

chap. 10 Energy methods

- 2 virtual work principles
 - i) PVW --- entirely equivalent to the equilibrium eqns
however, does not provide any information about the other
2 sets of eqns. { strain-displacement relationship
 | constitutive laws
 - ii) PCVW -- " the strain-displacement relationships
 " { equilibrium eqns
 | constitutive laws

Type of forces

- in virtual work principles, various categories of forces are clearly defined and used.
 - ① internal, external forces
 - ② reaction forces --- can be eliminated from the formulation since the work they perform vanishes when using kinematically admissible virtual displacements
But, when arbitrary virtual displacements are used, the virtual work does not vanish. → become an integral part of the formulation

Conservative forces

- the work they perform always vanishes for a closed path displacement
- total mechanical energy of the system is preserved
- if the externally applied forces are conservative, they can be derived from a potential → further simplify the calculation of VW
- if the strain energy of an elastic component exists, the corresponding elastic forces can be derived from this strain energy → "

• combination of } PVW
 } strain energy
 } potential of external forces } → principle of minimum
 total potential energy

PVW is always valid

PMTPE is limited to systems involving conservative forces

10.1 Conservative forces

\underline{r} : position vector of a particle

\underline{F} : force acting the particle, depends only upon the position of the particle, $\underline{F} = F(\underline{r})$

- Fig. 10.1 ... two arbitrary paths ACB , ADB

• Definition

- \underline{F} is conservative iff the work it performs along any path joining the same initial and final points is identical.

$$W = \int_{ACB} \underline{F} \cdot d\underline{r} = \int_{ADB} \underline{F} \cdot d\underline{r} \quad (10.1)$$

- work done along path ADB = (-) that along BDA

- work over the closed path $ACBPA$ = 0

$$W = \oint_{\text{any path}} \underline{F} \cdot d\underline{r} = \oint_C \underline{F} \cdot d\underline{r} = 0 \quad (10.2)$$

• Potential of a conservative force

- Stoke's theorem

$$\oint_C \underline{F} \cdot d\underline{r} = \iint_A \bar{n} \cdot \nabla \times \underline{F} \, dA = 0 \quad (10.3)$$

A : area enclosed by curve C

\bar{n} : outward normal to area A (Fig. 10.2)

$$\rightarrow \nabla \times \underline{F} = 0 \quad \rightarrow \nabla \times \nabla \Phi = 0 \quad (\Phi : \text{arbitrary scalar fn.})$$

- sol. of eqn. $\nabla \cdot \underline{F} = 0$ ---

$$\underline{F} = -\nabla \Phi \quad (10.4)$$

justified later "potential"

$$\underline{\Phi} = -\nabla \Phi = -\frac{\partial \Phi}{\partial x_1} \hat{i}_1 - \frac{\partial \Phi}{\partial x_2} \hat{i}_2 - \frac{\partial \Phi}{\partial x_3} \hat{i}_3 \quad (10.5)$$

- work done by a conservative force

$$W = \int_{r_1}^{r_2} \underline{F} \cdot d\underline{r} = - \int_{r_1}^{r_2} \nabla \Phi \cdot d\underline{r}$$

$$= - \int_{r_1}^{r_2} \left(\frac{\partial \Phi}{\partial x_1} dx_1 + \frac{\partial \Phi}{\partial x_2} dx_2 + \frac{\partial \Phi}{\partial x_3} dx_3 \right) = - \int_{r_1}^{r_2} d\Phi = \Phi(r_1) - \Phi(r_2)$$

depends only on the position of initial / final points

can be evaluated as the difference between the values of the potential fn.

$$W = \Phi(r_1) - \Phi(r_2) = -\Delta \Phi \quad (10.6)$$

Examples of conservative forces

i) gravity force --- $\underline{F} = mg \underline{r} \cdot \hat{i}_3 = mgx_3$

$$\underline{F}_g = -\nabla \Phi = -\frac{\partial \Phi}{\partial x_3} \hat{i}_3 = -mg \hat{i}_3$$

$$W = \int_{x_{3a}}^{x_{3b}} \underline{F}_g \cdot d\underline{x} = - \int_{x_{3a}}^{x_{3b}} \frac{\partial \Phi}{\partial x_3} dx_3 = \Phi(x_{3a}) - \Phi(x_{3b})$$

ii) restoring force of an elastic spring ---

restoring force $-ku$,

potential $A(u) = \frac{1}{2}ku^2$ --- "strain energy"

elastic force $F_s = -\frac{\partial A}{\partial u} = -ku$

$$W = \int_{u_a}^{u_b} F_s du = - \int_{u_a}^{u_b} \frac{\partial A}{\partial u} du = A(u_a) - A(u_b)$$

10.1.1 Potential for internal and external forces

- in PVW, a distinction is made between { internal forces
externally applied loads

- In elastic systems, internal forces { stresses acting in a body
elastic forces in structural components

→ potential of internal forces = "strain energy", "deformation energy",
 "internal energy" ... A

$$W_I = -\Delta A \quad (10.7)$$

- potential of external forces ... Φ

$$W_E = -\Delta \Phi \quad (10.8)$$

- total potential energy

$$\Pi = A + \Phi \quad (10.9)$$

- total work done by both internal and external forces

$$W = W_I + W_E = -\Delta A - \Delta \Phi = -\Delta \Pi \quad (10.10)$$

... "for conservative systems, the work done by the internal and external forces = negative change in total potential energy"

- adding an arbitrary constant to the potential Π will not alter the work done

10.1.2 Calculation of the potential fns

- potential of internal forces ... "strain energy", $A = A(\epsilon)$
 it is convenient to select $A(\epsilon = 0) = 0$, undeformed or untrained state.

$$W_I = -\Delta A = -[A(\epsilon) - A(\epsilon = 0)] = -A(\epsilon)$$

(10.11)

$$A(\epsilon) = -W_I$$

- it is cumbersome to compute the work done within a solid as the negative product of the internal stress component acting through strains or deformations. → alternative approach

Eq. (7.19), $W_I = -W_E \rightarrow$

$$A(\epsilon) = W_E \quad (10.12)$$

... if the internal forces in a solid are conservative, the work done by the externally applied forces = strain energy stored in a body

- assumption ... the forces are applied slowly, in a quasi-steady manner
associated kinetic energy is negligible

- potential of the externally applied loads, Φ ... negative of the
work done by the external force acting through the displacements.

