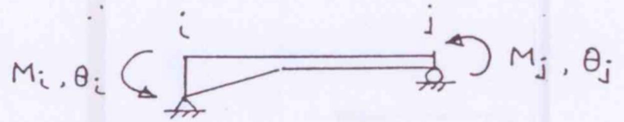
457.649 Advanced Structural Analysis

Part IV:
Modeling Options

Structural Design Lab.(Prof. Ho-Kyung Kim)
Dept. of Civil & Environmental Eng.
Seoul National University

End Releases



$$\begin{Bmatrix} M_i \\ M_j \end{Bmatrix} = \begin{bmatrix} K_{ii} & K_{ij} \\ K_{ji} & K_{jj} \end{bmatrix} \begin{Bmatrix} \theta_i \\ \theta_j \end{Bmatrix} \qquad \left(K_{ii} = C_{ii} \frac{EI_{ref}}{L} \right)$$

If node j is released, then $M_j \equiv 0$.

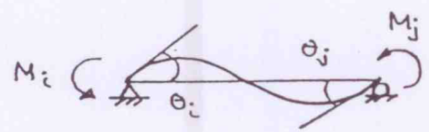
$$M_j = K_{ij}\,\theta_i + K_{jj}\,\theta_j = 0 \rightarrow \theta_j = -\frac{K_{ij}}{K_{jj}}\,\theta_i$$

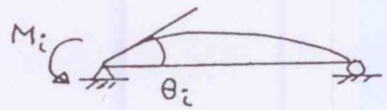
$$M_i = K_{ii}\,\theta_i + K_{ij}\,\theta_j = \left(K_{ii} - \frac{K_{ij}^2}{K_{jj}} \right) \theta_i$$

Hence

$$\begin{Bmatrix} M_i \\ M_j \end{Bmatrix} = \begin{bmatrix} K_{ii} - \frac{K_{ij}^2}{K_{jj}} & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} \theta_i \\ \theta_j \end{Bmatrix}$$

- For prismatic beam:





$$\begin{Bmatrix} M_i \\ M_j \end{Bmatrix} = \begin{bmatrix} \frac{4EI}{L} & \frac{2EI}{L} \\ \frac{2EI}{L} & \frac{4EI}{L} \end{bmatrix} \begin{Bmatrix} \theta_i \\ \theta_j \end{Bmatrix} \qquad \begin{Bmatrix} M_i \\ M_j \end{Bmatrix} = \begin{bmatrix} \frac{3EI}{L} & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} \theta_i \\ \theta_j \end{Bmatrix}$$

$$4 - \frac{2^2}{4} = 3$$

Shear Deformations in Beams

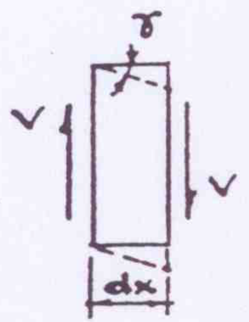


Fig. 1

If shear force were to produce uniform shear stress (and hence uniform shear strain) over the depth of a beam, the shear deformation would be as shown. The action-deformation relationship for shear would then be:

$$\gamma = V/GA$$

where A = cross section area.

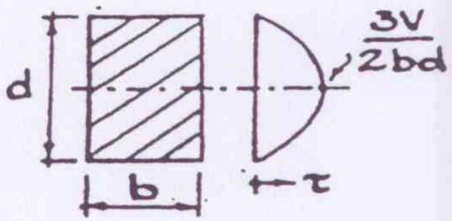


Fig. 2

In fact, however, the shear stress is generally not uniform. For example, for a rectangular section it is as shown in Fig. 2. Hence, the shear strain must also vary over the depth, and the cross section must warp as shown in Fig. 3. For computing shear deformation effects in beams, the shear deformation is now measured by the <u>effective</u> shear strain, γ_{eff}.

