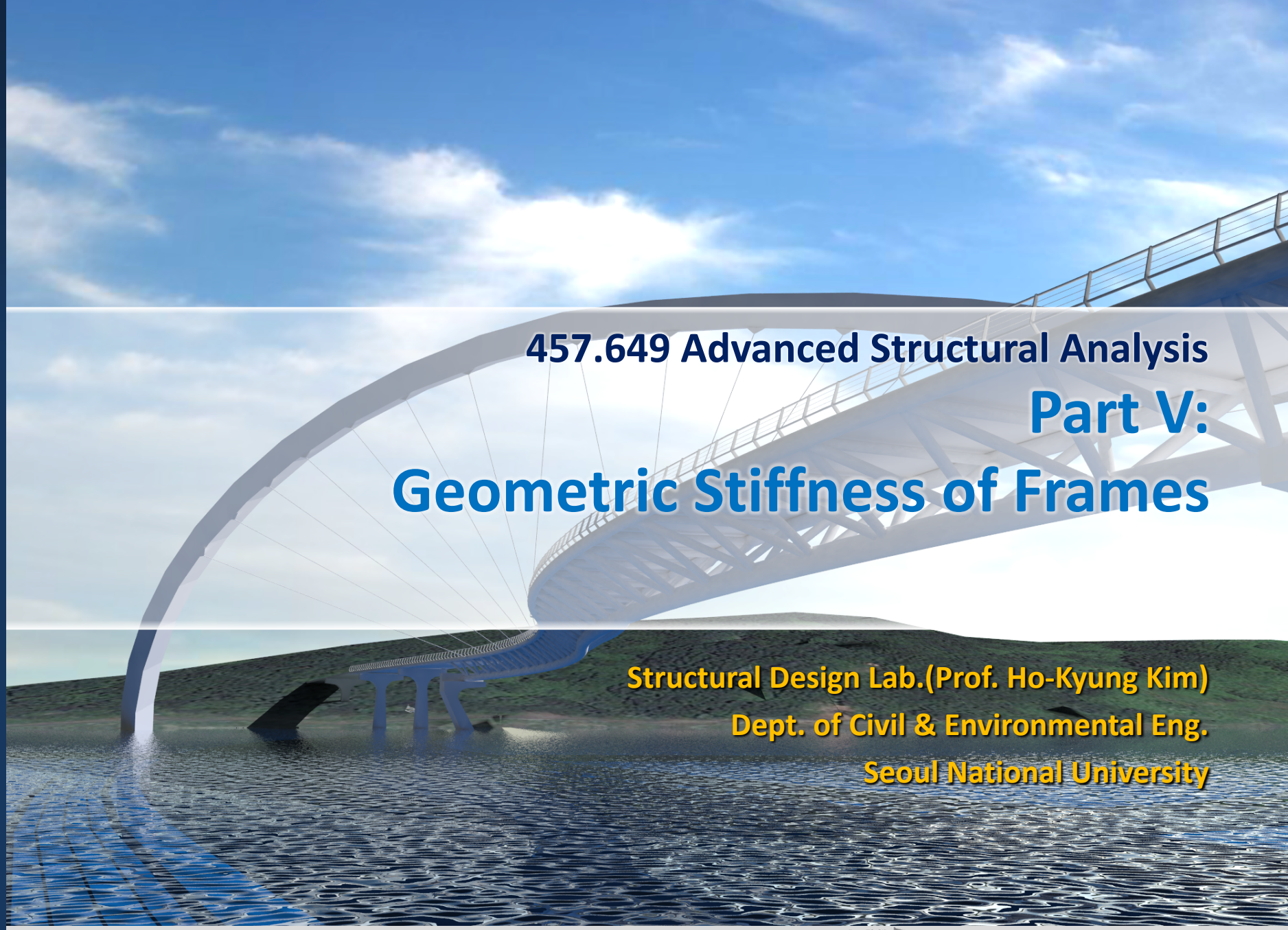


457.649 Advanced Structural Analysis

Part V:
Geometric Stiffness of Frames

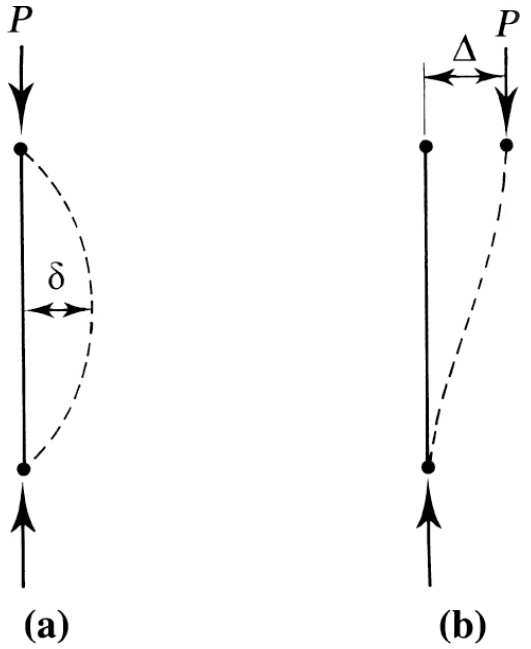
Structural Design Lab.(Prof. Ho-Kyung Kim)
Dept. of Civil & Environmental Eng.
Seoul National University