N_p forces, P_i , const. magnitude, line of action fixed in space \rightarrow "dead
loads"

$$\Phi = -W_F = -\sum_{i=1}^{N_p} P_i d_i - \sum_{j=1}^{N_d} Q_j \phi_j \quad (10.13)$$

- Non-conservative forces

i) aerodynamic force ... lift \propto AOA, non-conservative, cannot be
derived from potential

ii) follower force ... const. magnitude, but the orientation of their line
of action changes with the rotation of structures
ex) thrust of a rocket jet engine

10.2 Principle of minimum total potential energy

- system represented by N generalized coord. $\underline{q} = \{q_1, q_2, \dots, q_N\}^T$

- if the system is conservative, strain energy $A = A(\underline{q})$

potential of the externally applied loads $\Phi = \Phi(\underline{q})$

\rightarrow infinitesimal increment

$$dA = \frac{\partial A}{\partial q_1} dq_1 + \frac{\partial A}{\partial q_2} dq_2 + \dots + \frac{\partial A}{\partial q_N} dq_N = \sum_{i=1}^N \frac{\partial A}{\partial q_i} dq_i \quad (10.14)$$

$$d\Phi = \frac{\partial \Phi}{\partial q_1} dq_1 + \frac{\partial \Phi}{\partial q_2} dq_2 + \dots + \frac{\partial \Phi}{\partial q_N} dq_N = \sum_{i=1}^N \frac{\partial \Phi}{\partial q_i} dq_i$$

- VW done by the internal forces $\delta W_I = -\delta A(\underline{q})$

external ... $\delta W_E = -\delta \Phi(\underline{q})$

$$\delta W_I = -\delta A = -\sum_{i=1}^N \frac{\partial A}{\partial q_i} \delta q_i \quad (10.15)$$

$$\delta W_E = -\delta \Phi = -\sum_{i=1}^N \frac{\partial \Phi}{\partial q_i} \delta q_i$$

- comparing Eq. (9.24) and (10.13),

$$Q_i^I = -\frac{\partial A}{\partial q_i}, \quad Q_i^E = -\frac{\partial \Phi}{\partial q_i} \quad (10.16)$$

- $\sim PVW : Q_i^I + Q_i^E = 0$, by introducing Eq. (10.16)

$$-\frac{\partial A}{\partial q_i} - \frac{\partial \Phi}{\partial q_i} = \frac{\partial (A + \Phi)}{\partial q_i} = \frac{\partial \text{total potential}}{\partial q_i} = 0 \quad (10.17)$$

$$\delta W = -\delta \Pi$$

- Principle 4 : a system is in static equilibrium iff the sum of the Π done by the internal and external forces vanishes for all arbitrary virtual displacements, $\rightarrow \delta W = -\delta \Pi = 0$

$$\rightarrow \delta \Pi = 0 \quad (10.18)$$

$$\delta \Pi = \sum_{i=1}^N \left[\frac{\partial \Pi}{\partial q_i} \right] \delta q_i = 0 \quad (10.19)$$

||
0 → Eq. (10.17)

- \sim Principle 8 : A conservative system is in equilibrium iff virtual changes in the total PE vanish for all virtual displacements.

"Principle of stationary TPE"

- Kinematically admissible virtual displacements are used \rightarrow reaction forces are eliminated from the formulation.

Arbitrary virtual displacements \rightarrow reaction forces must be treated as externally applied loads

- Graphical illustration of Principle 8 (Fig. 10.3)
 \dots TPE is stationary at points A, B and C.

- \sim increments in TPE by Taylor series

$$d\Pi \approx \sum_{i=1}^N \frac{\partial \Pi}{\partial q_i} dq_i + \sum_{i=1}^N \sum_{j=1}^N \frac{\partial^2 \Pi}{\partial q_i \partial q_j} dq_i dq_j$$

in the neighborhood of static equilibrium,

$$d\pi \simeq \sum_{i=1}^N \sum_{j=1}^N \frac{\partial^2 \pi}{\partial q_i \partial q_j} dq_i dq_j \quad (10-20)$$

- ① $\underline{\Delta} > 0$ for all $dq_i \rightarrow TPE$ is minimum at equilibrium
 \rightarrow "stable" (A) --- TPE cannot increase without an external source of E
 - ② $\underline{\Delta} = 0 \rightarrow$ "neutrally stable" (B)
 - ③ $\underline{\Delta} < 0 \rightarrow$ "unstable" (C) --- released PE is converted to KE,
 leading to spontaneous motion of the system
 - Principle 9 --- A conservative system is in a "stable" state of equilibrium iff the TPE is a min. w.r.t. changes in the generalized coord.
 - 10.2.1 Non-conservative external forces
 - if the externally applied loads are not conservative
 - $\delta W = \delta W_L + \delta W_E = - \delta A + \delta W_E^{nc} = 0$
 - \rightarrow Principle 10 --- A system is in equilibrium iff virtual changes in the strain energy equal the VW done by the externally applied loads for all arbitrary virtual displacements