Shear Deformations in Beams

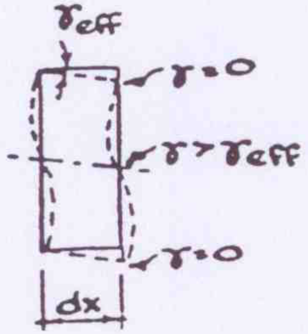


Fig. 3

To obtain γ_{eff}, use is made of strain energy. The work done by the shear force per unit length of beam is

$$W = 0.5\ V.\gamma_{eff}$$

The corresponding strain energy is

$$U = \int_{vol} 0.5\,\tau\gamma\ dV = \int_{vol} \frac{\tau^2}{2G}\,dV$$

Equating U = W allows γ_{eff} to be calculated.

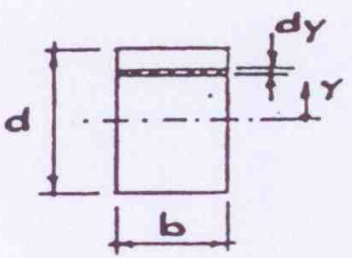


Fig. 4

For example, for a rectangular section:

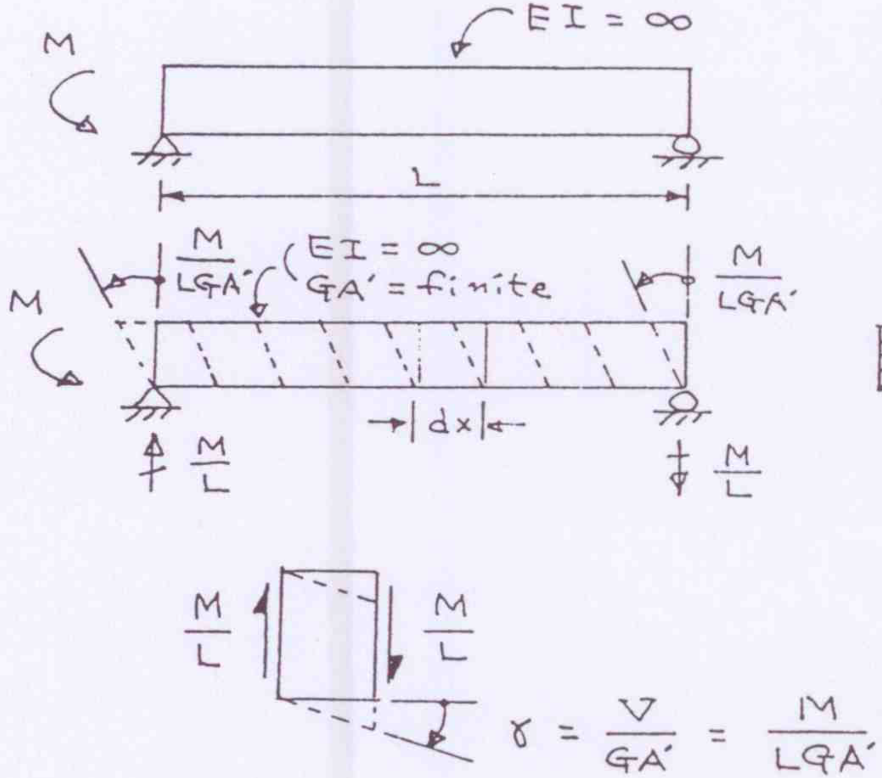
$$U = \int_{y=-d/2}^{d/2} \left[\frac{3V}{2bd} \left(1 - \frac{4y^2}{d^2}\right)\right]^2 \cdot \frac{1}{2G} \cdot bdy = W = \frac{1}{2} V\gamma_{eff}$$

Hence

$$\gamma_{eff} = \frac{6V}{5Gbd} = \frac{V}{GA'}$$

where $A' = 5bd/6$ = effective shear area.

