form:

$$\psi_m = (1 - \zeta)\psi_1 + \zeta\psi_2 \tag{8.21}$$

Example 8.2 Rectangular section subjected to bending moment

Calculate the mean curvature in a reinforced concrete member of a rectangular cross-section (Fig. 8.4) due to a bending moment M = 250 kN-m (221 kip-ft), excluding creep effect and employing the following data: $b = 400$ mm (16 in); $h = 800$ mm (32 in); $d = 750$ mm (30 in); $d' =$ 50 mm (2 in); $A_s = 2120\,\text{mm}^2$ ($3.29\,\text{in}^2$); $A'_s = 760\,\text{mm}^2$ ($1.18\,\text{in}^2$); $E_s =$ 200 GPa (29 000 ksi); $E_c = 30$ GPa (4350 ksi); $f_{ct} = 2.5$ MPa (360 psi); $\beta_1 = 1$ and $\beta_2 = 0.5$.

Assuming the spacing between cracks $s_{rm} = 300$ mm (12 in), find the width of a crack.

The moment of inertia and the section modulus of transformed uncracked section are (graphs of Fig. 3.5 may be employed):

$$I_1 = 0.0191\,\text{m}^4 \qquad W_1 = 0.0488\,\text{m}^3.$$

Equation (8.15) gives $M_r = 122\,\text{kN-m}$ (90.0 kip-ft). Substitution in Equation (8.17) gives $\zeta = 0.88$.

Depth of compression zone in state 2 (by Equation (7.16) or the graphs of Fig. 8.4):

$$c = 0.191\,\text{m} \quad (7.52\,\text{in}).$$

The centroid of the transformed fully cracked section coincides with the neutral axis. The moment of inertia (calculated from first principles or by use of graphs of Fig. 7.6) is

$$I_2 = 0.00543\,\text{m}^4.$$

The curvatures due to $M = 250\,\text{kN-m}$, assuming the section to be in states 1 and 2 (Equations (8.23) and (8.24)) are:

$$\psi_1 = 437 \times 10^{-6}\,\text{m}^{-1} \quad \psi_2 = 1530 \times 10^{-6}\,\text{m}^{-1}.$$

The mean curvature (Equation (8.21)) is

$$\psi_m = [(1 - 0.88)437 + 0.88 \times 1530]10^{-6} = 1400 \times 10^{-6}\,\text{m}^{-1}.$$

The strain in steel in state 2 is

$$\varepsilon_{s2} = \psi_2 y_s = 1530 \times 10^{-6}(0.75 - 0.191) = 856 \times 10^{-6}.$$

The width of a crack (Equation (8.14)) is

$$w_m = 300 \times 0.88 \times 856 \times 10^{-6} = 0.23\,\text{mm}\ (0.0091\,\text{in}).$$

8.5 Curvature due to bending and axial force

The eccentricity of the axial force is:

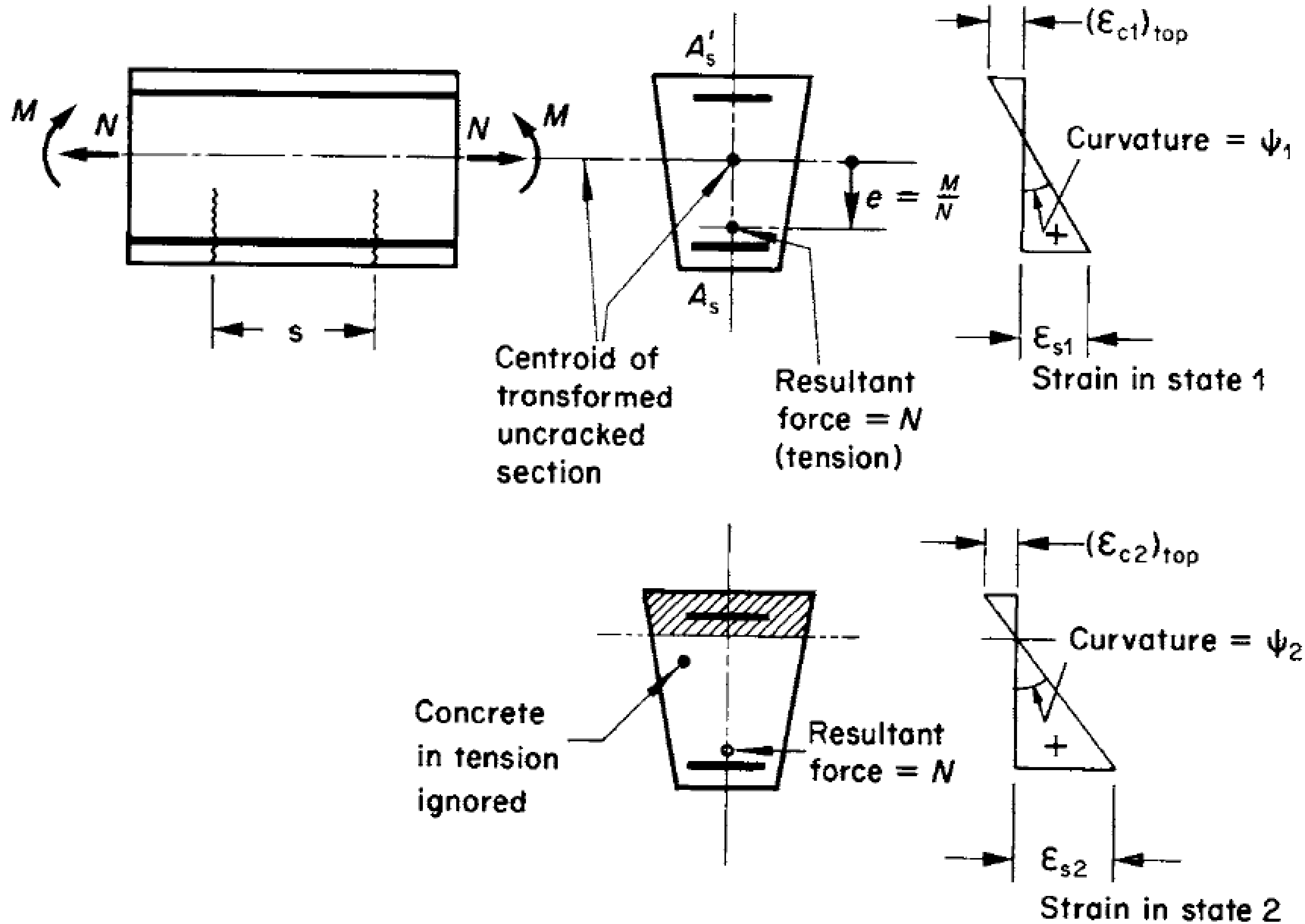
$$e = M/N \tag{8.25}$$

Our sign convention is as follows: N is positive when tensile and M is positive when it produces tension at the bottom fibre. It thus follows that e is positive when the resultant of M and N is situated below the centroid of the transformed uncracked section (Fig. 8.6).