10.3 Strain energy in springs

- strain energy --- function of deformation of the structure
 $A = A(\varepsilon)$

deformation field \rightarrow function of {the displacement field
generalized coord.

spring { rectilinear spring
torsional / rotational spring

10.2.1 Rectilinear springs

- 2 primary lumped properties { stiffness constant
un-stretched length : u_0
- force applied to the spring : F , force in the spring : F_s
constitutive behavior : $F = F_s(\Delta)$, $\Delta = u - u_0$: extension

$$F(\Delta = 0) = F(u = u_0) = 0$$

• Linearly elastic spring

- relationship between an applied load and the resulting extension is linear ($F = k\Delta$) \rightarrow spring is linear
- k : stiffness constant, unit: force / length, N/m
- strain energy in the spring (10-21)
- $A = W_E = \int_{u_0}^u F du = \int_{u_0}^u k\Delta du = \int_0^\Delta k\Delta d\Delta = \frac{1}{2}k\Delta^2 = \frac{1}{2}F\Delta$
- positive-definite fn. of Δ , i.e., $A > 0$ for any (+) or (-) Δ
vanishes only when $\Delta = 0$
- internal force in the spring $F_s = - \frac{\partial A}{\partial u} = -k\Delta$
 - (-) : force in the spring opposes the externally applied force
- constitutive law: straight line in the force vs. extension plt (Fig. 10.5)
- strain energy (A): shaded area under the curve
- complementary strain energy (A'), stress energy: shaded area to the left of the straight line, "force energy"
- $$A' = \int_0^F (u - u_0) dF = \int_0^F \Delta dF = \int_0^F \frac{F}{k} dF = \frac{1}{2} \frac{F^2}{k} = \frac{1}{2} F_s \Delta \quad (10.22)$$

$$A' = \frac{1}{2} \frac{F^2}{k} = \frac{1}{2} F \Delta = \frac{1}{2} k \Delta^2 = A$$

Fig. 10.5 $A = A' = \frac{1}{2} F \Delta$, $A + A' = F \Delta$ (10.23)

Nonlinearly elastic springs

- metals (aluminum, copper) ... slight amount of nonlinearly elastic behavior prior to yield point
- elastomers ... quite obvious nonlinearly elastic behavior
- analytical models, the simplest form

$$F = F_0 \tanh\left(\frac{\Delta}{u_0}\right) \quad (10.24)$$

F_0 : ref. force, u_0 : ref. displacement

- Fig. 10.6 ... aluminum, no sharp transition from linear to nonlinear behavior

$$k = \frac{\partial F}{\partial \Delta} = \frac{F_0}{u_0} \operatorname{sech}^2\left(\frac{\Delta}{u_0}\right) = k_0 \operatorname{sech}^2\left(\frac{\Delta}{u_0}\right)$$

k_0 ... stiffness of the spring at zero elongation

- strain energy

$$A = \int_0^\Delta F \, du = F_0 u_0 \int_0^\Delta \tanh \bar{\Delta} \, d\bar{\Delta} = F_0 u_0 \ln(\cosh \bar{\Delta})$$

complementary strain energy

$$A' = \int_0^F \Delta \, dF = F_0 u_0 \int_0^F \operatorname{arctanh}(\bar{F}) \, d\bar{F} = u_0 F_0 (\bar{F} \operatorname{arctanh} \bar{F} + \ln \sqrt{1 - \bar{F}^2})$$

- In contrast to the linearly elastic spring, $A \neq A'$, however, $A + A' = F \Delta$

- elastic force in the spring

$$F = \frac{\partial A}{\partial \Delta} = \frac{1}{u_0} \frac{\partial}{\partial \Delta} [F_0 u_0 \ln(\cosh \bar{\Delta})] = F_0 \tanh\left(\frac{\Delta}{u_0}\right) \quad (10.25)$$

- Fig. 10.7, upper ... strain energy or potential
middle ... force-extension relationship

→ "softening spring", decreasing stiffness at higher extensions

10.3.2 Torsional springs

- angular motion, θ , under the action of an externally applied torque, M (Fig. 10.9)
- linearly elastic torsional spring: $M = k\theta$
 k : unit $\text{N}\cdot\text{m}/\text{rad}$, $\text{N}\cdot\text{m}/\text{deg}$

10.3.3 Bars

- strain energy

$$A = \frac{1}{2} ke^2 = \frac{1}{2} \cdot \frac{EA}{L} e^2 \quad (10.29)$$

e : bar elongation

10.4 Strain energy in beams

10.4.1 Beam under axial loads

- beam subjected only to axial loads (Fig. 5.6)
- infinitesimal slice, left face displacement \bar{u}_1
 right " $\bar{u}_1 + \left(\frac{d\bar{u}_1}{dx_1}\right) dx_1$
- left face, axial force N_1 , displacement from 0 to \bar{u}_1 , work:
 - $\frac{1}{2} N_1 \bar{u}_1$, (-) due to that displacement and force are created positive in opposite directions
- right face, work $= \frac{1}{2} N_1 \left[\bar{u}_1 + \left(\frac{d\bar{u}_1}{dx_1}\right) dx_1 \right]$
- total work $= \frac{1}{2} N_1 \left(\frac{d\bar{u}_1}{dx_1}\right) dx_1 = \frac{1}{2} N_1 \bar{\epsilon}_1 dx_1$
- external work: $dW_E = \frac{1}{2} N_1 \bar{\epsilon}_1 dx_1 = \frac{1}{2} S \bar{\epsilon}^2 dx_1 \quad (10.33)$

$$a(\bar{\epsilon}_1) = \frac{1}{2} S \bar{\epsilon}^2 \quad (10.34)$$

: "strain energy density function"

- potential of the axial force, $N_1 = - \frac{\partial a(\bar{\epsilon}_1)}{\partial \bar{\epsilon}_1} = \frac{1}{2} S \bar{\epsilon}$
 internal force in the beam