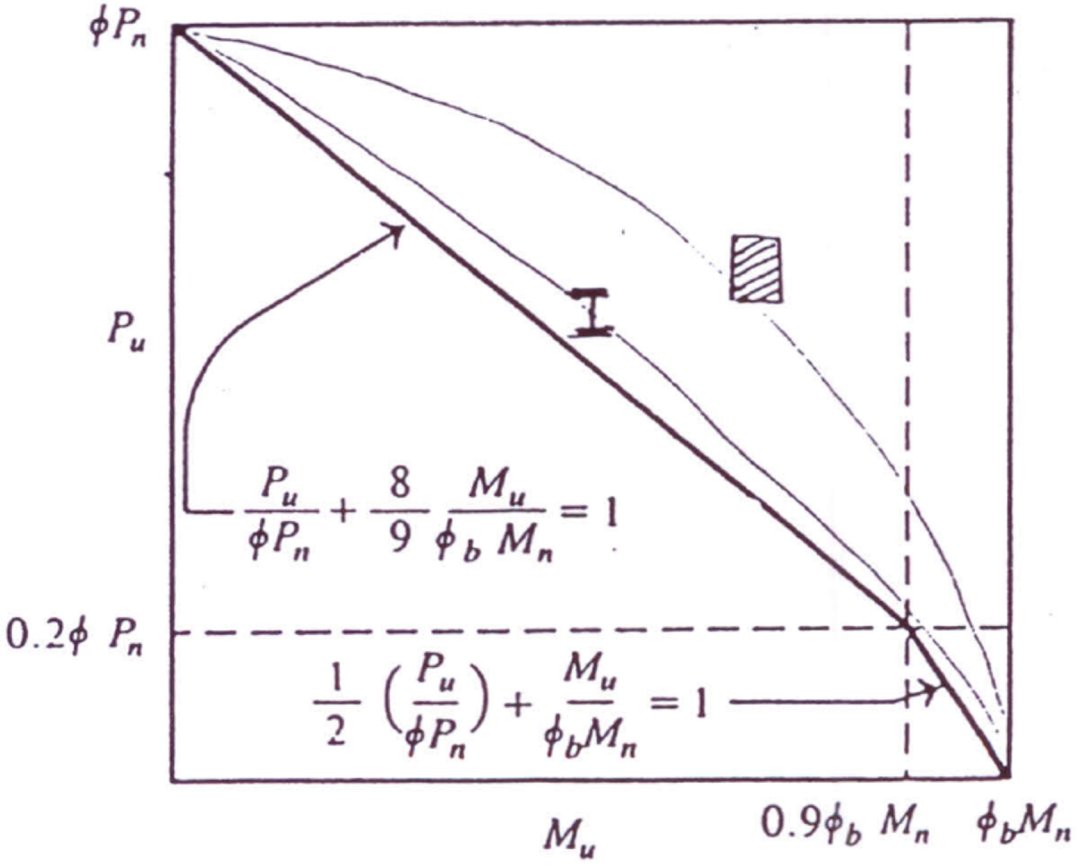
Source of Nonlinearity

- **Geometric effects:**
  - Initial imperfections
  - P-Δ effect
  - P-δ effect

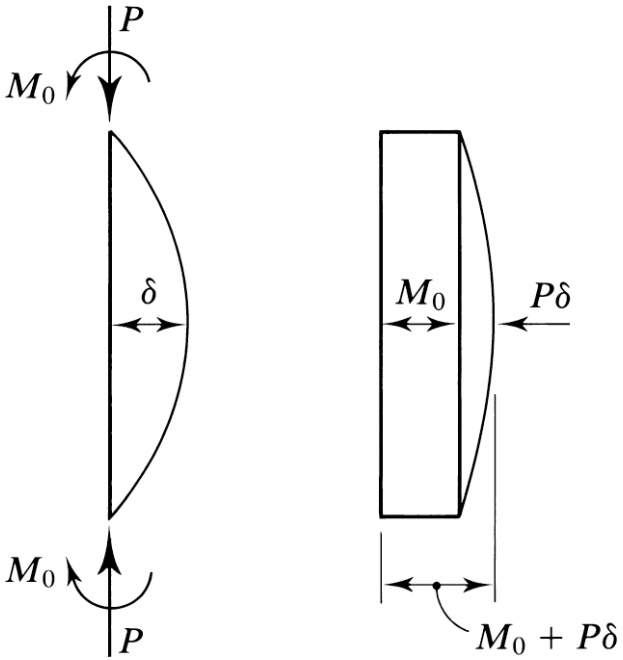
- **Material effects:**
  - Plastic deformation of steel structures
  - Cracking or creep of reinforced concrete structures
  - Inelastic interaction of axial force, bending, shear and torsion

P-δ and P-Δ Effects

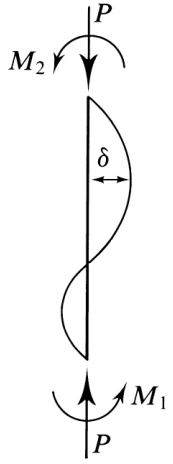


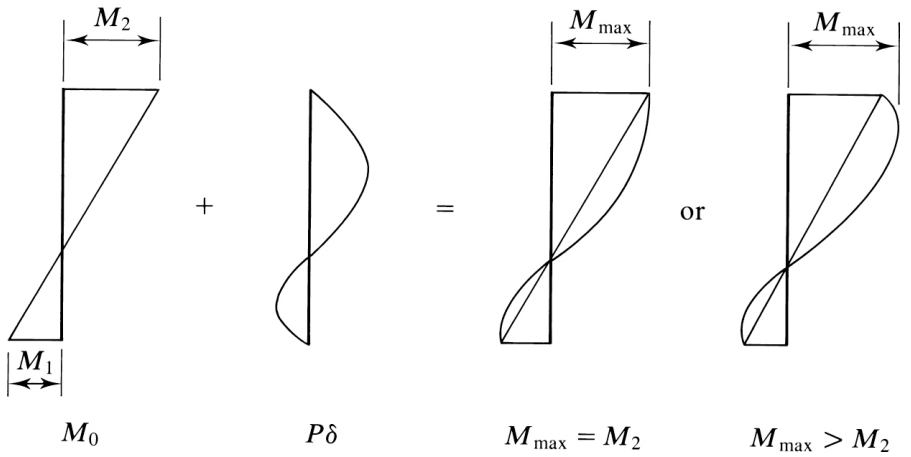


Moment Amplification or Second-Order Analysis for P-δ



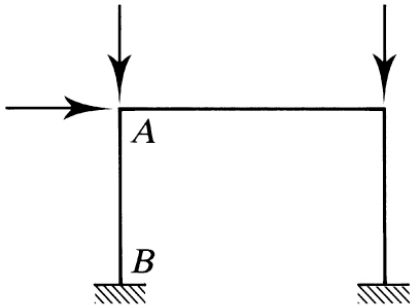
$$M_{\max} = M_0 \frac{C_m}{1-(P/P_e)}$$