Without change in eccentricity, we can find the values of N_r and the corresponding M_r that produce at the bottom fibre a tensile stress f_{ct}, the strength of concrete in tension:

$$N_r = f_{ct}\left(\frac{1}{A} + \frac{e}{W_{bot}}\right)_1^{-1} \tag{8.26}$$

$$M_r = eN_r \tag{8.27}$$



M
N
N
M
s
A_s'
A_s
$e = \frac{M}{N}$
Centroid of transformed uncracked section
Resultant force = N (tension)
$(\varepsilon_{c1})_{top}$
Curvature = ψ_1
+
ε_{s1}
Strain in state 1
Concrete in tension ignored
Resultant force = N
$(\varepsilon_{c2})_{top}$
Curvature = ψ_2
+
ε_{s2}
Strain in state 2

When $N > N_r$ and $M > M_r$, cracking occurs and the mean strain in the reinforcement can be calculated by:

$$\varepsilon_m = (1 - \zeta)\varepsilon_{s1} + \zeta\varepsilon_{s2} \tag{8.28}$$

where

$$\zeta = 1 - \beta_1\beta_2\left(\frac{\sigma_{sr}}{\sigma_{s2}}\right)^2 \tag{8.29}$$

or

$$\zeta = 1 - \beta_1\beta_2\left(\frac{M_r}{M}\right)^2 \tag{8.30}$$

$$\zeta = 1 - \beta_1\beta_2\left(\frac{N_r}{N}\right)^2 \tag{8.31}$$

It is to be noted that in a fully cracked section, the position of the neutral axis depends on the eccentricity $e = M/N$, not on the separate values of M and N. Because e is assumed to be unchanged, $(M/N) = (M_r/N_r)$ and

$$\frac{\sigma_{sr}}{\sigma_{s2}} = \frac{M_r}{M} = \frac{N_r}{N} \tag{8.32}$$

Assuming that the cracks are spaced at a distance s_{rm}, the width of a crack

$$w_m = s_{rm} \zeta \varepsilon_{s2} \tag{8.33}$$

The mean curvature in the cracked member

$$\psi_m = (1 - \zeta)\psi_1 + \zeta \psi_2 \tag{8.34}$$

Example 8.3 Rectangular section subjected to M and N

Calculate the mean curvature for the reinforced concrete section of Example 8.2 subjected to $M = 250$ kN-m (184 kip-ft) combined with an axial force $N = -200$ kN (–45 kip) at mid-height. All other data are the same as in Example 8.2. Assuming spacing between cracks, s_{rm} = 300 mm, find the width of a crack.

The area of the transformed section in state 1

$$A_1 = 0.336\,\text{m}^2.$$

The centroid of A_1 is very close to mid-height; the eccentricity is considered to be measured from mid-height:

$$e = \frac{250}{-200} = -1.25\,\text{m}.$$

Substitution in Equations (8.26) and (8.27) (with $f_{ct} = 2.5\,\text{MPa}$, $W_1 = 0.0488\,\text{m}^3$; see Example 8.2) gives:

$$M_r = 138\,\text{kN-m}.$$

Substitution in Equation (8.30) gives

$$\zeta = 0.85.$$

The presence of N does not change the curvature in state 1 from what is calculated in Example 8.2. Thus,

$$\psi_1 = 437 \times 10^{-6}\,\text{m}^{-1}.$$

Solution of Equation (7.20) or use of graphs in Fig. 7.4 gives the depth of the compression zone:

$$c = 0.241\,\text{m}\ (9.49\,\text{in}).$$

Distance from the top fibre to the centroid of the transformed section in state 2 (Fig. 7.5) is

$$\bar{y} = 0.195\,\mathrm{m}\ (7.68\,\mathrm{in}).$$

The area and the moment of inertia of the transformed section in state 2 about an axis through its centroid (Fig. 7.6) are

$$A_2 = 0.115\,\mathrm{m}^2, \quad I_2 = 0.00544\,\mathrm{m}^4.$$

The applied forces $N = -200\,\mathrm{kN}$ at mid-height combined with $M = 250\,\mathrm{kN\text{-}m}$ may be replaced by an equivalent system of $N' = -200\,\mathrm{kN}$ at the centroid of the transformed section in state 2 combined with $M' = 209\,\mathrm{kN\text{-}m}$.

The curvature in state 2 is

$$\psi_2 = \frac{209 \times 10^3}{30 \times 10^9 \times 0.00544} = 1280 \times 10^{-6}\,\mathrm{m}^{-1}.$$

The mean curvature (Equation (8.34)) is

$$\psi_m = [(1 - 0.85)437 + 0.85 \times 1280]10^{-6}$$
$$= 1150 \times 10^{-6}\,\mathrm{m}^{-1}\ (29.2 \times 10^{-6}\,\mathrm{in}^{-1}).$$

The axial strain at the centroid of the fully cracked section is

$$\varepsilon_{O2} = -\frac{200 \times 10^3}{30 \times 10^9 \times 0.115} = -58.0 \times 10^{-6}.$$

The strain in the bottom steel in state 2 is

$$\varepsilon_{s2} = 10^{-6}[-58.0 + 1280(0.75 - 0.195)] = 652 \times 10^{-6}.$$

Crack width (Equation (8.33)) is

$$w_m = 300 \times 0.85 \times 652 \times 10^{-6} = 0.17\,\mathrm{mm}\ (0.0067\,\mathrm{in}).$$

Summary

Axial tension (Fig. 8.1)

The mean axial strain

$$\varepsilon_{\mathrm{Om}} = (1-\zeta)\varepsilon_{\mathrm{O1}} + \zeta\varepsilon_{\mathrm{O2}} \tag{8.36}$$

where

$$\zeta = 1 - \beta_1\beta_2\left(\frac{N_{\mathrm{r}}}{N}\right)^2 \tag{8.37}$$

The mean curvature

$$\psi_{\mathrm{m}} = (1-\zeta)\psi_1 + \zeta\psi_2 \tag{8.40}$$

where

$$\zeta = 1 - \beta_1\beta_2\left(\frac{M_{\mathrm{r}}}{M}\right)^2 \tag{8.41}$$

Bending moment combined with axial force (Fig. 8.6)