- total strain energy by the axial force distribution

$$A(\bar{E}) = \int_0^L a(\bar{E}_1) dx_1 = \frac{1}{2} \int_0^L S \bar{E}_1^2 dx_1, \quad (10.35)$$

- in term of the axial force \rightarrow "total stress E"

$$A(\bar{E}) = \int_0^L \frac{N_1^2}{2S} dx_1 = A'(N_1) \text{ "complementary E" } (10.36)$$

$a'(N_1) = \frac{N_1}{ZS}$: "stress energy density function"

"complementary strain energy density"

10.4.2 Beam under transverse loads

- beams subjected to transverse loads (Fig. 5.14)

- left face rotation : $\frac{d\bar{u}_2}{dx_1}$

right " $\frac{d\bar{u}_2}{dx_1} + \left(\frac{d^2\bar{u}_2}{dx_1^2} \right) dx_1$

- work by bending moment M_3 at left face : $-\frac{1}{2} M_3 \frac{d\bar{u}_2}{dx_1}$

(-) due to that rotation and moment are counted positive in opposite directions

$$\text{right " : } \frac{1}{2} M_3 \left[\frac{d\bar{u}_2}{dx_1} + \left(\frac{d^2\bar{u}_2}{dx_1^2} \right) dx_1 \right]$$

- total work : $\frac{1}{2} M_3 \left(\frac{d^2\bar{u}_2}{dx_1^2} \right) dx_1 = \frac{1}{2} M_3 K_3 dx_1$, $\text{c } \text{sectional curvature}$

- external work : $dW_E = \frac{1}{2} M_3 K_3 dx_1 = \frac{1}{2} H_{33}^C K_3^2 dx_1 \quad (10.37)$

$a(K_3) = \frac{1}{2} H_{33}^C K_3^2$: "strain energy density fn" $\quad (10.38)$

- potential of the bending moment $M_3 = - \frac{\partial a(K_3)}{\partial K_3} = -H_{33}^C K_3$
internal moment in the beam

- total strain E by the bending moment distribution

$$A(K_3) = \int_0^L a(K_3) dx_1 = \frac{1}{2} \int_0^L H_{33}^C K_3^2 dx_1, \quad (10.39)$$

or $A(u_2(x_1)) = \frac{1}{2} \int_0^L H_{33}^C \left(\frac{d^2\bar{u}_2}{dx_1^2} \right)^2 dx_1, \quad (10.40)$

or $A(M_3) = \int_0^L \frac{M_3^2}{2H_{33}^C} dx_1 = A'(M_3) \quad (10.41)$

$a'(M_3) = \frac{1}{2} \frac{M_3^2}{H_{33}^C}$: "stress E density fn"

10.4.3 Beam under torsional loads

- circular cylindrical beam subjected to torsion
- rotation of the left face: ϕ_1
- .. right " : $\phi_1 + \left(\frac{d\phi_1}{dx_1}\right) dx_1$
- work by the torque M_1 at the left face: $-\frac{1}{2} M_1 \phi_1$
- (-) due to that rotation and torque are counted positive in opposite dir.
- .. right " : $\frac{1}{2} M_1 [\phi_1 + \left(\frac{d\phi_1}{dx_1}\right) dx_1]$
- total work: $\frac{1}{2} M_1 \left(\frac{d\phi_1}{dx_1}\right) dx_1 = \frac{1}{2} M_1 \kappa_1 dx_1$
↑ sectional twist rate
- external work: $dW_E = \frac{1}{2} M_1 \kappa_1 dx_1 = \frac{1}{2} H_{11} \kappa_1^2 dx_1$ (10.42)
- $a(\kappa_1) = \frac{1}{2} H_{11} \kappa_1^2$: "strain E density fm" (10.43)
- potential of the torque, $M_1 = -\frac{\partial a(\kappa_1)}{\partial \kappa_1} = -H_{11} \kappa_1$
↑ internal torque in the beam
- total strain energy by the torque distribution

$$A(\kappa_1) = \int_0^L a(\kappa_1) dx_1 = \frac{1}{2} \int_0^L H_{11} \kappa_1^2 dx_1 \quad (10.44)$$

or

$$A(M_1) = \int_0^L \frac{M_1^2}{2H_{11}} dx_1 = A'(M_1) \quad \begin{matrix} \text{"total complementary"} \\ \text{"strain E stored"} \end{matrix} \quad (10.45)$$

$$a'(M_1) = \frac{M_1^2}{2H_{11}} \quad \text{"stress E density fm"}$$

10.4.4 Relationship with VW

- internal VW by a bending moment M_3 : $dW_I = -M_3 \kappa_3 dx_1$, Eq. (9.69)
- $dW_E = -dW_I = M_3 \kappa_3 dx_1$

However, in Sec. 10.4, strain E stored in beam is

$$dW_E = \frac{1}{2} M_3 \kappa_3 dx_1$$

↑ $\frac{1}{2}$ factor difference

- internal VW: bending moment is assumed to remain constant while undergoing a curvature

$$dW_E = \left[\int_0^{x_3} M_3 dx_3 \right] dx_1 = \left[M_3 \int_0^{x_3} dx_3 \right] dx_1 = M_3 x_3 dx_1,$$

- strain E stored in beam : bending moment is assumed grow in proportion to the curvature

$$dW_E = \left[\int_0^{x_3} M_3 dx_3 \right] dx_1 = \left[\int_0^{x_3} k x_3 dx_3 \right] dx_1 = \frac{1}{2} k x_3^2 dx_1, \\ = \frac{1}{2} M_3 x_3 dx_1$$