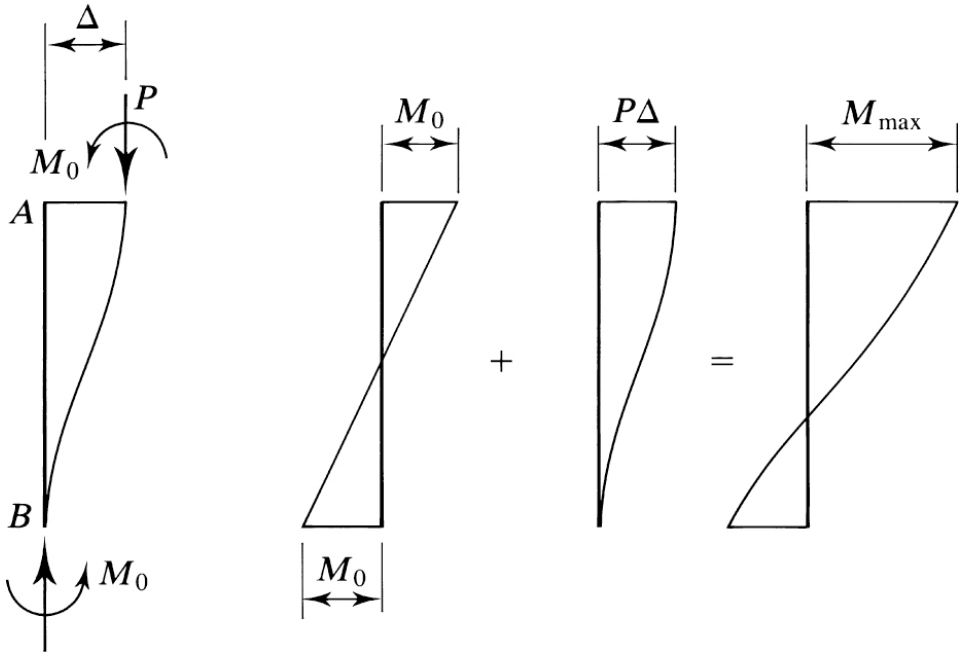




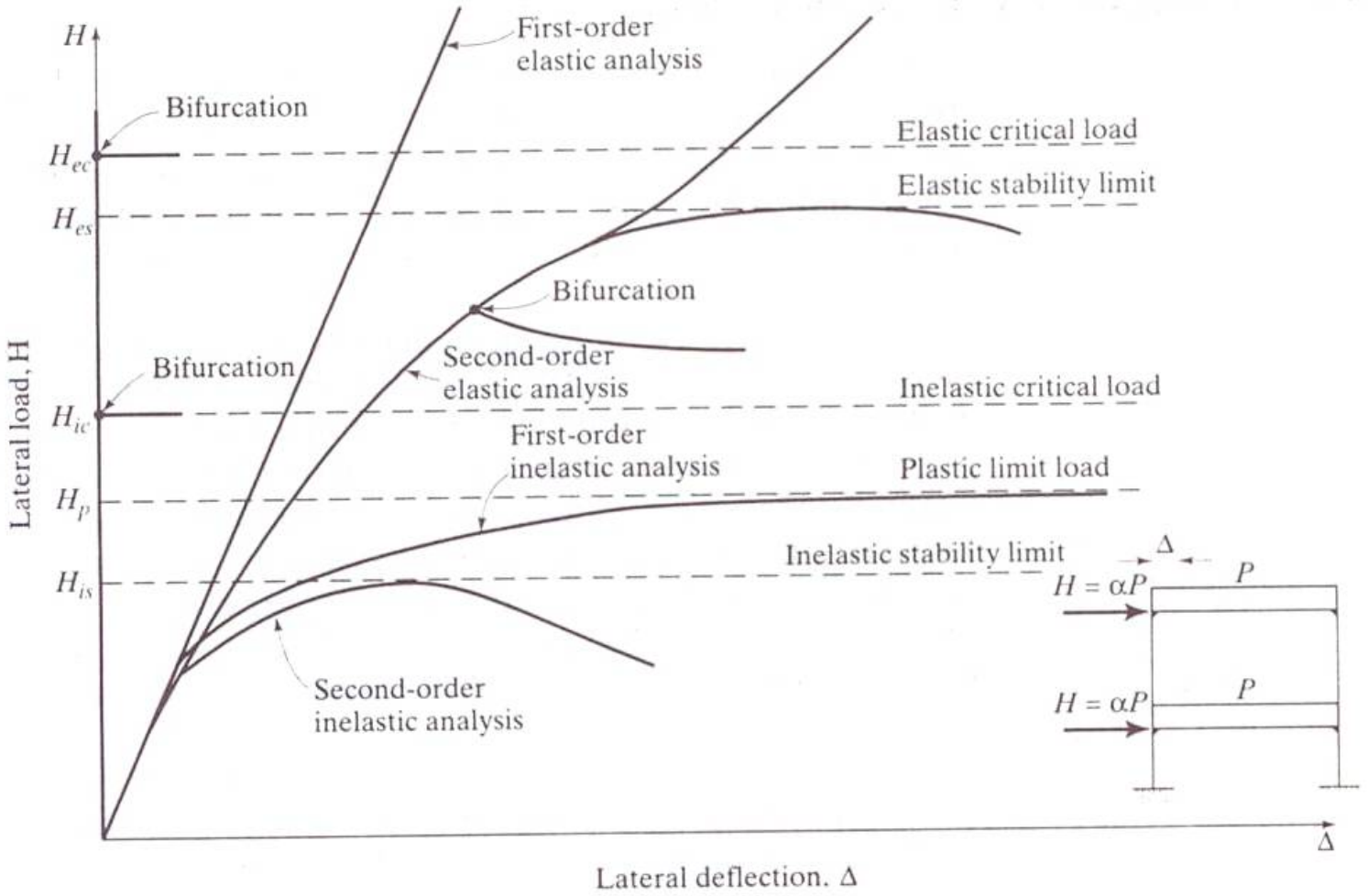
Moment Amplification or Second-Order Analysis for P-Δ

$$B_2 = \frac{1}{1 - \dfrac{\sum P_{nt}}{\sum P_{e2}}} \geq 1$$





Types of Analysis



Matrix Representation

- **Linear Elastic Analysis**

$$\left[\mathbf{K}_{\mathrm{e}}\right]\{\Delta\}=\{\mathbf{P}\}$$

- **Incremental form with tangential stiffness matrix**

$$\left[\mathbf{K}_{\mathrm{t}}\right]\{\mathbf{d}\Delta\}=\{\mathbf{dP}\}$$

- **Second-order Elastic Analysis**

$$\left[\mathbf{K}_{e}+\mathbf{K}_{g}\right]\{\mathbf{d}\Delta\}=\{\mathbf{dP}\}$$

