7.12 SUMMARY: THE EQUATIONS OF LINEAR ELASTICITY

We are now in a position to summarize the equations of the linear theory of elasticity. The unknown quantities are the linear strain tensor, ϵ_{mn}, the displacement tensor, u_m, and the stress tensor, σ_{mn}. These tensors are a function of the coordinates y_1, y_2, y_3 of the structure. Their tensor properties are denoted by the following transformation laws:

$$\tilde{\epsilon}_{mn}(\tilde{y}_1, \tilde{y}_2, \tilde{y}_3) = \epsilon_{rs}(y_1, y_2, y_3) l_{r\tilde{m}} l_{s\tilde{n}}, \tag{7.12.1}$$

$$\tilde{u}_m(\tilde{y}_1, \tilde{y}_2, \tilde{y}_3) = u_r(y_1, y_2, y_3) l_{\tilde{m}r}, \tag{7.12.2}$$

$$\tilde{\sigma}_{mn}(\tilde{y}_1, \tilde{y}_2, \tilde{y}_3) = \sigma_{rs}(y_1, y_2, y_3) l_{\tilde{m}r} l_{\tilde{n}s}. \tag{7.12.3}$$

The objective of solid mechanics is the complete specification of these three tensors throughout any given structure under a prescribed loading condition and/or a geometrical boundary condition. The equations which govern the behavior of ϵ_{mn}, u_m, and σ_{mn} are as follows:

Linear strain-displacement relations

$$\epsilon_{mn} = \frac{1}{2}\left(\frac{\partial u_m}{\partial y_n} + \frac{\partial u_n}{\partial y_m}\right). \tag{7.12.4}$$

Force equilibrium equations for linear strains

$$\frac{\partial \sigma_{mn}}{\partial y_m} + F_n = 0. \tag{7.12.5}$$

Constitutive relations for anisotropic materials (without temperature changes)

$$\sigma_{mn} = E_{mnpr}\epsilon_{pr}, \tag{7.12.6}$$

$$\epsilon_{pr} = S_{mnpr}\sigma_{pr}. \tag{7.12.7}$$

These last relations, which are different forms of the generalized Hooke's law, introduce the elasticity tensor E_{mnpr} and the compliance tensor S_{mnpr}. The components of the elasticity tensor are a property of the material, and are determined experimentally. Since most of the high-strength engineering materials are isotropic, the bulk of the literature in solid mechanics is devoted to this class of materials. The number of independent components of the elasticity tensor reduces to two for isotropy, and Eqs. (7.12.6) and (7.12.7) become much simpler.

The *stress-strain relations* for isotropic materials with changes in temperature are

$$\epsilon_{mn} = \frac{1}{E}[(1 + \nu)\sigma_{mn} - \nu\, \delta_{mn}\sigma_{rr}] + \alpha\, \delta_{mn}\, \Delta T, \tag{7.12.8}$$

or alternatively,

$$\sigma_{mn} = 2\mu\epsilon_{mn} + \lambda\,\delta_{mn}\epsilon_{rr} - (3\lambda + 2\mu)\,\delta_{mn}\alpha\,\Delta T, \tag{7.12.9}$$

where

$$\alpha = \text{Coefficient of thermal expansion,}$$
$$E = \text{Young's modulus,}$$
$$\nu = \text{Poisson's ratio,}$$
$$\mu = \frac{E}{2(1+\nu)}, \tag{7.12.10}$$
$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}. \tag{7.12.11}$$

The *boundary conditions* which Eqs. (7.12.4), (7.12.5), and either (7.12.8) or (7.12.9) must satisfy are of two types:

(a) *Geometric constraints on a region A_1 of the surface.* These mean the prescription of

$$\mathbf{u} = \mathbf{u}^*, \tag{7.12.12}$$

where $\mathbf{u}^*$ is a prescribed displacement. At the points of support the boundary displacements are zero.

(b) *Applied surface loads on a region A_2 of the surface.* This specification means that the interior stresses in region A_2 must be in equilibrium with the externally applied surface loads. This restriction is expressed as

$$\sigma_{mn}n_m\mathbf{i}_n = \boldsymbol{\sigma}^*, \tag{7.12.13}$$

where $\boldsymbol{\sigma}^*$ is the prescribed stress vector acting on region A_2.

We note that there are a total of fifteen unknowns, consisting of six components of the strain tensor, three components of the displacement tensor, and six components of the stress tensor. Balancing these unknowns are fifteen equations consisting of three equations of equilibrium, six strain-displacement relations, and six stress-strain relations. A remark should be made here concerning the six strain-displacement relations. We have shown in Section 5.13 that the six strain-displacement relations for linear strain also lead to a set of six compatibility relations between the components of the strain:

$$\frac{\partial^2 \epsilon_{nk}}{\partial y_m\,\partial y_l} + \frac{\partial^2 \epsilon_{ml}}{\partial y_n\,\partial y_k} - \frac{\partial^2 \epsilon_{nl}}{\partial y_m\,\partial y_k} - \frac{\partial^2 \epsilon_{mk}}{\partial y_n\,\partial y_l} = 0. \tag{7.12.14}$$

Thus instead of using the strain-displacement relations one may verify the efficacy of a solution for the strain components by determining whether the compatibility equations are satisfied. However, Eqs. (7.12.4) must be employed in order to determine the displacements of the body.