- same reasoning for torsion

internal, external VW : $dW_E = -dW_I = M_1 x_1 dx_1$

strain E : $dW_E = \frac{1}{2} H_1 x_1^2 dx_1$
difference

- when computing VW and CW : virtual displacements do not affect the forces or stresses in the system

strain E stored in the structure : internal forces and moments increase in proportion to the deformation

10.5 strain energy in solids

10.5.1 3-D solid

- Sec. 9.7.3, work done by the constant, external stress

$$W_E = \int_V \underline{\sigma}^T \underline{\epsilon} dV \quad (9.76)$$

- Then, if the stresses increase in proportion to the deformations

$$W_E = \frac{1}{2} \int_V \underline{\sigma}^T \underline{\epsilon} dV \quad (10.46)$$

- Hooke's law, $\underline{\sigma} = C \underline{\epsilon}$ (2.13)

$$C = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & & & & & \\ \nu & 1-\nu & \nu & & & & & \\ \nu & \nu & 1-\nu & & & & & \\ & & & \ddots & & & & \\ & & & & \frac{1-2\nu}{2} & 0 & 0 & \\ & & & & 0 & \frac{1-2\nu}{2} & 0 & \\ & & & & & 0 & \frac{1-2\nu}{2} & \\ & & & & & & 0 & \end{bmatrix} \quad (2.14)$$

$$W_E = \frac{1}{2} \int_V \frac{E}{(1+\nu)(1-2\nu)} \left[(1-\nu)(\epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2) + 2\nu(\epsilon_1 \epsilon_2 + \epsilon_1 \epsilon_3 + \epsilon_2 \epsilon_3) \right. \\ \left. + \frac{1-2\nu}{2} (\gamma_{23}^2 + \gamma_{31}^2 + \gamma_{12}^2) \right] dV = \int_V a(\epsilon) dV = A(\epsilon)$$

$\alpha(\epsilon)$: "strain E density fn for a 3-D solid"

- more compact form

$$\alpha(\epsilon) = \frac{1}{2} \frac{E}{(1+\nu)(1-2\nu)} [(1-\nu) I_1^2 - \nu (1-2\nu) I_2] \quad (10.48)$$

I_1, I_2 : first 2 invariants of the strain tensor, Eqs (1.26)

$$- \alpha(\epsilon) = \frac{1}{2} \underline{\epsilon}^T \underline{\epsilon} \quad (10.49)$$

- Hooke's law is a linear relationship $\Rightarrow \alpha(\epsilon) = \alpha'(\sigma)$

- complementary strain E density

$$\begin{aligned} \alpha'(\sigma) = & \frac{1}{2E} [\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu (\sigma_{12} + \sigma_{13} + \sigma_{23}) \\ & + \nu (1+\nu) (\tau_{11}^2 + \tau_{22}^2 + \tau_{33}^2)] \end{aligned} \quad (10.50)$$

$$- \underline{\epsilon} = \underline{\underline{\sigma}} \underline{\underline{\sigma}} \quad (2.10)$$

$$\underline{\underline{\sigma}} = \frac{1}{E} \left[\begin{array}{ccc} 1 & -\nu & -\nu \\ -\nu & 1 & -\nu \\ -\nu & -\nu & 1 \\ \hline 0 & 0 & 0 \end{array} \right] \quad (2.12)$$

$$- \alpha'(\sigma) = \frac{1}{2} \underline{\sigma}^T \underline{\underline{\sigma}} \underline{\sigma} \quad (10.52)$$

10.5.2 3-D beams

- Eq. (9.78): internal w done by const. stress resultants in 3-D beams

- w done by the same stress resultants when they increase in proportion to the deformation

$$W_E = \frac{1}{2} \int_0^L (N_1 \bar{\epsilon}_1 + M_2 \bar{x}_2 + M_3 \bar{x}_3) dx, \quad (10.53)$$

Hooke's law \rightarrow sectional constitutive laws, Eq. (6.12)

$$A = \frac{1}{2} \int_0^L (\sigma_1^2 + H_{22}^c \bar{x}_2^2 - 2H_{23}^c \bar{x}_2 \bar{x}_3 + H_{33}^c \bar{x}_3^2) dx, \quad (10.54)$$

- complementary strain E --- using the compliance form, Eq. (6.13)

$$A' = \frac{1}{Z} \int_0^L \left(\frac{N_i^2}{S} + \frac{H_{33}^c}{\Delta H} M_3^2 + Z \frac{H_{23}^c}{\Delta H} M_2 M_3 + \frac{H_{22}^c}{\Delta H} M_2^2 \right) dx, \quad (10.15)$$

where, $\Delta H = H_{22}^c H_{33}^c - H_{23}^c {}^2$

assuming that the origin must be located at the section's centroid

10.6 Applications to trusses and beams

10.6.1 Application to trusses

- 3-bar, hypostatic truss (Fig. 10.16)

- bar length : $L_1 = L_3 = L/\cos\theta, L_2 = L$

bar elongations : Eq. (9.27), $e_1 = u_1 \cos\theta + u_2 \sin\theta, e_2 = u_2,$
 $e_3 = -u_1 \cos\theta + u_2 \sin\theta$

- bar strain E : $A = \frac{1}{Z} k e^2, \text{Eq. (10.29)}, k = \frac{EA}{L}$ (bar stiffness)

$$A = \frac{1}{Z} \left(\frac{EA \cos\theta}{L} e_1^2 + \frac{EA}{L} e_2^2 + \frac{EA \cos\theta}{L} e_3^2 \right)$$

$$= \frac{1}{Z} \frac{EA}{L} \left[(u_1 \cos\theta + u_2 \sin\theta)^2 \cos\theta + u_2^2 + (-u_1 \cos\theta + u_2 \sin\theta)^2 \cos\theta \right]$$

$$= \frac{1}{Z} \frac{EA}{L} \left[Z u_1^2 \cos^3\theta + (1 + Z \sin^2\theta \cos\theta) u_2^2 \right]$$