- **First-order Inelastic Analysis**

$$\left[\mathbf{K}_{\mathrm{e}}+\mathbf{K}_{m}\right]\{\mathbf{d}\Delta\}=\{\mathbf{dP}\}$$

- **Second-order Inelastic Analysis**

$$\left[\mathbf{K}_{\mathrm{e}}+\mathbf{K}_{g}+\mathbf{K}_{m}\right]\{\mathbf{d}\Delta\}=\{\mathbf{dP}\}$$

- **Eigenvalue Problem**

$$\left[\mathbf{K}_{\mathrm{e}}+\lambda\hat{\mathbf{K}}_{g}\right]\{\Delta\}=\{0\}\qquad\frac{1}{\lambda}\{\Delta_{f}\}=-\left[\mathbf{K}_{ef}\right]^{-1}\left[\hat{\mathbf{K}}_{gf}\right]\{\Delta_{f}\}$$

Large Displacement

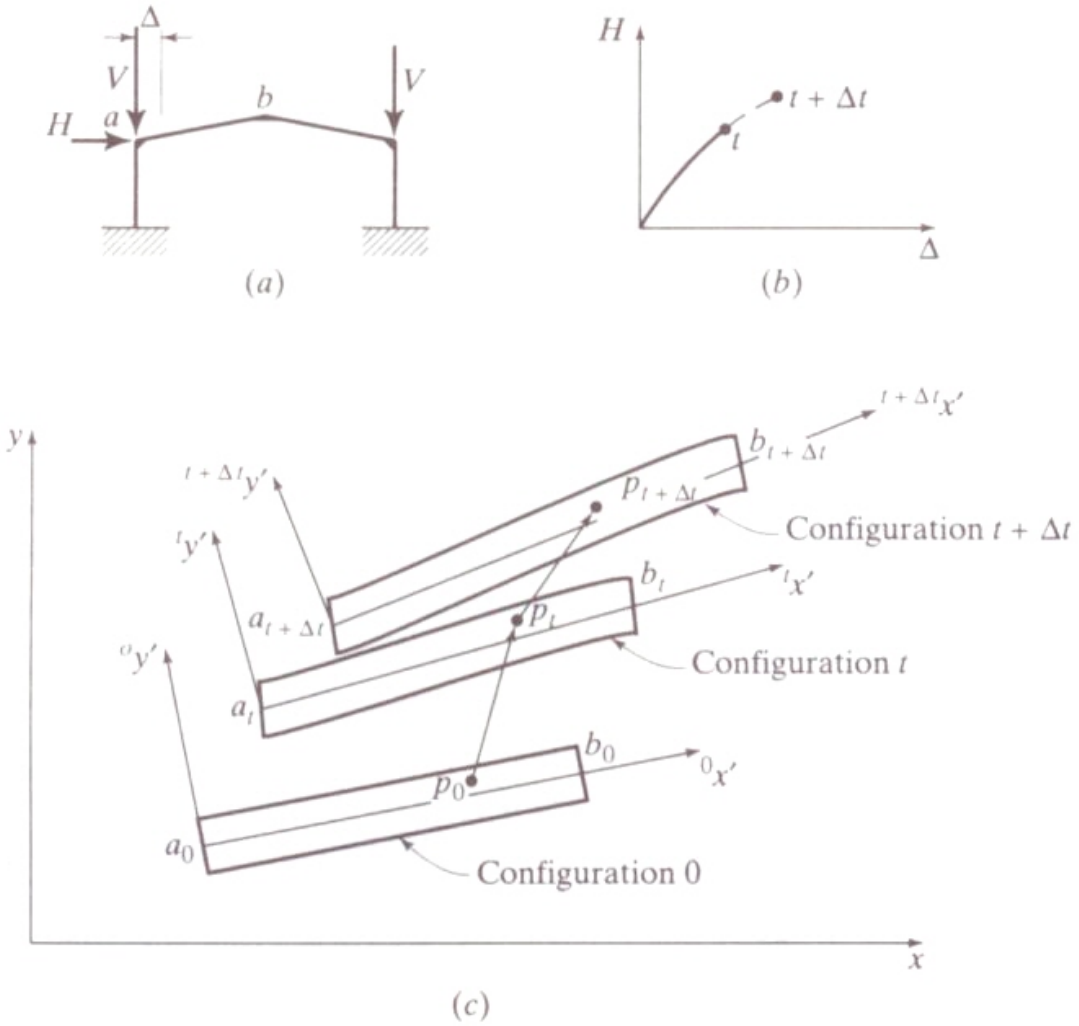


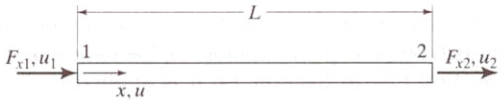
Figure 8.4 Element deformations.