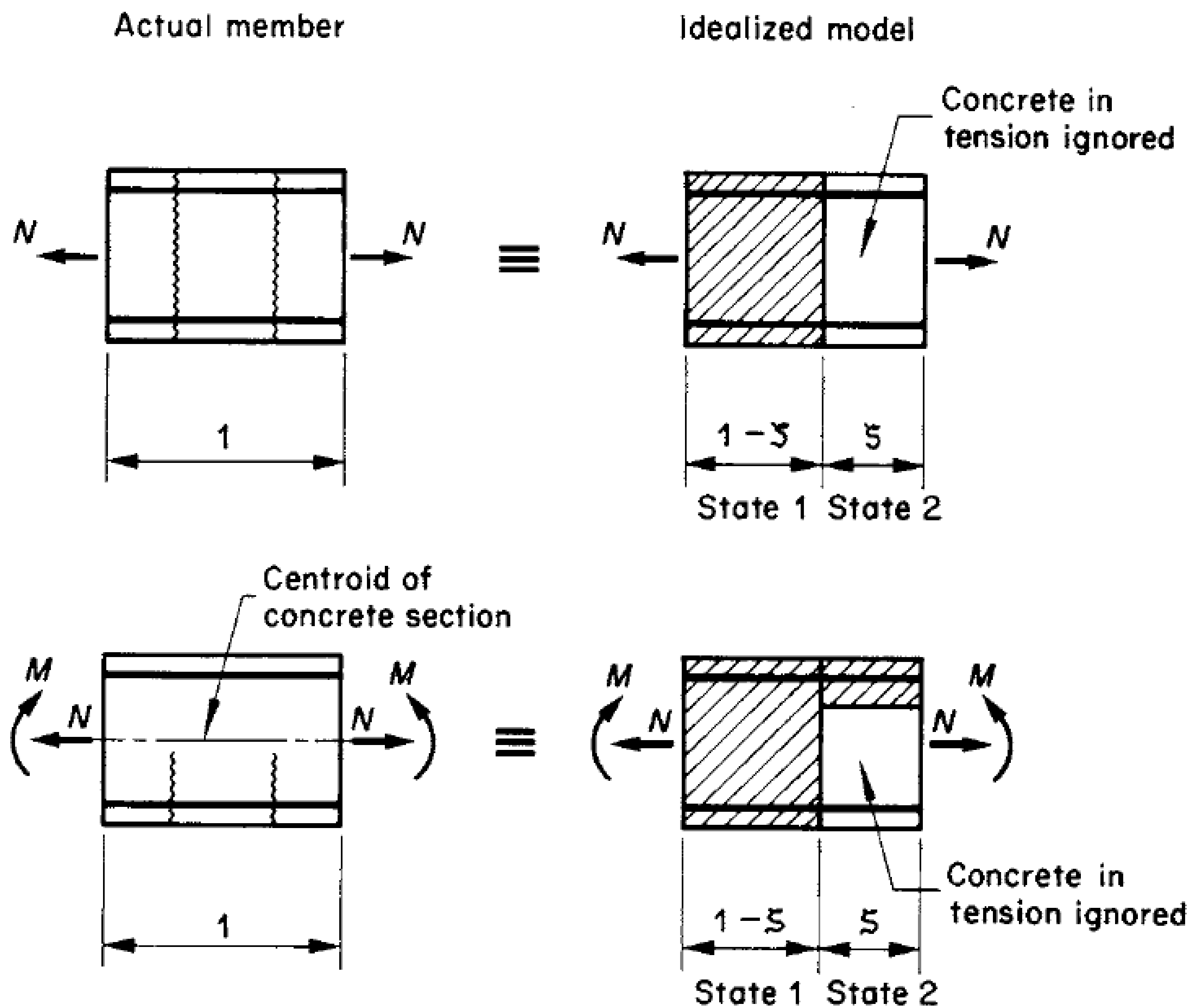
The mean axial strain and curvature

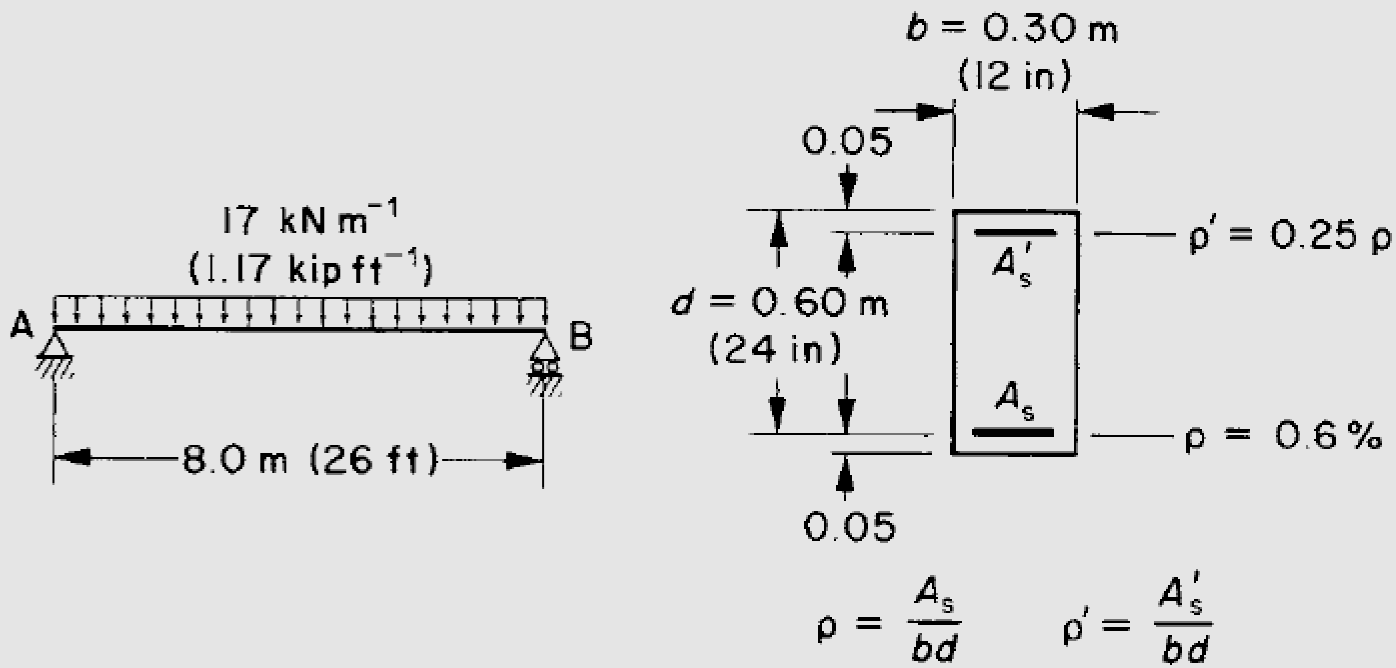
$$\varepsilon_{\mathrm{Om}} = (1 - \zeta)\varepsilon_{\mathrm{O1}} + \zeta\varepsilon_{\mathrm{O2}} \tag{8.43}$$