The exact solution of the complete set of equations has not been accomplished except for a very few simple cases. The real challenge of solid mechanics to the engineer lies, then, in devising suitably accurate approximate solutions. Great ingenuity is required, first to construct a suitable mathematical model of the structure, and second, to solve the simplified equations which describe the mathematical model. Finally, engineering judgment is required to evaluate the validity of the analysis in the light of past experience and new experiments.

Our further studies in solid mechanics will be devoted to the solutions of the equations of elasticity. There are many different mathematical tools which can be used, and these range over a broad spectrum of mathematics from numerical analysis through the calculus of variations to the complex variable. The development of these analytical tools and their application to elasticity is a fascinating experience.

7.13 SIMPLE EXAMPLES OF SOLUTIONS FOR EQUATIONS OF ELASTICITY

As we have observed, for a solid of general geometry and arbitrary boundary conditions it is not possible to obtain a close solution of the fifteen equations of elasticity and the corresponding boundary conditions listed in the preceding section. For some important structural components of simplified geometry and loading, however, the number of equations can be reduced. Here for purposes of illustration we will present two simple examples for which the fifteen equations are satisfied. In addition we will derive the expressions of strain energy for the simple structural elements.

(a) Tension of a prismatic bar. Let us consider a uniform prismatic bar under uniform tension stress, as shown in Fig. 7.8. If body forces are not present, the equations of equilibrium (Eq. 7.12.5) are satisfied by the following components of stress:

$$\begin{aligned} \sigma_{11} &= S = \text{Constant}, \\ \sigma_{22} &= \sigma_{33} = \sigma_{12} = \sigma_{23} = \sigma_{31} = 0. \end{aligned} \tag{7.13.1}$$

It is seen that the lateral surface around the bar is free of stress. Since all stress

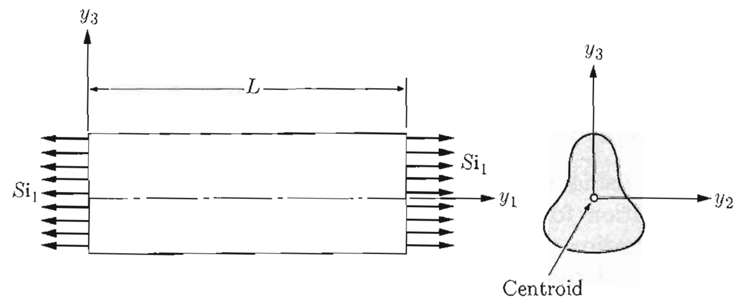

FIG. 7.8 Prismatic bar under uniaxial stress.

components except σ_{11} are zero, the boundary conditions given by Eq. (7.12.13) are satisfied. At the two ends, however, the boundary condition requires that the prescribed boundary stress vector $\boldsymbol{\sigma}^*$ be given by the normal stress $S\mathbf{i}_1$.

Assuming that the material is isotropic, we can calculate the state of strain using Eq. (7.12.8). The result is

$$\begin{aligned} \epsilon_{11} &= \sigma_{11}/E, \\ \epsilon_{22} &= \epsilon_{33} = -\nu\sigma_{11}/E, \\ \epsilon_{12} &= \epsilon_{23} = \epsilon_{31} = 0. \end{aligned} \tag{7.13.2}$$

It is obvious that all the compatibility relations (7.12.14) are satisfied, since all strain components are constants. The strain-energy density U^* is simply [from Eq. (7.10.3)]

$$U^* = (1/2)\sigma_{11}\epsilon_{11}, \tag{7.13.3}$$

which can be written as either

$$U^* = (1/2E)\sigma_{11}^2, \tag{7.13.4}$$

or

$$U^* = (E/2)\epsilon_{11}^2. \tag{7.13.5}$$

For a bar of cross-sectional area A, length L, and total axial force P, we have

$$\sigma_{11} = P/A, \tag{7.13.6}$$

and hence the total strain energy in the bar is

$$U = \iiint U^*\, dV = P^2L/2AE. \tag{7.13.7}$$

If the elongation of the bar is Δ, we have

$$\epsilon_{11} = \Delta/L, \tag{7.13.8}$$

and

$$U = AE\Delta^2/2L. \tag{7.13.9}$$

(b) Pure bending of a uniform beam. Consider a uniform beam bent in its plane of symmetry by two equal and opposite bending moments M, as shown in Fig. 7.9. Let the y_1-axis lie along the centroid of the cross section. The bending moment as shown is about the y_2-axis. According to elementary bending theory the stress components are given by

$$\begin{aligned} \sigma_{11} &= Cy_3, \\ \sigma_{22} = \sigma_{33} = \sigma_{12} &= \sigma_{23} = \sigma_{31} = 0, \end{aligned} \tag{7.13.10}$$

where C is a constant.