- potential of externally applied load, $P_1 \rightarrow \underline{\Phi} = -P_1 u,$

total potential $\Pi = A + \underline{\Phi} = A - P_1 u,$

- 2 D.O.F.'s, PMTPE, Eq. (10.17) \rightarrow

$$\frac{\partial \Pi}{\partial u_1} = \frac{EA}{L} Z u_1 \cos^3\theta - P_1 = 0$$

$$\frac{\partial \Pi}{\partial u_2} = \frac{EA}{L} (1 + Z \sin^2\theta \cos\theta) u_2 = 0$$

- Matrix form : two linear eqns for the Z generalized coord.

$$\begin{bmatrix} Z \cos^3\theta & 0 \\ 0 & 1 + Z \sin^2\theta \cos\theta \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \frac{L}{EA} \begin{Bmatrix} P_1 \\ 0 \end{Bmatrix}$$

$$\rightarrow u_1 = \frac{P_1 L}{ZEA \cos^3\theta}, \quad u_2 = 0$$

elongations : $\frac{e_1}{L} = \frac{1}{2\cos^2\theta} \frac{P}{EA}$, $e_2 = 0$, $\frac{e_3}{L} = -\frac{1}{2\cos^2\theta} \frac{P}{EA}$

- bar forces : $\frac{F_1}{P_1} = \frac{1}{2\cos\theta}$, $F_2 = 0$, $\frac{F_3}{P_1} = -\frac{1}{2\cos\theta}$

- PMTPE ... does NOT make special provisions for the fact that the 3-bar truss is a hyperstatic structure.

- various loading condition -- Right of Fig. 10-16

$$\Phi = -P_1 u_1 - P_2 u_2$$

$$\begin{bmatrix} 2\cos^2\theta & 0 \\ 0 & 1+2\sin^2\theta\cos\theta \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \frac{L}{EA} \begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix}$$

$$\rightarrow u_1 = \frac{P_1 L}{2EA\cos^2\theta}, \quad u_2 = \frac{P_2 L}{EA(1+2\sin^2\theta\cos\theta)}$$

General procedure

- ① find length L_i , stiffness $k_i = \frac{(EA)_i}{L_i}$ of each bar.
- ② select the generalized coord. of the problem, except those at the supports
- ③ find bar extensions e_i in terms of the joint displacements, Eq. (9.27)
- ④ determine the total strain E , $A = \frac{1}{2} \sum_{i=1}^{N_b} k_i e_i$
- ⑤ write Φ , using Eq. (10.13), $\Phi = -\sum_{j=1}^{N_p} P_j d_j$
- ⑥ PMTPE \rightarrow governing eqns. by Eq. (10.17), linear set of N eqns for N gen coord.
- ⑦ solve the eqns
- ⑧ determine bar elongations
- ⑨ " bar forces from $F_i = k_i e_i$

both iso- and hyperstatic problems can be solved in the same manner.

10.6.3 Applications to beams

- beam under a distributed transverse load, $p_2(x_1)$, Fig. 5.14
 - potential of the externally applied loads

$$\Phi = - \int_0^L p_2(x_1) \bar{u}_2(x_1) dx,$$

(10.3d)

- total potential Π of the beam --- from Eq. (10.9)

$$\Pi = A + \Phi = \frac{1}{2} \int_0^L H_{33} \left(\frac{d^2 \bar{u}_2}{dx_1^2} \right)^2 dx_1 - \int_0^L p \bar{u}_2 dx_1$$

Eq. (10.40)

--- now $\Pi = \Pi(\bar{u}_2(x_1))$, a function of another function \Rightarrow "functional"

\Rightarrow beam problems are infinite dimensional or continuous problems since determination of the transverse displacement field, $\bar{u}_2(x_1)$

\hookrightarrow planar truss w/ ZN unknowns, "finite dimensional, discrete"

• minimization of the TPE of finite dimension \rightarrow standard calculus functional \rightarrow calculus of variations

• Reduction of infinite # of DoFs \rightarrow finite # --- by choosing specific fns for $u_2(x_1) \rightarrow$ chap. 11

3-D beam under complex loading condition

--- distributed loads $p_1(x_1), p_2(x_1), p_3(x_1)$
concentrated .. P_1, P_2, P_3

distributed moment $q_1(x_1), q_2(x_1), q_3(x_1)$
concentrated .. Q_1, Q_2, Q_3

$$\rightarrow \Phi = - \int_0^L p_1 \bar{u}_1 dx_1 - P_1 \bar{u}_1(\alpha L) - \int_0^L q_1 \bar{\theta}_1 dx_1 - Q_1 \bar{\theta}_1(\alpha L)$$

(11.59)

$$- Q_2 \frac{d\bar{u}_3}{dx_1}(\alpha L)$$

$$- Q_3 \frac{d\bar{u}_2}{dx_1}(\alpha L)$$

Euler-Bernoulli assumption --- $\bar{\theta}_3 = \frac{d\bar{u}_2}{dx_1}, -\bar{\theta}_3 \bar{\theta}_3(\alpha L) \rightarrow -Q_3 \frac{d\bar{u}_2}{dx_1}(\alpha L)$

$$\bar{\theta}_2 = -\frac{d\bar{u}_3}{dx_1}, -Q_2 \bar{\theta}_2(\alpha L) \rightarrow Q_2 \frac{d\bar{u}_3}{dx_1}(\alpha L)$$

10.8 P of Minimum CE

Sec. 10.2 --- P of VW \rightarrow P of MTPF

two assumptions ... ① internal forces are conservative \leftarrow strain E
 ② external " also " \leftarrow potential of the externally applied loads