Displacement in terms of nodal displacements

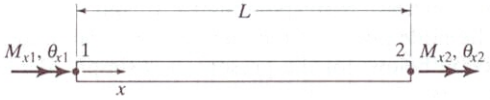
$$\Delta = N_1\Delta_1 + N_2\Delta_2 + \cdots + N_i\Delta_i + \cdots + N_n\Delta_n$$

$$= \sum_{i=1}^{n} N_i\Delta_i = [N]\{\Delta\}$$

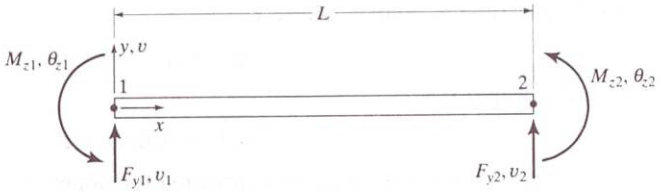
- **e.g. for axial force element,**
  - $u = N_1u_1 + N_2u_2$



- **for pure torsion element,**
  - $\theta_x = N_1\theta_{x1} + N_2\theta_{x2}$



- **for flexural element,**
  - $v = N_1v_1 + N_2v_2 + N_3\theta_{z1} + N_4\theta_{z2}$



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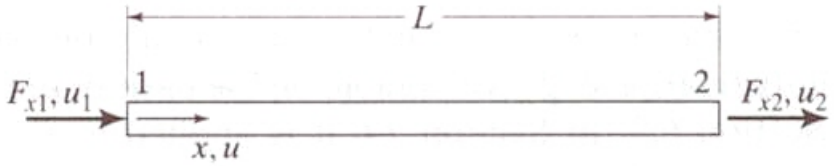
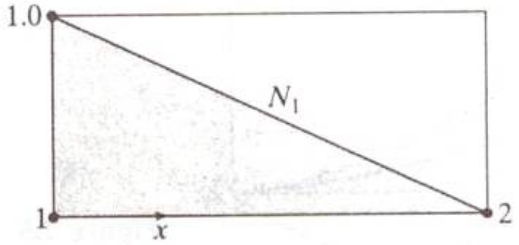
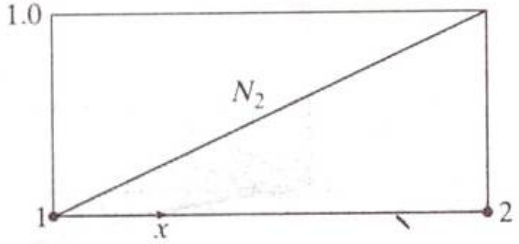
Shape Functions for Axial Member



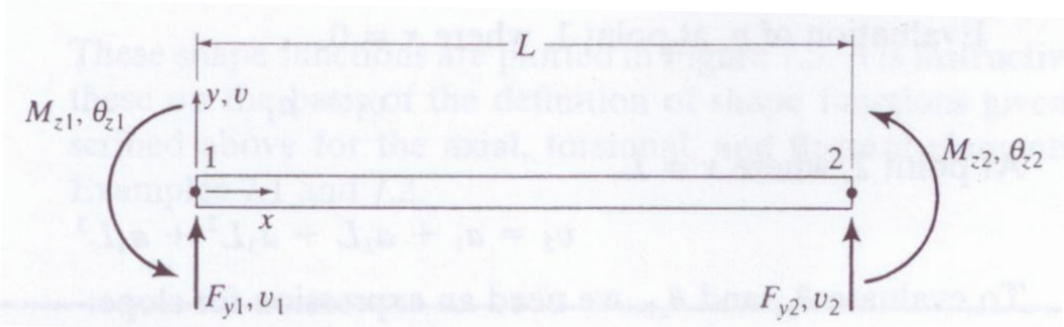
$$e_x = \frac{du}{dx} \Rightarrow u = a_1 + a_2 x$$



$$\begin{aligned} u &= N_1 u_1 + N_2 u_2 \\ &= \left(1 - \frac{x}{L}\right) u_1 + \frac{x}{L} u_2 \\ &= (1 - \xi) u_1 + \xi u_2 \end{aligned}$$



Shape Functions for Flexural Member



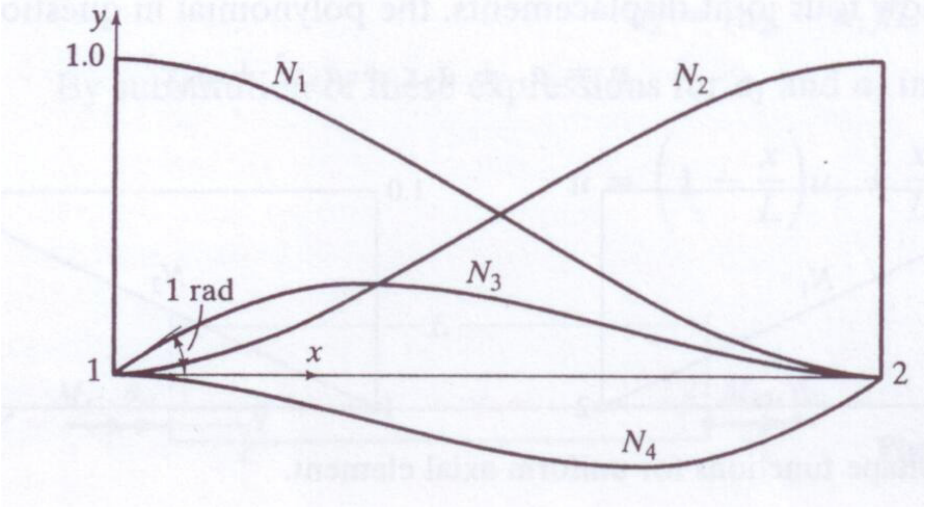
$$v = N_1 v_1 + N_2 v_2 + N_3 \theta_{z1} + N_4 \theta_{z2}$$

$$N_1 = 1 - 3\left(\frac{x}{L}\right)^2 + 2\left(\frac{x}{L}\right)^3$$

$$N_2 = x\left(1 - \frac{x}{L}\right)^2$$

$$N_3 = 3\left(\frac{x}{L}\right)^2 - 2\left(\frac{x}{L}\right)^3$$

$$N_4 = x\left[\left(\frac{x}{L}\right)^2 - \frac{x}{L}\right]$$



Element Stiffness Matrix

$$\delta W = \delta W_{ext} - \delta W_{int} = 0$$

$$\delta W_{int} = \int_{vol} [\delta \mathbf{e}][\mathbf{E}]\{\mathbf{e}\} d(vol)$$