Shear Deformations – Effect on $[k_v]$



Zero deformations unless $GA' \neq \infty$

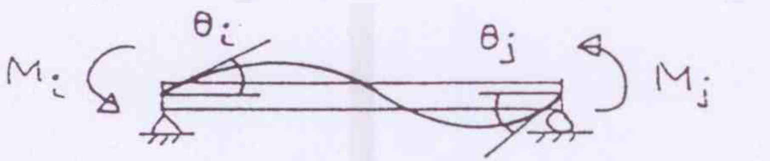
$A' = \frac{5}{6} A$

(Shear stress is not uniform along the section.)

$\gamma = \frac{V}{GA'} = \frac{M}{LGA'}$

Pure shearing deformation

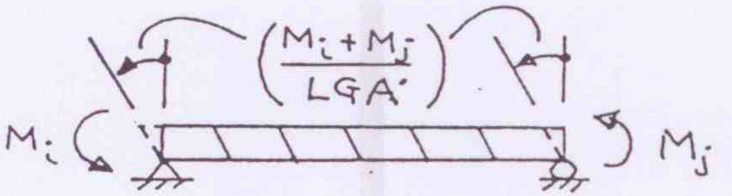
Simplest way : add flexibilities



EI = finite, $GA' = \infty$ } bending only

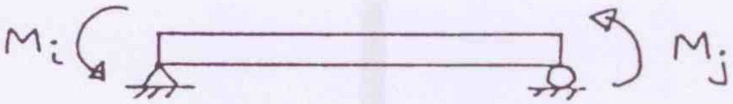
$$\begin{Bmatrix} \theta_i \\ \theta_j \end{Bmatrix} = \frac{L}{6EI} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{Bmatrix} M_i \\ M_j \end{Bmatrix}$$

Shear Deformations – Effect on $[k_v]$



$EI = \infty$, $GA' =$ finite } shear only

$$\begin{Bmatrix}\theta_i\\ \theta_j\end{Bmatrix} = \frac{1}{LGA'}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\begin{Bmatrix}M_i\\ M_j\end{Bmatrix}$$



$EI =$ finite

$GA' =$ finite

$$\begin{Bmatrix}\theta_i\\ \theta_j\end{Bmatrix} = \begin{bmatrix}\frac{L}{3EI}+\frac{1}{LGA'} & -\frac{L}{6EI}+\frac{1}{LGA'}\\ \text{sym.} & \frac{L}{3EI}+\frac{1}{LGA'}\end{bmatrix}\begin{Bmatrix}M_i\\ M_j\end{Bmatrix}$$

(1) Invert algebraically, then program $[k_v]$

(2) Program $[f]$, then invert numerically. (DIPSEE)

$$\begin{bmatrix}a & b\\ c & d\end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix}d & -b\\ -c & a\end{bmatrix}$$

Symmetry and Antisymmetry

- The analysis of symmetric structures can often be greatly simplified by substituting for the given loading condition an equivalent system consisting of the sum of a symmetric and an antisymmetric loading condition. In certain cases the given loading itself may be symmetric or antisymmetric. While rigid frame type structure will be used as example in this discussion, the same basic principles can be applied to any type of structure, for example a truss, slab or shell.
- Any structure can be defined by specifying its geometric and stiffness properties with reference to a rectangular Cartesian coordinate system (x, y, z). If desired, the same structure can be described by reference to a second rectangular coordinate system (x', y', z'). Depending on the relationship of the structure as defined in the two coordinate systems it will be shown that in certain cases the structure may have one or more of the following three types of structural symmetry.

1. **Axial symmetry**
2. **Planar symmetry**
3. **Point symmetry**

- It should be noted that no structure will be defined as being antisymmetric since this leads to ambiguities and is unnecessary.

Symmetry and Antisymmetry

Axial, Planar, and Point Symmetry

Figs. 3a, 4a, and 5a illustrate three dimensional frames which are general examples of the three possible types of structural symmetry. In each case the structure has identical geometric and stiffness properties when described in either of two coordinate systems (x, y, z) or (x', y', z'). Symmetric pairs of points such as a-a', b-b', c-c', etc. are easily located from the above definition.