$$\psi_{\mathrm{m}} = (1 - \zeta)\psi_1 + \zeta\psi_2 \tag{8.44}$$

where

$$\zeta = 1 - \beta_1\beta_2\left(\frac{M_{\mathrm{r}}}{M}\right)^2 = 1 - \beta_1\beta_2\left(\frac{N_{\mathrm{r}}}{N}\right)^2 \tag{8.45}$$


Actual member
Idealized model
Concrete in tension ignored
N
N
N
N
≡
1
1 − ζ
ζ
State 1
State 2
Centroid of concrete section
M
M
M
M
N
N
N
N
≡
1
1 − ζ
ζ
Concrete in tension ignored
State 1
State 2



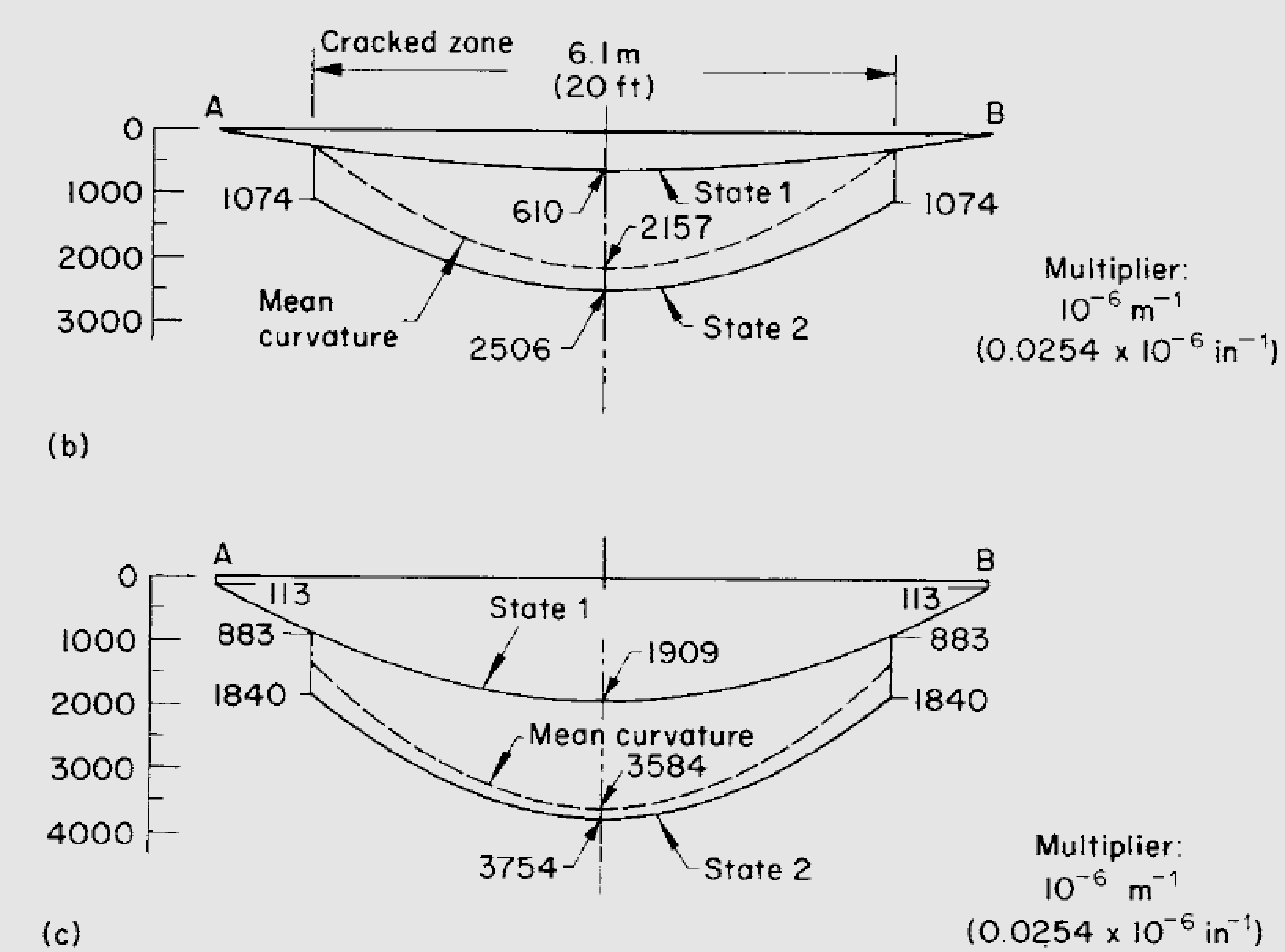
(a)

Example 8.4 Non-prestressed simple beam: variation of curvature over span

The reinforced concrete simple beam of the constant cross-section shown in Fig. 8.8(a) has bottom and top steel area ratios, $\rho = 0.6$ per cent and $\rho' = 0.15$ per cent. At time t_0, uniform load $q = 17.0\,\text{kN/m}$ (1.17 kip/ft) is applied. It is required to find the curvatures at t_0 and at a later time t and to draw sketches of the variations of the curvature over the span. The following data are given:

$E_s = 200\,\text{GPa}$ (29 000 ksi); $E_c(t_0) = 30.0\,\text{GPa}$ (4350 ksi); $f_{ct} = 2.5\,\text{MPa}$ (0.36 ksi); $\beta_1 = 1.0$; $\beta_2 = 1.0$ for calculation of instantaneous curvature and 0.5 for long-term curvature; creep coefficient $\varphi(t, t_0) = 2.5$; aging coefficient, $\chi(t, t_0) = 0.8$; free shrinkage, $\varepsilon_{cs}(t, t_0) = -250 \times 10^{-6}$.