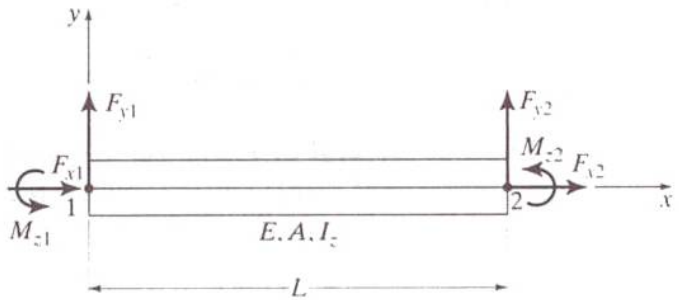
$$\delta W_{ext} = \sum_{i=1}^{n} \delta \Delta_i F_i = [\delta \mathbf{\Delta}]\{\mathbf{F}\}$$

$$\left[\int_{vol} [\delta \mathbf{\Delta}]\{\mathbf{N}'\}\,[\mathbf{E}][\mathbf{N}']\{\mathbf{\Delta}\}\; d\,(vol)\right] = [\delta \mathbf{\Delta}]\{\mathbf{F}\}$$

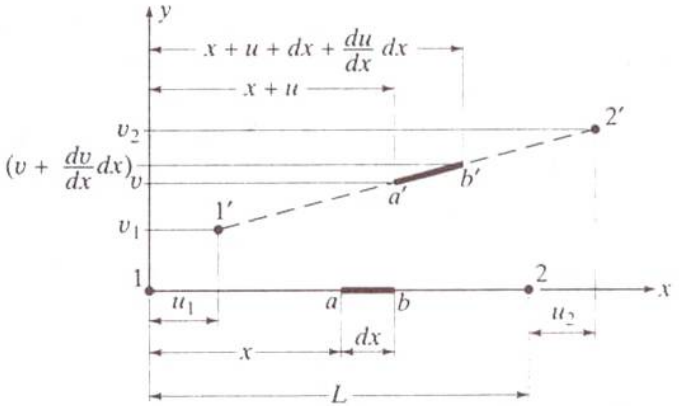
$$[k] = \left[\int_{vol} \{\mathbf{N}'\}\,[\mathbf{E}][\mathbf{N}']\; d\,(vol)\right]$$

Read Chapters 6 and 7 in “Matrix Structural Analysis” (McGuire et al.).

Geometric Stiffness Matrices: Axial Force Member



(a)



(*b*) stretching and rigid body rotation of axis

$$a'b' = \left[1 + 2\frac{du}{dx} + \left(\frac{du}{dx}\right)^2 + \left(\frac{dv}{dx}\right)^2\right]^{1/2} dx$$

$$\frac{a'b'}{dx} = 1 + \frac{du}{dx} + \frac{1}{2}\left[\left(\frac{du}{dx}\right)^2 + \left(\frac{dv}{dx}\right)^2\right]$$

$$e_{\text{fin}} = \frac{du}{dx} + \frac{1}{2}\left[\left(\frac{du}{dx}\right)^2 + \left(\frac{dv}{dx}\right)^2\right] \tag{9.1}$$

$$\delta W_{\text{int}} = \int_{\text{vol}} \sigma_x \, \delta e_{\text{fin}} \, d(\text{vol}) \tag{9.2}$$

$$\delta W_{\text{int}} = \int_0^L \sigma_x A\left(\frac{d\delta u}{dx}\right) dx + \frac{1}{2}\int_0^L \sigma_x A\left[\delta\left(\frac{du}{dx}\right)^2 + \delta\left(\frac{dv}{dx}\right)^2\right] dx \tag{9.3}$$

$$\delta W_{\text{int}} = \int_0^L \left(\frac{du}{dx}\right) EA\left(\frac{d\delta u}{dx}\right) dx + \frac{1}{2} F_{x2}\int_0^L \left[\delta\left(\frac{du}{dx}\right)^2 + \delta\left(\frac{dv}{dx}\right)^2\right] dx \tag{9.4}$$

$$\delta W_{\text{int } g} = F_{x2}\int_0^L \left[\left(\frac{d\delta u}{dx}\frac{du}{dx}\right) + \left(\frac{d\delta v}{dx}\frac{dv}{dx}\right)\right] dx \tag{9.5}$$

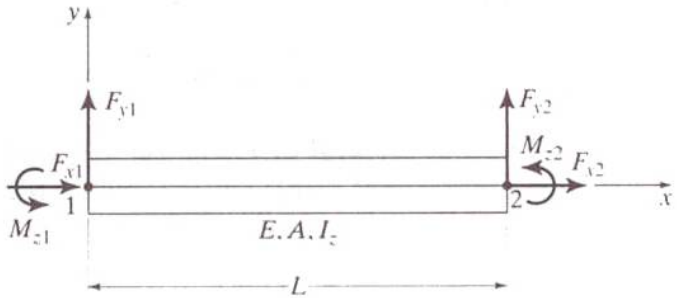
$$[\mathbf{k_g}] = F_{x2}\int_0^L [\{\mathbf{N}'_u\}\lfloor \mathbf{N}'_u \rfloor + \{\mathbf{N}'_v\}\lfloor \mathbf{N}'_v \rfloor] \, dx \tag{9.6}$$

$$u = (1 - \xi)u_1 + \xi u_2 \quad \text{and} \quad v = (1 - \xi)v_1 + \xi v_2 \tag{9.7}$$

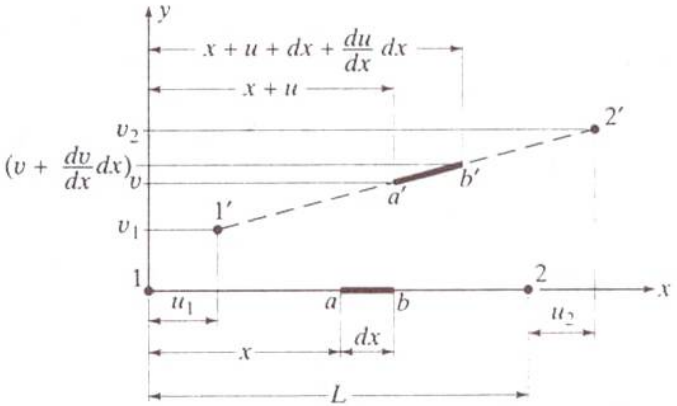
$$\lfloor \mathbf{N}'_u \rfloor = \begin{vmatrix} -\frac{1}{L} & \frac{1}{L} \end{vmatrix} \quad \text{and} \quad \lfloor \mathbf{N}'_v \rfloor = \begin{vmatrix} -\frac{1}{L} & \frac{1}{L} \end{vmatrix} \tag{9.8}$$

$$[\mathbf{k_g}] = \frac{F_{x2}}{L}\begin{array}{c} \begin{matrix} u_1 & v_1 & u_2 & v_2 \end{matrix} \\ \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \end{array} \tag{9.9}$$