Axial structural symmetry, Fig. 3a, exists if the relationship between the two coordinate systems is such that x' = -x, y' = y, and z' = -z. The y-axis is the axis of structural symmetry for this case. Note that both (x, y, z) and (x', y', z') are right hand coordinate systems. A simple visual tool for identifying axial symmetry is to rotate the structure 180^o about the assumed axis of symmetry and see if the same structure is obtained. This is sometimes called the "principle of rotation." All planar structure which possess either planar or point symmetry can be shown to also have an axis of symmetry.

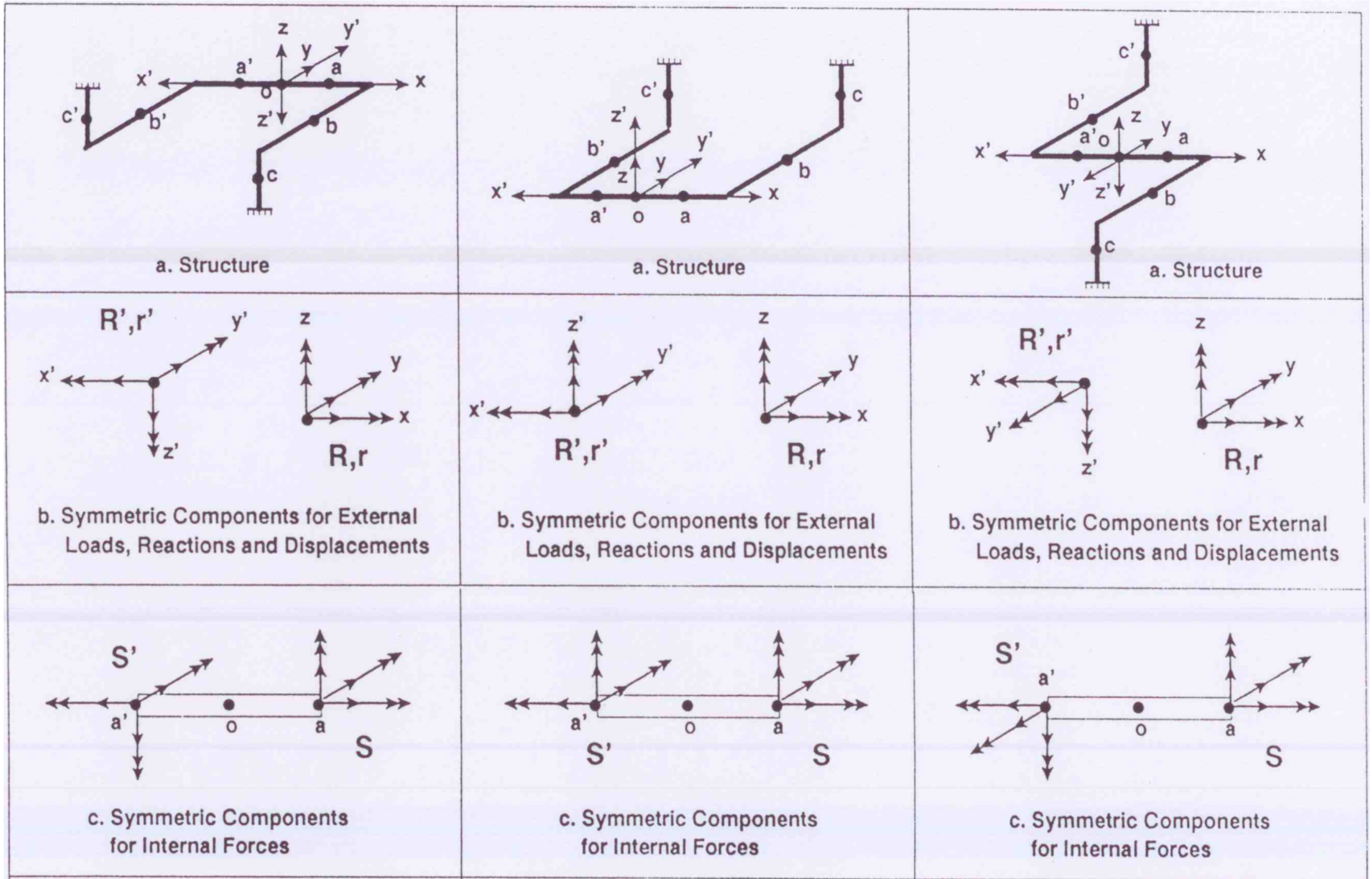
Planar structural symmetry, Fig. 4a, exists if the relationship between the two coordinate systems is such that x' = -x, y' = y, and z' = z. The x = 0 (yz plane at origin) is the plane of structural symmetry for this case. Note that (x, y, z) is a right hand coordinate system while (x', y', z') is a left hand coordinate system. Planar symmetry is generally recognized at a visual glance using the "principle of reflection." Simply imagine a mirror at plane x = 0 and if the mirror image reflection obtained is identical with the actual structure, planar symmetry exists.