Fig. 10.27 ... constitutive relationship \rightarrow strain E
 2nd assumption not shown

P of MCE \rightarrow P of CVW

2 assumptions --- ① complementary strain E fn.
 ② prescribed displacements can be derived from a potential \rightarrow Sec 10.1.1

10.8.1 potential of the prescribed displacements

Fig. 10.28 ... 3-bar truss, prescribed displacement Δ at B

driving force D, unknown quantity

- P of CVW, Eq. (9.57) ... $\delta W_E' = \Delta \delta D$

now it is assumed that the prescribed displacement can be derived from a potential, Φ' "potential of the prescribed displacement" or

$$\Delta = - \frac{\partial \Phi'(D)}{\partial D} \quad \text{"dislocation potential"} \quad (10.101)$$

$$\delta W_F' = \Delta \delta D = - \frac{\partial \Phi'}{\partial D} \delta D = - \delta \Phi'(D) \quad (10.102)$$

10.9.2 constitutive laws for elastic materials

strain E for a bar ... $A = \frac{1}{2} k e^2$, $k = \frac{EA}{L}$

$$\text{bar forces } F = \frac{\partial A(e)}{\partial e} = ke$$

complementary strain E ... $A' = \frac{1}{2} \frac{1}{k} F^2$, $\frac{1}{k}$: compliance

$$\text{elongation } e = \frac{\partial A(F)}{\partial F} = \frac{1}{k} F$$

linearly elastic material, $A = A'$, $A(e) = \frac{1}{2} k e^2$

$$A'(F) = \frac{1}{2} \frac{1}{k} F^2$$

- elastic, but not linear

$$\text{Eq. (10.23)} \rightarrow A(e) + A'(F) = eF$$

differentiate, $\left(\frac{\partial A}{\partial e}\right) de + \left(\frac{\partial A'}{\partial F}\right) dF = F de + e dF$

Regrouping, $\left[F - \frac{\partial A}{\partial e}\right] de + \left[e - \frac{\partial A'}{\partial F}\right] dF = 0$

2 bracketed terms must vanish

$$F = \frac{\partial A(e)}{\partial e}, \quad e = \frac{\partial A'(F)}{\partial F} \quad (10.103)$$

-- same constitutive laws { in stiffness } form
{ in compliance }

- existence of the { strain E fn } \Leftrightarrow assumption of a constitutive
complementary counterpart law

10.2.3 P of Min. CE

$$\therefore P \text{ of CVW} \dots \delta W' = \delta W_E' + \delta W_I' = 0$$

3-bar truss, Fig. 10.28

$$\delta W_I' = -e_A \delta F_A - e_B \delta F_B - e_C \delta F_C$$

- assuming elastic material, existence of complementary strain E fn.

Eq. (10.103 b) \rightarrow

$$\delta W_I' = -\frac{\partial A'_A(F_A)}{\partial F_A} \delta F_A - \frac{\partial A'_B(F_B)}{\partial F_B} \delta F_B - \frac{\partial A'_C(F_C)}{\partial F_C} \delta F_C$$

$$= -\delta A'_A - \delta A'_B - \delta A'_C = -\delta A'$$

$$A' = A'_A + A'_B + A'_C : \text{total C strain E}$$

- prescribed displacement at B -- can be derived from a potential

$$\delta W_E' = -\delta \Phi'(D)$$

- P of CVW \rightarrow

$$\delta W' = \delta W_E' + \delta W_I' = -\delta A' - \delta \Phi' = -\delta(A' + \Phi') = 0$$

- total C E, $\Pi' \dots \Pi' = A' + \Phi'$ (10.104)

- statement $\dots \delta\Pi' = 0$ (10.105)

- Principle 11 (P of stationary C E) A conservative system undergoes compatible deformations iff the total C E vanishes for all statically admissible virtual forces.

- stationary = minimum value for stable equilibrium
 \rightarrow P of Min. C E

- Principle 12 (P of Min. C E)
 iff the total complementary E is a minimum w.r.t. arbitrary changes in statically admissible forces.

10.2.4 P of least work

total C E = system's C strain E + potential of the prescribed displacement
 if prescribed displacement = 0, total C E = C strain E
 \rightarrow P of least work

- Principle 13 (P of least work) In the absence of prescribed displacement, a conservative system undergoes compatible displacements iff the C strain E is a min. w.r.t. arbitrary changes in statically admissible forces.

• P 14 (P of least work)
 a linearly elastic system .. the strain E

10.9 Energy theorems

- Fig. 10.40 --- properly constrained elastic body subjected to various concentrated loads and couples

$P_i, i=1, 2, \dots, N \rightarrow$ displacement Δ_i

$\Theta_j, j=1, \dots, M \rightarrow$ rotation Φ_j

10.9.1 Clapeyron's theorem

- Eq. (10.12) ... strain E stored in the body = work done by the external forces as they are increased quasi-statically from zero to final values

$$A = W_E = \sum_{i=1}^N \int_0^{\Delta_i} P_i du_i + \sum_{j=1}^M \int_0^{\theta_j} Q_j d\theta_j$$

- linearly elastic ... applied loads are proportional to the displacements

$$P_i \propto u_i, \quad Q_j \propto \theta_j$$

$$A = W_E = \sum_{i=1}^N \frac{1}{2} P_i \Delta_i + \sum_{j=1}^M \frac{1}{2} Q_j \theta_j \quad (10.107)$$

- Clapeyron's theorem \rightarrow useful for evaluating the strain E as well as computing the deflection, A , at the point of application of a load, P

\Leftrightarrow Eq. (10.13) ... difference by a factor of $\frac{1}{2}$.