What is the deflection at mid-span at time t?


Cracked zone
6.1m
(20 ft)
A
B
0
1000
2000
3000
1074
610
State 1
2157
1074
Mean
curvature
2506
State 2
Multiplier:
10^{-6} m^{-1}
(0.0254×10^{-6} in^{-1})
(b)
A
B
0
1000
2000
3000
4000
113
883
1840
State 1
1909
113
883
1840
Mean curvature
3584
3754
State 2
Multiplier:
10^{-6} m^{-1}
(0.0254×10^{-6} in^{-1})
(c)

(a) Curvature at time t_0

The following sections' properties will be used in the analysis of curvatures at t_0:

Transformed uncracked section (state 1) Area, $A_1 = 0.2027\,\text{m}^2$; centroid O_1 is at 0.331 m below top edge; moment of inertia about an axis through O_1, $I_1 = 7.436 \times 10^{-3}\,\text{m}^4$; section modulus $W_1 = 23.33 \times 10^{-3}\,\text{m}^3$.

Transformed cracked section (state 2) Depth of compression zone (Equation (7.16)), $c = 0.145\,\text{m}$; centroid O_2 lies on neutral axis; moment of inertia about an axis through O_2, $I_2 = 1.809 \times 10^{-3}\,\text{m}^4$.

The bending moment at mid-span = $17 \times 8^2/8 = 136\,\text{kN-m}$. The bending moment which produces cracking (Equation (8.15))

$$M_r = 23.33 \times 10^{-3} \times 2.5 \times 10^6 = 58.3\,\text{kN-m}.$$

The interpolation coefficient for instantaneous curvature (Equation (8.41)) is

$$\zeta = 1 - 1.0 \times 1.0' \left(\frac{58.3}{136}\right)^2 = 0.82.$$

The interpolation coefficient for long-term curvature (Equation (8.41)) is

$$\zeta = 1 - 1.0 \times 0.5 \left(\frac{58.3}{136}\right)^2 = 0.91.$$

The curvature at t_0, assuming states 1 and 2 (Equations (8.23) and (8.24)):

State 1

$$\psi_1(t_0) = \frac{136 \times 10^3}{30 \times 10^9 \times 7.436 \times 10^{-3}} = 610 \times 10^{-6}\ \text{m}^{-1}$$

State 2

$$\psi_2(t_0) = \frac{136 \times 10^3}{30 \times 10^9 \times 1.809 \times 10^{-3}} = 2506 \times 10^{-6}\ \text{m}^{-1}$$

Interpolation

Mean curvature at time t_0 (Equation (8.40))

$$\psi(t_0) = (1 - 0.82)610 \times 10^{-6} + 0.82 \times 2506 \times 10^{-6} = 2157 \times 10^{-6}\ \text{m}^{-1}.$$

With parabolic variation of the bending moment over the span, the value $M_r = 58.3$ kN-m is reached at distance 0.98 m from the support. Thus, cracking occurs over the central 6.05 m (19.8 ft) of the span.

Fig. 8.8(b) shows the variation of the curvatures at time t_0, with the assumptions of states 1 and 2; the mean curvature is also shown with the broken curve.

(b) Curvatures at time t

The age-adjusted modulus of elasticity of concrete (Equation (1.31))

$$\bar{E}_c(t, t_0) = \frac{30 \times 10^9}{1 + 0.8 \times 2.5} = 10\,\text{GPa}.$$

$$\bar{\alpha} = \frac{E_s}{\bar{E}_c(t, t_0)} = \frac{200}{10} = 20.$$

The following sections' properties are required for the age-adjusted transformed sections in states 1 and 2.

Age-adjusted transformed section in state 1 $\bar{A}_1 = 0.2207\,\mathrm{m}^2$; centroid $\bar{O}_1$ is at 0.344 m below top edge. Moment of inertia about an axis through $\bar{O}_1$, $\bar{I}_1 = 8.724 \times 10^{-3}\,\mathrm{m}^4$; y = coordinate of the centroid of the concrete area (measured downwards from $\bar{O}_1$); $y_c = -0.020\,\mathrm{m}$; area of concrete, $A_c = 0.1937\,\mathrm{m}^2$; moment of inertia of A_c about an axis through $\bar{O}_1$, $I_c = 6.937 \times 10^{-3}\,\mathrm{m}^4$; $r_c^2 = I_c/A_c = 35.34 \times 10^{-3}\,\mathrm{m}^2$.

The curvature reduction factor (Equation (3.18)) is

$$\kappa_1 = \frac{6.937 \times 10^{-3}}{8.724 \times 10^{-3}} = 0.795.$$

$$\eta = A_c/\bar{A}$$

$$\kappa = I_c/\bar{I}$$

Age-adjusted transformed section in state 2 $\bar{A}_2 = 70.1 \times 10^{-3}\mathrm{m}^2$; centroid $\bar{O}_2$ is at 0.233 m below top edge; moment of inertia about an axis through $\bar{O}_2$, $\bar{I}_2 = 4.277 \times 10^{-3}\,\mathrm{m}^4$; y-coordinate of centroid of concrete area in compression (measured downwards from $\bar{O}_2$); $y_c = -0.161\,\mathrm{m}$; area of the compression zone; $A_c = 0.0431\,\mathrm{m}^2$; moment of inertia of A_c about an axis through $\bar{O}_2$, $I_c = 1.190 \times 10^{-3}\mathrm{m}^4$; $r_c^2 = I_c/A_c = 27.62 \times 10^{-3}\,\mathrm{m}^2$.

The curvature reduction factor (Equation (7.31)) is

$$\kappa_2 = \frac{1.190}{4.277} = 0.278.$$

$$\eta = A_c/\bar{A}$$

$$\kappa = I_c/\bar{I}$$

Changes in curvature due to creep and shrinkage

State 1

The curvature at $t_0 = 610 \times 10^{-6}\ \mathrm{m}^{-1}$; the corresponding axial strain at $\overline{O}_1$ $= 610 \times 10^{-6}\,(0.344 - 0.331) = 8 \times 10^{-6}$.