Point structural symmetry, Fig. 5a, exists if the relationship between the two coordinate systems is such that x' = -x, y' = -y, and z' = -z. The origin point O is the point of structural symmetry. Note that (x, y, z) is a right hand coordinate system while (x', y', z') is a left hand coordinate system. No simple visual tool exists for identifying point symmetry.

Once the type of structural symmetry has been identified, Theorems I and II of Section 1 can be applied using the directions for symmetric components at symmetric points or symmetric cut sections as shown in Figs. 3b, 3c, 4b, 4c, and 5b, 5c. Single arrow-head vectors represent forces and translational displacements and double arrow-head vectors represent moments and rotational displacements. The directions of these vectors are identical with the pairs of coordinate systems for each corresponding case of structural symmetry. Note to establish the curl of moment or rotational displacement vectors the right hand rule must be used for right hand coordinate systems and the left hand rule must be used for left hand coordinate systems.

Antisymmetric complements of external loadings, reactions, displacements or internal forces are simply obtained by reversing directions on one side of Figs. 3b, 3c, 4b, 4c, and 5b, 5c.

Symmetry and Antisymmetry



a. Structure / b. Symmetric Components for External Loads, Reactions and Displacements / c. Symmetric Components for Internal Forces

Fig. 3 Axial Symmetry about y Axis

Fig. 4 Planar Symmetry about yz Plane

Fig. 5 Point Symmetry about Origin Point O

Symmetry and Antisymmetry

Internal forces and external displacements at origin

For the structures shown in Figs. 3, 4, and 5 it can be deduced from symmetry that certain internal forces S at a cut section through the origin O and certain external displacements r at the origin point must be zero. This often is a great aid in reducing the number of unknowns in the solution of an indeterminate structure.

If one imagines the points a and a' to be brought closer and closer together until they finally meet at point O, symmetry will still require that the directions of S on either side of the cut section at the origin must have the directions shown in Figs. 3c, 4c, and 5c. However, statics require that the internal forces S must be equal and opposite. Where these two requirements contradict each other they can only be satisfied by having the particular internal force to be zero.

A similar argument can be used for the external displacements at origin point O. Symmetry requires that as points a and a' are brought closer and closer together until they finally meet at point O that the directions of r must have the directions shown in Figs. 3b, 4b, and 5b. However, geometry requires that displacement vectors r must be equal and in the same direction. Where these two requirements contradict each other they can only be satisfied by having the particular displacement to be zero.

Using the above logic Table 1 summarizes the conditions existing at the origin point O for each type of structural symmetry under symmetric and antisymmetric loadings. A (0) denotes the quantity must be zero and a (?) indicates the quantity remains an unknown. The following comments can be made from a study of the table.

1. Quantities which are zero (0) under symmetric loading are unknowns (?) under antisymmetric loading and vise-versa.
2. For a particular case and loading, if an internal force S is zero (0) the corresponding external displacement r is unknown (?) and vise-versa.