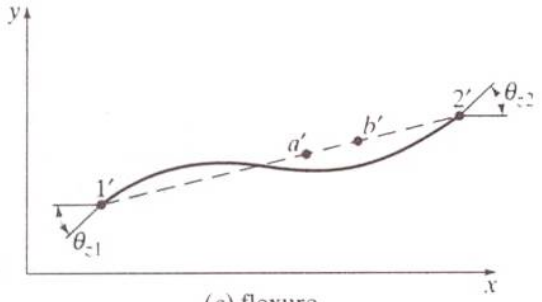
Geometric Stiffness Matrices: Combined Bending and Axial Force



(a)



(b) stretching and rigid body rotation of axis



(c) flexure

$$e_{fin} = \frac{du}{dx} - y\left(\frac{d^2v}{dx^2}\right) + \frac{1}{2}\left[\left(\frac{du}{dx}\right)^2 + \left(\frac{dv}{dx}\right)^2\right] \tag{9.10}$$

$$\delta W_{int} = \int_0^L \sigma_x A\left(\frac{d\delta u}{dx}\right) dx + \int_0^L M_z\left(\frac{d^2\delta v}{dx^2}\right) dx + \frac{1}{2}\int_0^L \sigma_x A\left[\delta\left(\frac{du}{dx}\right)^2 + \delta\left(\frac{dv}{dx}\right)^2\right] dx \tag{9.11}$$

$$\delta W_{int} = \int_0^L \left(\frac{du}{dx}\right) EA\left(\frac{d\delta u}{dx}\right) dx + \int_0^L \left(\frac{d^2v}{dx^2}\right) EI_z\left(\frac{d^2\delta v}{dx^2}\right) dx + \frac{1}{2}F_{x2}\int_0^L \left[\delta\left(\frac{du}{dx}\right)^2 + \delta\left(\frac{dv}{dx}\right)^2\right] dx \tag{9.12}$$

$$u = (1 - \xi)u_1 + \xi u_2$$

$$v = (1 - 3\xi^2 + 2\xi^3)v_1 + (1 - 2\xi + \xi^2)x\theta_{z1} + (3\xi^2 - 2\xi^3)v_2 - (\xi - \xi^2)x\theta_{z2}$$

$$[\mathbf{k_g}] = \frac{F_{x2}}{L}\begin{array}{c} \\ 1 \\ \\ \\ \\ \\ \end{array}\begin{array}{c} \begin{array}{cccccc} u_1 & v_1 & \theta_{z1} & u_2 & v_2 & \theta_{z2} \end{array} \\ \begin{bmatrix} 0 & 0 & -1 & 0 & 0 \\ \frac{6}{5} & \frac{L}{10} & 0 & -\frac{6}{5} & \frac{L}{10} \\ & \frac{2L^2}{15} & 0 & -\frac{L}{10} & -\frac{L^2}{30} \\ & & 1 & 0 & 0 \\ & \text{Sym.} & & \frac{6}{5} & -\frac{L}{10} \\ & & & & \frac{2L^2}{15} \end{bmatrix} \end{array} \tag{9.14}$$

Combined Torsion and axial force

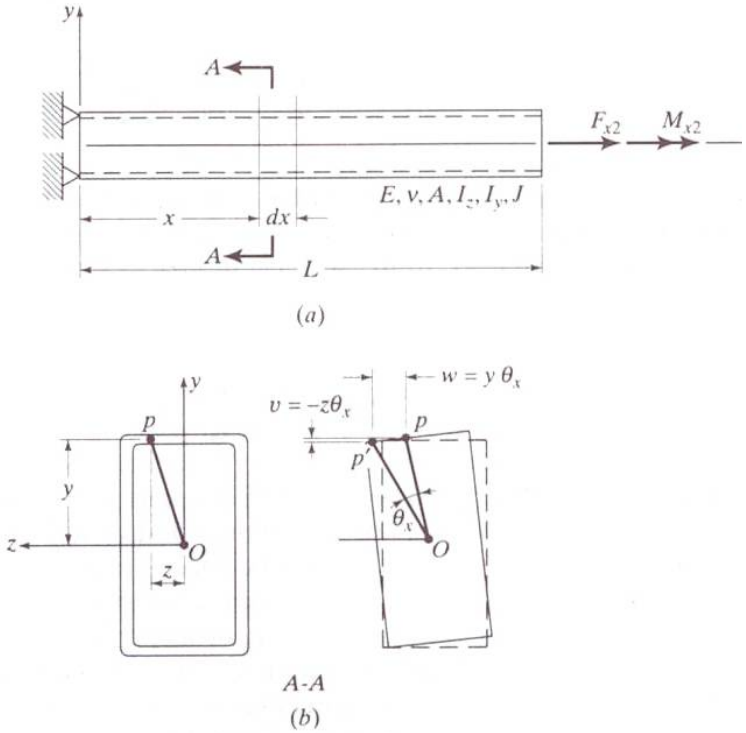


Figure 9.2 Combined Torsion and Axial Force.