The change in curvature during the period t_0 to t (Equation (3.16)),

$$\Delta\psi = 0.795\left[2.5\left(610 \times 10^{-6} + 8 \times 10^{-6}\,\frac{-0.020}{35.34 \times 10^{-3}}\right)\right.$$

$$\left. + (-250 \times 10^{-6})\,\frac{-0.020}{35.34 \times 10^{-3}}\right]$$

$$= 1299 \times 10^{-6}\ \mathrm{m}^{-1}.$$

The curvature at time t (state 1)

$$\psi_1(t) = (610 + 1299)10^{-6} = 1909 \times 10^{-6}\ \mathrm{m}^{-1}.$$

State 2

The curvature at $t_0 = 2506 \times 10^{-6}\ \mathrm{m}^{-1}$; the corresponding axial strain at $\overline{O}_2 = 2506 \times 10^{-6}\,(0.233 - 0.145) = 222 \times 10^{-6}$.

The change in curvature during the period t_0 to t (Equation (7.27))

$$\Delta\psi = 0.278\left[2.5\left(2506 \times 10^{-6} + 222 \times 10^{-6}\,\frac{-0.161}{27.62 \times 10^{-3}}\right)\right.$$

$$\left. + (-250 \times 10^{-6})\,\frac{-0.161}{27.62 \times 10^{-3}}\right]$$

$$= 1248 \times 10^{-6}\ \mathrm{m}^{-1}.$$

The curvature at time t (state 2)

$$\psi_2(t) = (2506 + 1248)10^{-6} = 3754 \times 10^{-6}\ \mathrm{m}^{-1}.$$

Interpolation

Mean curvature at time t (Equation (8.40))

$$\psi(t) = (1 - 0.91)1909 \times 10^{-6} + 0.91 \times 3754 \times 10^{-6}$$

$$= 3584 \times 10^{-6}\ \mathrm{m}^{-1}$$

$$= 91.13 \times 10^{-6}\ \mathrm{in}^{-1}.$$

The curvature at the end section is caused only by shrinkage and may be calculated by Equation (3.16). However, if we ignore this value and calculate the deflection by assuming parabolic variation of curvature, with zero at ends and maximum at the centre, we obtain (Equation (C.8)):

$$\text{Deflection at centre} = 3584 \times 10^{-6}\,\frac{8^2}{96}$$

$$= 0.0239\,\text{m}$$

$$= 23.9\,\text{mm}\ (0.948\,\text{in}).$$

By numerical integration, a more accurate value of the deflection at the centre is 23.5 mm (0.925 in).

It can be seen in Fig. 8.8(b) and (c)[6] that once M_r is exceeded, the line

Example 8.5 Pre-tensioned simple beam: variation of curvature over span

Find the mean curvature at a section at mid-span of a partially prestressed beam shown in Fig. 8.9(a), after application of a live

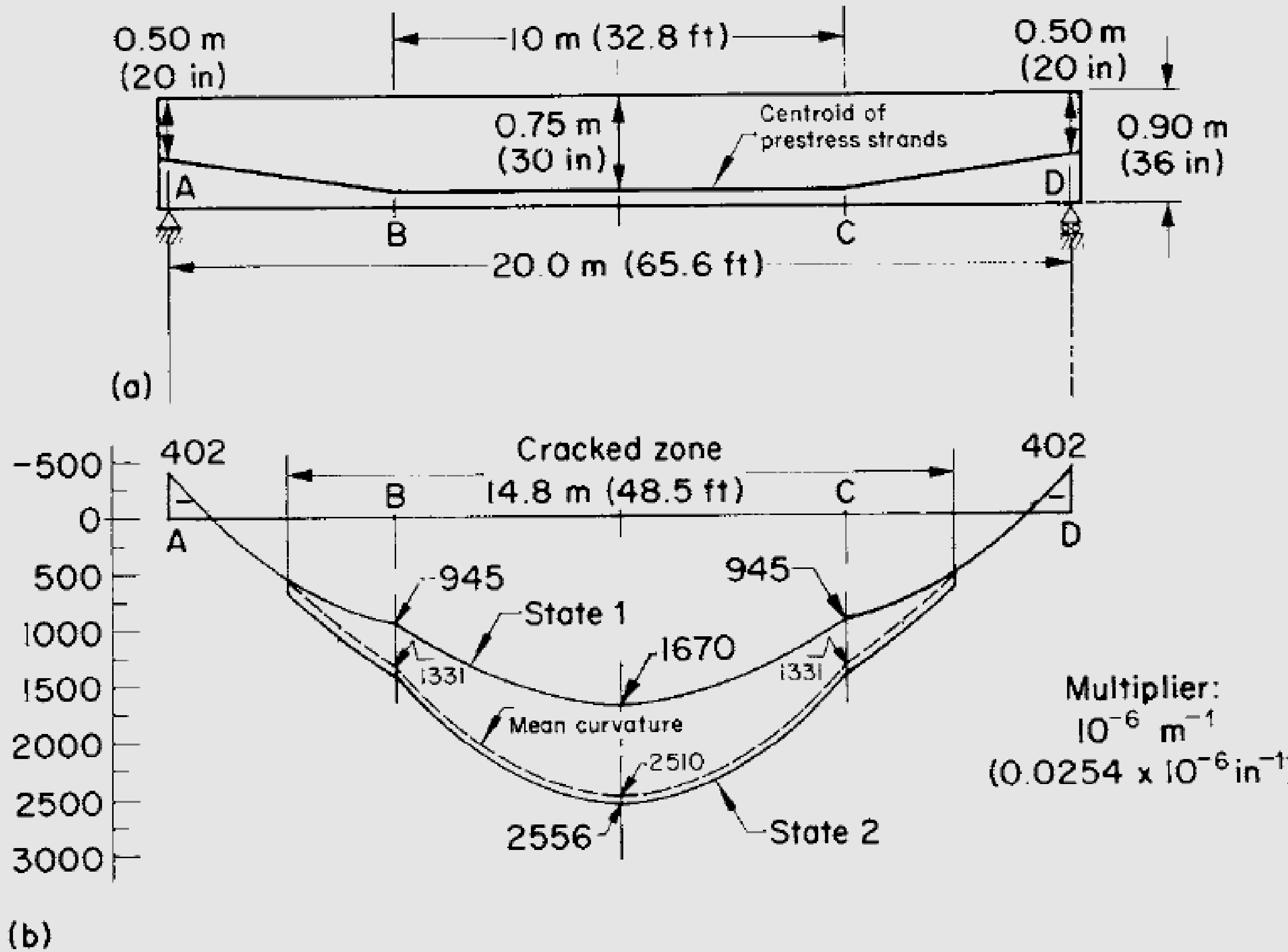


Fig. 7.12(a) shows the cross-section at mid-span. The section is constant over the span, with the exception of the location of the prestressed steel. The beam is pretensioned with a tendon depressed at points B and C, resulting in the profile shown in Fig. 8.9(a). The beam carries uniform dead and live loads of intensities 14.0 and 8.0 kN/m, respectively (0.96 and 0.55 kip/ft), resulting in bending moments at mid-span of 700 and 400 kN-m (6200 and 3540 kip-in). Assume a high-bond quality of reinforcement and tensile strength of concrete $f_{ct} = 2.5$ MPa. Other data are the same as in Example 7.5.