- load P is assumed to remain constant

difference in the nature of the applied loading

10.9.2 Castigliano's first theorem

- Eq. (10.10) ... $\Pi = A + \Phi = A - \sum_{i=1}^N \cancel{(P_i \Delta_i)}$ dead loads

P of MTPE \rightarrow stationarity of the total Π , Eq. (10.17)

$$\frac{\partial \Pi}{\partial \Delta_j} = \frac{\partial A}{\partial \Delta_j} - \frac{\partial}{\partial \Delta_j} \sum_{i=1}^N P_i \Delta_i = \frac{\partial A}{\partial \Delta_j} - P_j = 0$$

$$\rightarrow P_i = \frac{\partial A}{\partial \Delta_i} : \text{Castigliano's 1st theorem} \quad (10.108)$$

- * All theorems are valid only for elastic structures

Clapeyron's theorem
Castigliano's 2nd theorem } \leftarrow further limited to linearly elastic structures

10.9.3 Crotti-Engesser theorem

- Clapeyron's, Castigliano's 1st theorems \leftarrow P of MTPE

→ parallel developments based on P of MCE

- Eq. (10.104) : $\Pi' = A' + \Phi'$

$\Phi' = - \sum_{i=1}^N p_i \Delta_i$, p_i : driving forces required to obtain the prescribed displacements

$$\rightarrow \Pi' = A' + \Phi' = A' - \sum_{i=1}^N p_i \Delta_i$$

- statically admissible stress field $\rightarrow A' = A'(p_i)$

of MCE $\rightarrow \frac{\partial \Pi'}{\partial p_j} = \frac{\partial A'}{\partial p_j} - \frac{\partial}{\partial p_j} \sum_{i=1}^N p_i \Delta_i = \frac{\partial A'}{\partial p_j} - \Delta_j = 0$

$$\Rightarrow \underline{\Delta_i} = \frac{\partial A'}{\partial p_i} : \text{Crotti-Engesser theorem} \quad (10.109)$$

-- can be applied to multiple applied loads

10.9.4 Castigliano's 2nd theorem

- in the derivation of the Crotti-Engesser theorem, existence of CE is assumed for elastic material

If linearly elastic, $A = A'$

$$\rightarrow \underline{\Delta_i} = \frac{\partial A}{\partial p_i} : \text{Castigliano's 2nd theorem} \quad (10.110)$$

prescribed deflection

10.9.5 Applications of energy theorems

• Castigliano's 2nd theorem -- also useful for hyperstatic problems

- cantilevered beam w/ a tip support

-- a prescribed tip displacement, which is required to vanish

p_i : driving force, \rightarrow reaction force at the support

Castigliano's 2nd theorem $\Rightarrow \Delta_i = 0, \frac{\partial A}{\partial p_i} = 0$

-- compatibility eqn. at the tip support

\rightarrow principle of least work (Principle 13)

10.9.6 The dummy load method

- Is it possible to use Castiglano's 2nd theorem to compute the deflection at a point where no load is applied?
- 1st step ... a fictitious or "dummy load" P is applied to the structure at the point where the displacement is to be computed.
- 2nd " ... $\hat{A} = \frac{\partial A}{\partial P}$ by Castiglano's 2nd theorem
- last " ... $A = \lim_{P \rightarrow 0} \hat{A}$
- $\therefore A = \lim_{P \rightarrow 0} \frac{\partial A}{\partial P}$ (10.11)
- if elastic, but nonlinear, A' must be used instead of A .

Example 10.19 Tip deflection of a cantilevered beam

$$\hat{A} = \frac{\partial A}{\partial P} = \frac{1}{ZH_{33}^c} \left(\frac{P_0 L^4}{4} + \frac{ZPL^3}{3} \right)$$

$$A = \lim_{P \rightarrow 0} \hat{A} = \frac{P_0 L^4}{ZH_{33}^c}$$

or, can be obtained by taking the limit before carrying out the integrations

$$A = \left[\frac{\partial}{\partial P} \int_0^L \frac{M_3^2}{ZH_{33}^c} dx_1 \right]_{P=0} = \int_0^L \frac{M_3}{H_{33}^c} \left[\frac{\partial M_3}{\partial P} \right]_{P=0} dx_1$$

10.9.7 Unit load method revisited

- P of CVW \rightarrow unit load method
- dummy load method $\xrightarrow{\uparrow (?)}$
- Castiglano's 2nd theorem \leftarrow P of Min CE
- dummy load method ... strain E in an isostatic beam

$$A = \int_0^L \frac{M_3^2}{ZH_{33}^c} dx_1$$

$M_3(x_1)$... bending moment distribution generated by the externally applied loads and dummy load

- Castiglione's 2nd theorem

$$\Delta = \lim_{P \rightarrow 0} \frac{\partial A}{\partial P} = \lim_{P \rightarrow 0} \int_0^L \frac{M_3}{H_{33}^c} \frac{\partial M_3}{\partial P} dx, \quad (10.113)$$

$\lim_{P \rightarrow 0} M_3 = M_3$ = bending moment due to externally applied loads
only

$\lim_{P \rightarrow 0} \frac{\partial M_3}{\partial P} = \hat{M}_3$ = bending moment due to a unit load only

- Eq. (10.113) \rightarrow unit load method, Eq. (7.23)

$$\Delta = \int_0^L \frac{\hat{M}_3 M_3}{H_{33}^c} dx, \quad (10.114)$$

M_3 is identical for {unit
dummy load method}

however, \hat{M}_3 has a difference

{ dummy load method ... bending moment acting in the structure
subjected to a unit dummy load

unit load method ... "any statically admissible" bending moment
distribution in equilibrium with unit load

→ not necessarily the actual bending moment distribution acting in the
structure subjected to the unit load

→ more versatile, can result in a significant simplification of the
procedure