$$v = -z\theta_x \qquad \text{and} \qquad w = y\theta_x$$

$$\frac{dv}{dx} = -z\frac{d\theta_x}{dx} \qquad \text{and} \qquad \frac{dw}{dx} = y\frac{d\theta_x}{dx} \tag{9.15}$$

$$e_{\text{fin}} = \frac{du}{dx} + \frac{1}{2}\left[\left(\frac{du}{dx}\right)^2 + \left(\frac{dv}{dx}\right)^2 + \left(\frac{dw}{dx}\right)^2\right] \tag{9.1a}$$

$$e_{\text{fin g}} = \frac{1}{2}\left[z^2\left(\frac{d\theta_x}{dx}\right)^2 + y^2\left(\frac{d\theta_x}{dx}\right)^2\right] = \frac{1}{2}[z^2 + y^2]\left(\frac{d\theta_x}{dx}\right)^2$$

$$\delta W_{\text{int g}} = \frac{1}{2}\int_{\text{vol}} \sigma_x\, \delta\left(\frac{d\theta_x}{dx}\right)^2 [z^2 + y^2]\, dA\, dx$$

$$\delta W_{\text{int g}} = \frac{1}{2}\frac{F_{x2}I_\rho}{A}\int_0^L \delta\left(\frac{d\theta}{dx}\right)^2 dx \tag{9.16}$$

$$[\mathbf{k_g}] = \frac{F_{x2}I_\rho}{AL}\begin{bmatrix} \theta_{x1} & \theta_{x2} \\ 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{9.17}$$

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Three Dimensional Geometric Nonlinear Analysis

$$
[\mathbf{k}_g] = \frac{F_{x2}}{L}
\begin{array}{c}
\begin{matrix} u_1 & v_1 & w_1 & \theta_{x1} & \theta_{y1} & \theta_{z1} & u_2 & v_2 & w_2 & \theta_{x2} & \theta_{y2} & \theta_{z2} \end{matrix} \\
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
 & \frac{6}{5} & 0 & 0 & 0 & \frac{L}{10} & 0 & -\frac{6}{5} & 0 & 0 & 0 & \frac{L}{10} \\
 & & \frac{6}{5} & 0 & -\frac{L}{10} & 0 & 0 & 0 & -\frac{6}{5} & 0 & -\frac{L}{10} & 0 \\
 & & & \frac{I_\rho}{A} & 0 & 0 & 0 & 0 & 0 & -\frac{I_\rho}{A} & 0 & 0 \\
 & & & & \frac{2L^2}{15} & 0 & 0 & 0 & \frac{L}{10} & 0 & -\frac{L^2}{30} & 0 \\
 & & & & & \frac{2L^2}{15} & 0 & -\frac{L}{10} & 0 & 0 & 0 & -\frac{L^2}{30} \\
 & & & & & & 1 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & & \frac{6}{5} & 0 & 0 & 0 & -\frac{L}{10} \\
 & & & & \text{Sym} & & & & \frac{6}{5} & 0 & \frac{L}{10} & 0 \\
 & & & & & & & & & \frac{I_\rho}{A} & 0 & 0 \\
 & & & & & & & & & & \frac{2L^2}{15} & 0 \\
 & & & & & & & & & & & \frac{2L^2}{15}
\end{bmatrix}
\end{array}
\tag{9.18}
$$

Three Dimensional Geometric Nonlinear Analysis

$$
[\mathbf{k}_g] = \begin{array}{c}
\begin{array}{cccccccccccc}
u_1 & v_1 & w_1 & \theta_{x1} & \theta_{y1} & \theta_{z1} & u_2 & v_2 & w_2 & \theta_{x2} & \theta_{y2} & \theta_{z2}
\end{array} \\
\begin{bmatrix}
\frac{F_{x2}}{L} & 0 & 0 & 0 & 0 & 0 & -\frac{F_{x2}}{L} & 0 & 0 & 0 & 0 & 0 \\
 & \frac{6F_{x2}}{5L} & 0 & \frac{M_{y1}}{L} & \frac{M_{x2}}{L} & \frac{F_{x2}}{10} & 0 & -\frac{6F_{x2}}{5L} & 0 & \frac{M_{y2}}{L} & -\frac{M_{x2}}{L} & \frac{F_{x2}}{10} \\
 & & \frac{6F_{x2}}{5L} & \frac{M_{z1}}{L} & -\frac{F_{x2}}{10} & \frac{M_{x2}}{L} & 0 & 0 & -\frac{6F_{x2}}{5L} & \frac{M_{z2}}{L} & -\frac{F_{x2}}{10} & -\frac{M_{x2}}{L} \\
 & & & \frac{F_{x2}I_\rho}{AL} & -\frac{2M_{z1}-M_{z2}}{6} & \frac{2M_{y1}-M_{y2}}{6} & 0 & -\frac{M_{y1}}{L} & -\frac{M_{z1}}{L} & -\frac{F_{x2}I_\rho}{AL} & -\frac{M_{z1}+M_{z2}}{6} & \frac{M_{y1}+M_{y2}}{6} \\
 & & & & \frac{2F_{x2}L}{15} & 0 & 0 & -\frac{M_{x2}}{L} & \frac{F_{x2}}{10} & -\frac{M_{z1}+M_{z2}}{6} & -\frac{F_{x2}L}{30} & \frac{M_{x2}}{2} \\
 & & & & & \frac{2F_{x2}L}{15} & 0 & -\frac{F_{x2}}{10} & -\frac{M_{x2}}{L} & \frac{M_{y1}+M_{y2}}{6} & -\frac{M_{x2}}{2} & -\frac{F_{x2}L}{30} \\
 & & & & & & \frac{F_{x2}}{L} & 0 & 0 & 0 & 0 & 0 \\
 & & & & \text{Sym.} & & & \frac{6F_{x2}}{5L} & 0 & -\frac{M_{y2}}{L} & \frac{M_{x2}}{L} & -\frac{F_{x2}}{10} \\
 & & & & & & & & \frac{6F_{x2}}{5L} & -\frac{M_{z2}}{L} & \frac{F_{x2}}{10} & \frac{M_{x2}}{L} \\
 & & & & & & & & & \frac{F_{x2}I_\rho}{AL} & \frac{M_{z1}-2M_{z2}}{6} & -\frac{M_{y1}-2M_{y2}}{6} \\
 & & & & & & & & & & \frac{2F_{x2}L}{15} & 0 \\
 & & & & & & & & & & & \frac{2F_{x2}L}{15}
\end{bmatrix}
\end{array}
\tag{9.19}
$$