The stress and strain in the section at mid-span have been analysed in Example 7.5. The curvature in state 2 is obtained by summing up the values of curvatures shown in Fig. 7.12(b) and (c) and 7.13(c) and (e). This gives the following value of curvature in state 2:

$$\psi_2 = 2556 \times 10^{-6} \text{ m}^{-1}.$$

Cracking is produced at time t only after application of a live load. Immediately before application of the live load, after occurrence of prestress loss, the curvature at mid-span is $1167 \times 10^{-6}\ \mathrm{m}^{-1}$ (sum of curvature values indicated in Fig. 7.12(b) and (c)). Assuming no cracking (state 1), the live load would produce additional curvature of $499 \times 10^{-6}\ \mathrm{m}^{-1}$. This is calculated by dividing the live-load moment by $[E_c(t)I_1(t)]$, where $E_c(t) = 30\,\mathrm{GPa}$ is the modulus of elasticity of concrete at time t and $I_1(t) = 26.74 \times 10^{-3}\ \mathrm{m}^4$ is the centroidal moment of inertia of transformed uncracked section at time t. Thus, after live-load application, the total curvature in state 1 is

$$\psi_1 = (1167 + 499)10^{-6} = 1670 \times 10^{-6}\ \mathrm{m}^{-1}.$$

The stress at the bottom fibre due to the live-load moment on the uncracked section is 8.580 MPa. Addition of this value to the stress of 2.323 MPa existing before application of the live load (Fig. 7.13(b)) gives the stress at the bottom fibre after the live-load application with the assumptions of state 1

$$\sigma_{1\ max} = 2.323 + 8.580 = 10.903\,\mathrm{MPa}.$$

The interpolation coefficient between states 1 and 2 (Equation (8.49)) is

$$\zeta = 1 - \beta_1\beta_2\left(\frac{f_{\mathrm{ct}}}{\sigma_{1\,\max}}\right)^2$$

$$= 1 - 1.0 \times 1.0\left(\frac{2.5}{10.903}\right)^2 = 0.95.$$

$\beta_1 = 1.0$ because of the high-bond quality of the reinforcement and $\beta_2 = 1.0$, assuming that the deflection is calculated for non-repetitive loading.

The mean curvature at mid-span (Equation (8.44)) is

$$\psi_{\mathrm{m}} = (1 - 0.95)1670 \times 10^{-6} + 0.95 \times 2556 \times 10^{-6}$$

$$= 2510 \times 10^{-6}\ \mathrm{m}^{-1}(63.8 \times 10^{-6}\ \mathrm{in}^{-1}).$$

The curvature variation over the span is shown in Fig. 8.9(b).[7] The length of the zone where cracking occurs is 14.8 m. Over this zone, three lines are plotted for curvatures in states 1 and 2 and mean curvature.

If we assume parabolic variation and use the values of the mean curvature at the ends and the centre, we obtain by Equation (C.8):

$$\text{Deflection at the centre} = \frac{20^2}{96}\,[2(-402) + 10 \times 2510]10^{-6}$$

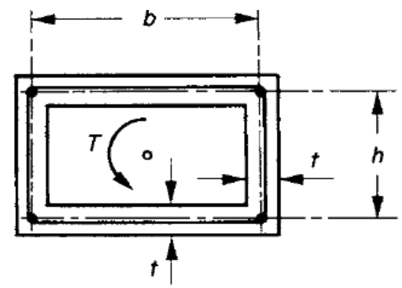
$$= 101.2\,\text{mm}\ (3.99\,\text{in}).$$

Using five sections instead of three and employing Equation (C.16) gives a more accurate value for the central deflection after application of live load of 86.2 mm (3.39 in). The dead-load deflection, including effects of creep, shrinkage and relaxation is 38.4 mm (1.51 in).

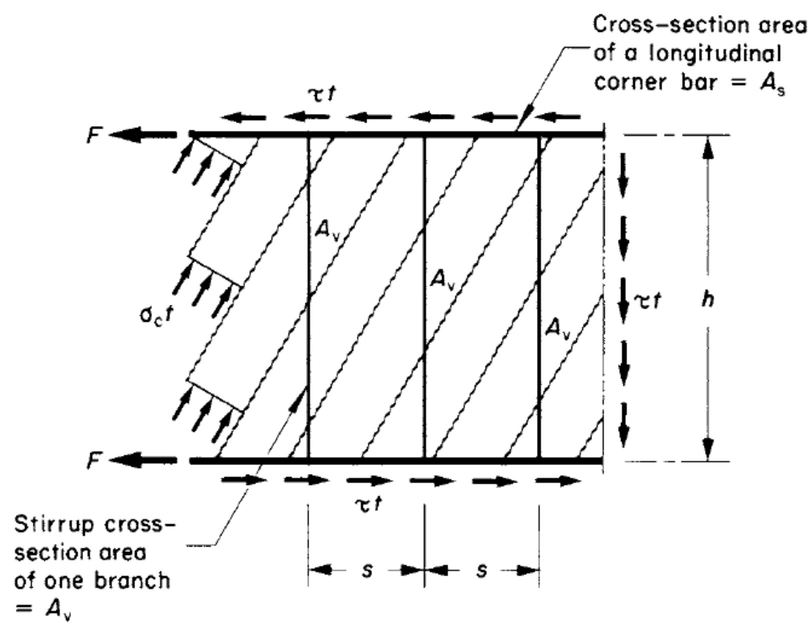
In the design of a partially prestressed cross-section, the amount of non-prestressed steel may be decreased and the prestressed steel increased such that the ultimate strength in flexure is unchanged. The amount of deflection is one criterion for the decision on the amounts of prestressed steel and non-prestressed reinforcement. The calculated deflection in this example may be considered excessive. Assuming that the yield stresses of the non-prestressed reinforcement and the prestressed steel are 400 and 1600 MPa (58 and 230 ksi), the area of the bottom non-prestressed reinforcement may be reduced from 1600 to 400 mm^2, with the addition of prestressed steel of area 300 mm^2 at the same level without substantial change in the flexural strength of the section. If the stress before transfer is the same in all prestressed steel as in the original design, the tension in the added prestressed steel before transfer is 312.5 kN.