Int. Force	Axial Symmetry about y Axis		Planar Symmetry about y-z Plane		Point Symmetry about Origin	
S	SL	AL	SL	AL	SL	AL
Fx	?	0	?	0	?	0
Fy	0	?	0	?	?	0
Fz	?	0	0	?	?	0
Mx	?	0	0	?	0	?
My	0	?	?	0	0	?
Mz	?	0	?	0	0	?

Ext. Disp.	Axial Symmetry about y Axis		Planar Symmetry about y-z Plane		Point Symmetry about Origin	
r	SL	AL	SL	AL	SL	AL
x-dsp	0	?	0	?	0	?
y-dsp	?	0	?	0	0	?
z-dsp	0	?	?	0	0	?
x-rot	0	?	?	0	?	0
y-rot	?	0	0	?	?	0
z-rot	0	?	0	?	?	0

Symmetry and Antisymmetry

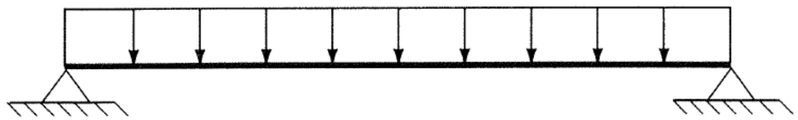


Figure 7.20 Simply supported beam with uniform load.

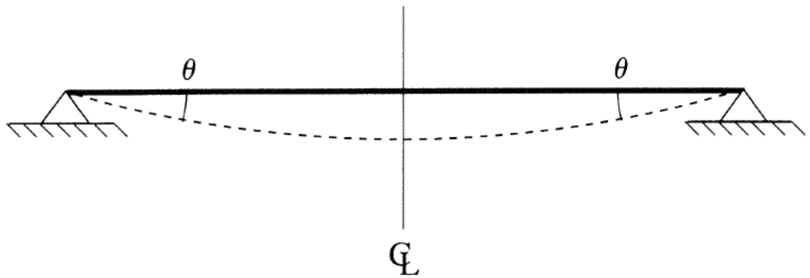


Figure 7.21 Displaced shape of uniformly loaded beam.

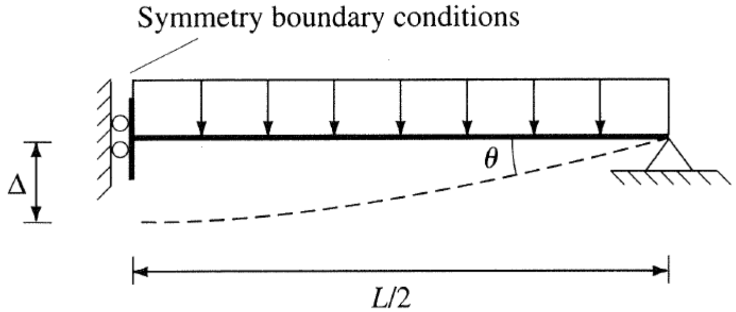


Figure 7.22 Symmetric portion of structure.

Symmetry and Antisymmetry

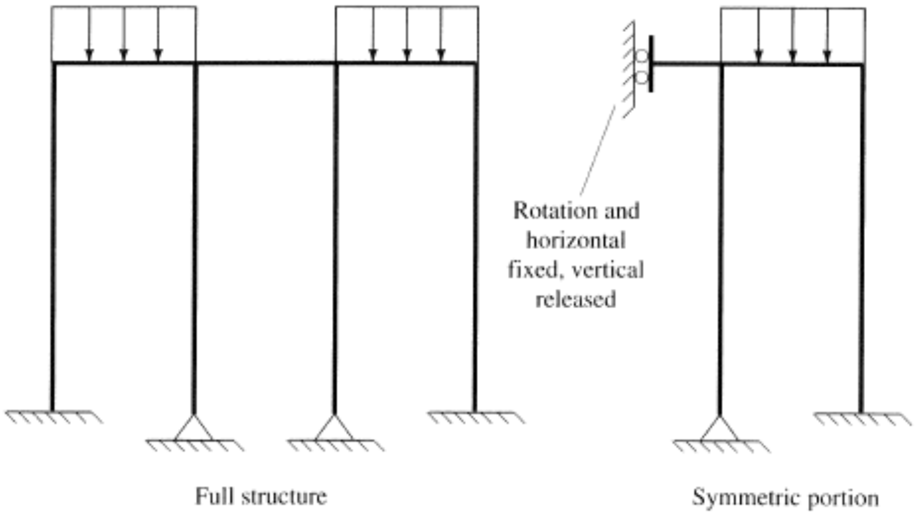


Figure 7.24 Symmetry frame example.