With the second design, the curvatures in states 1 and 2 at mid-span, after application of the live load, will respectively be 1109×10^{-6} and 1976×10^{-6} m^{-1} and the corresponding mean curvature will be 1897×10^{-6} m^{-1}. The deflection just before and after the application of the live load will respectively be 6.0 and 43.1 mm (0.24 and 1.70 in) and the length of the cracked zone after the live-load application will be 12.5 m (40.8 ft).

Torsional deformation



(a)



(b)

According to the theory of elasticity, the angle of twist per unit length is

$$\theta_1 = \frac{T}{G_c J_1} \tag{8.50}$$

where T is the twisting moment, G_c is the shear modulus of concrete and J_1 is the torsion constant. For a rectangular section,

$$J_1 = cb^3\left[\frac{1}{3} - 0.21\,\frac{b}{c}\left(1 - \frac{b^4}{12c^4}\right)\right] \tag{8.51}$$

where c and b are the two sides of the rectangle with $b \leqslant c$. The maximum shear stress is at the middle of the longer side c and its value

$$\tau_{\max} = \frac{T}{\mu b c^2} \tag{8.52}$$

where μ is a dimensionless coefficient which varies with the aspect ratio c/b as follows:[9]

c/b	1.0	1.5	1.75	2.0	2.5	3.0	4.0	6.0	8.0	10.0	∞
μ	0.208	0.231	0.239	0.246	0.258	0.267	0.282	0.299	0.307	0.313	0.333

For a closed hollow section

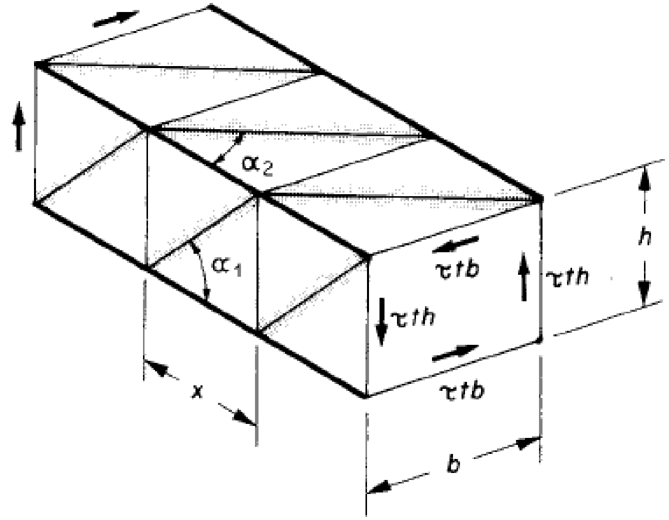
$$J_1 = 4A_0^2[\int(\mathrm{d}s/t)]^{-1} \tag{8.53}$$

where t is the wall thickness; A_0 is the area enclosed by a line through the centre of the thickness and the integral is carried out over the circumference.

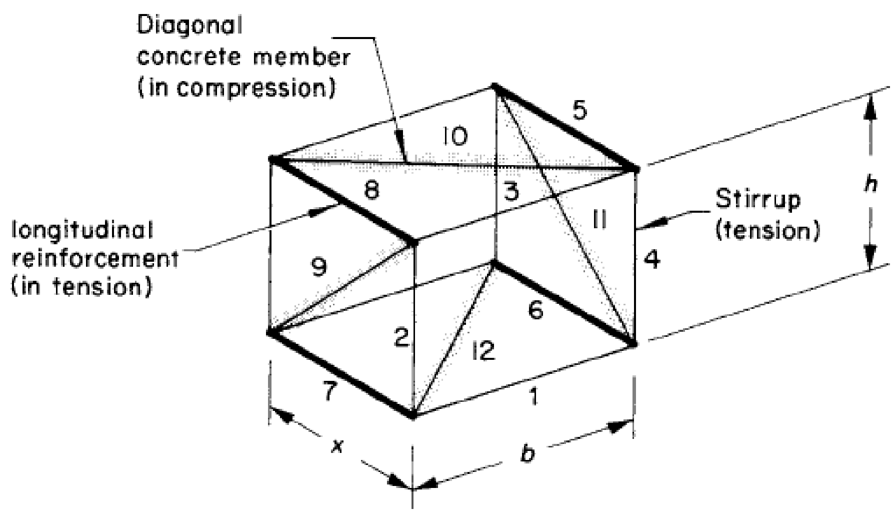
The shear flow (the shearing force per unit length of the circumference) is given by:

$$\tau t = T/2A_0 \tag{8.54}$$

where τ is the shear stress.


α₂
α₁
τtb
τth
τth
τtb
h
x
b

(a)


Diagonal
concrete member
(in compression)
10
5
8
3
11
Stirrup
(tension)
h
longitudinal
reinforcement
(in tension)
9
4
6
2
12
7
1
x
b

(b)

$$F_1 = F_3 = \frac{1}{2h} \qquad F_2 = F_4 = \frac{1}{2b}; \tag{8.56}$$

forces in the longitudinal bars

$$F_5 = F_6 = F_7 = F_8 = \frac{x}{2hb}; \tag{8.57}$$

forces in the diagonal members

$$F_9 = F_{11} = -\frac{1}{2b \sin \alpha_1} \qquad F_{10} = F_{12} = -\frac{1}{2h \sin \alpha_2}; \tag{8.58}$$

where α_1 and α_2 are angles defined in Fig. 8.11(a). It is suggested that the distance x in Fig. 8.11(b) be selected such that the angles α_1 and α_2 are close to 45 degrees.

The angle of twist per unit length of the cracked member is considered the same as the relative rotation of the two cross-sections of the panel in Fig. 8.11(b) divided by the distance x between them. Considering virtual work, the angle of twist per unit length is:

$$\theta = \frac{T}{x} \sum_{i=1}^{12} \left(\frac{F^2 l}{AE}\right)_i \tag{8.59}$$