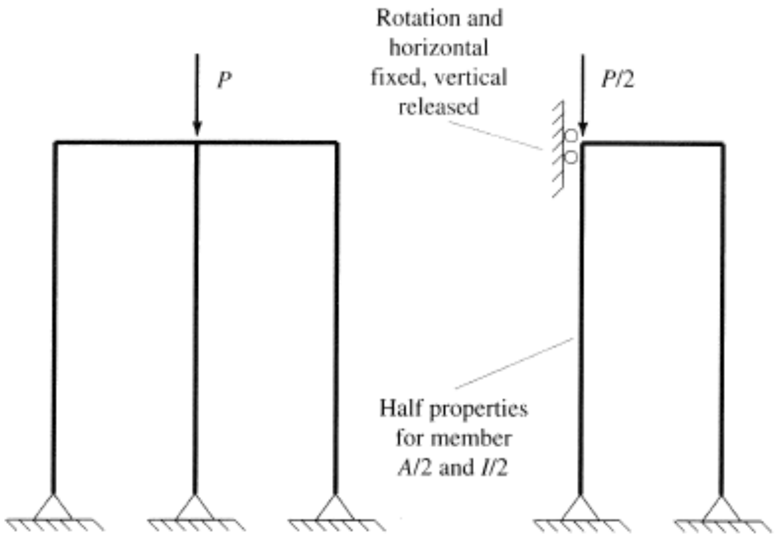


Figure 7.25 Symmetry conditions with member and load in plane.

Symmetry and Antisymmetry

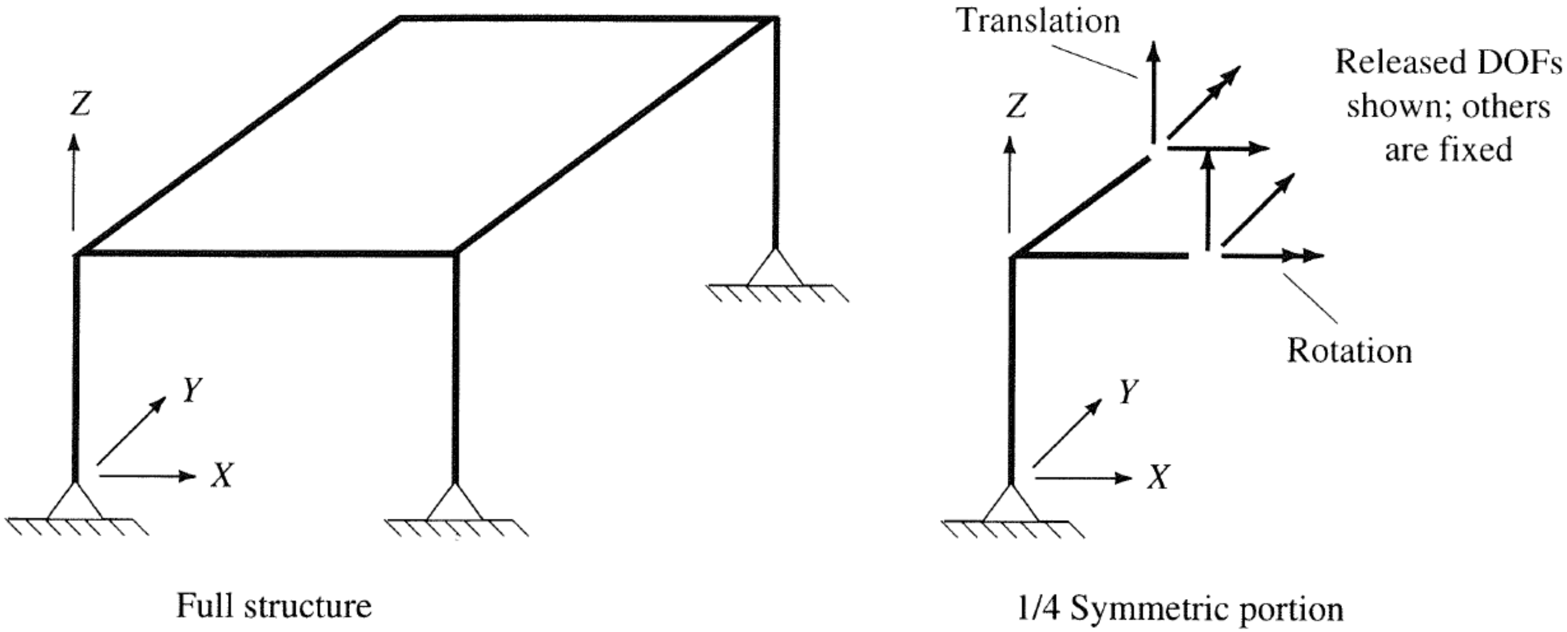


Figure 7.26 Three-dimensional structure with symmetry boundary conditions.

Symmetry and Antisymmetry

1. Loading

1. Symmetric loading is obtained by replacing each given force by one-half its original value plus the symmetric complement of this force.
2. Antisymmetric loading is obtained by replacing each given force by one-half its original value plus the antisymmetric complement of this force.

Two simple examples of this superposition are illustrated in Figs. 1 and 2:

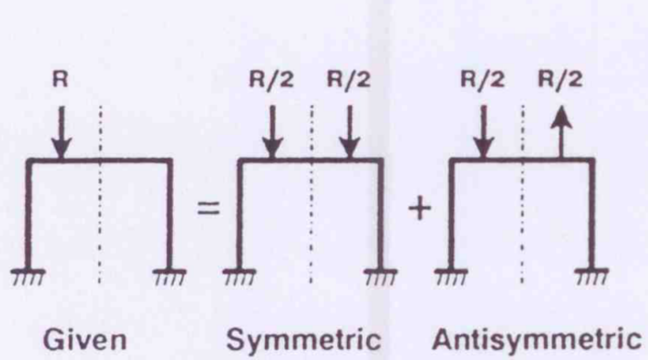


Fig. 1 Vertical Load Superposition for a Symmetrical Structure

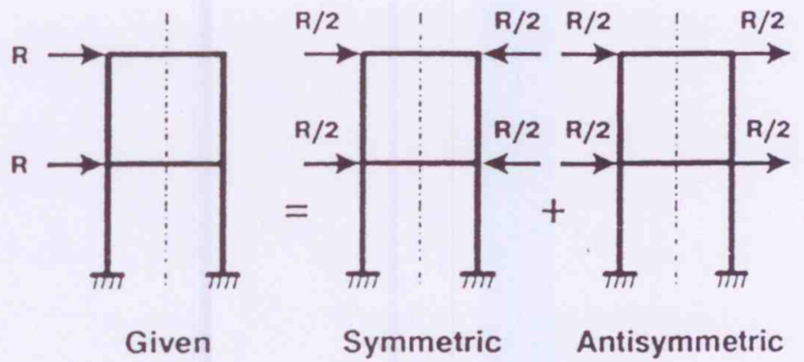


Fig. 2 Lateral Load Superposition for a Symmetrical Structure

For each of the three types of structural symmetry: axial; planar; and point; the following theorems are valid provided the proper definitions are used for symmetric and antisymmetric loadings, reactions, displacements and internal forces in each case.

I. Symmetric external loadings applied to a symmetric structure produce symmetric reactions, displacements and internal forces.

II. Antisymmetric external loadings applied to a symmetric structure produce antisymmetric reactions, displacements and internal forces.