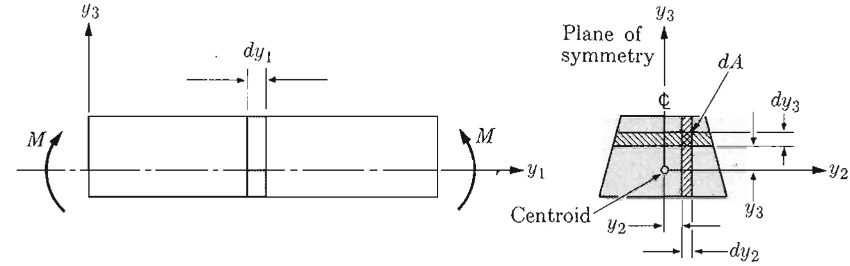

FIG. 7.9 Prismatic bar under pure bending.

We see that all the equations of equilibrium, Eq. (7.12.5), and the boundary conditions, Eq. (7.12.13), for the surface around the beam are satisfied. At the two ends of the beam the boundary condition requires that the prescribed boundary stress vector $\boldsymbol{\sigma}^*$ should consist only of normal stresses, $\sigma_{11}^* \mathbf{i}_1$. The variation of σ_{11}^* is also given by Cy_3, that is, it must be directly proportional to the distance from the y_2-axis. Since the prescribed boundary stresses σ_{11}^* must be equipollent to the applied bending moment M, we have

$$M = -\iint \sigma_{11}^* y_3 \, dA = -C \iint y_3^2 \, dA = -IC, \tag{7.13.11}$$

where I is the cross-sectional moment of inertia with respect to the y_2-axis: $I = \iint y_3^2 \, dA$. In order to satisfy the other conditions of equilibrium we need to verify that the resulting axial force and moment about the y_3-axis are zero. This is indeed true:

$$\iint \sigma_{11}^* \, dA = C \iint y_3 \, dA = 0, \tag{7.13.12}$$

and

$$\iint \sigma_{11}^* y_2 \, dy_2 \, dy_3 = C \iint y_2 y_3 \, dy_2 \, dy_3 = 0, \tag{7.13.13}$$

because the y_2-axis passes through the centroid, and y_3 is the axis of symmetry of the cross section. We can now determine the constant C from Eq. (7.13.11),

$$C = -M/I, \tag{7.13.14}$$

and substitute this value in Eq. (7.13.10).

$$\sigma_{11} = -My_3/I. \tag{7.13.15}$$

For an isotropic material the strain components are given by

$$\epsilon_{11} = -My_3/EI, \tag{7.13.16}$$

$$\epsilon_{22} = \epsilon_{33} = \nu My_3/EI, \tag{7.13.17}$$

$$\epsilon_{12} = \epsilon_{23} = \epsilon_{31} = 0. \tag{7.13.18}$$

The six compatibility equations are then satisfied, because the strain components are either zero or of only a first power in y_3.

Let us now consider the displacement of the beam under the condition of pure bending. The strain-displacement relations Eq. (7.12.4) yield the following differential equations:

$$\frac{\partial u_1}{\partial y_1} = \epsilon_{11} = -\frac{My_3}{EI}, \tag{7.13.19}$$

$$\frac{\partial u_2}{\partial y_2} = \epsilon_{22} = \nu\frac{My_3}{EI}, \tag{7.13.20}$$

$$\frac{\partial u_3}{\partial y_3} = \epsilon_{33} = \nu\frac{My_3}{EI}, \tag{7.13.21}$$

$$\frac{1}{2}\left(\frac{\partial u_1}{\partial y_2} + \frac{\partial u_2}{\partial y_1}\right) = \epsilon_{12} = 0, \tag{7.13.22}$$

$$\frac{1}{2}\left(\frac{\partial u_2}{\partial y_3} + \frac{\partial u_3}{\partial y_2}\right) = \epsilon_{23} = 0, \tag{7.13.23}$$

$$\frac{1}{2}\left(\frac{\partial u_1}{\partial y_3} + \frac{\partial u_3}{\partial y_1}\right) = \epsilon_{31} = 0. \tag{7.13.24}$$

The displacements u_1, u_2, and u_3 can be obtained by integrating these equations when the geometrical conditions, i.e., the constraints of the bar are given. Let us now focus our attention on the lateral deflection of the beam, i.e., the component u_3. From Eq. (7.13.19) we obtain

$$u_1 = -\frac{My_1y_3}{EI} + \bar{u}, \tag{7.13.25}$$

where $\bar{u}$ may be a function of y_2 and y_3. From Eq. (7.13.24) we obtain

$$\frac{\partial u_3}{\partial y_1} = -\frac{\partial u_1}{\partial y_3} = \frac{My_1}{EI} - \frac{\partial \bar{u}}{\partial y_3}. \tag{7.13.26}$$

Differentiation with respect to y_1 gives

$$\frac{\partial^2 u_3}{\partial y_1^2} = \frac{M}{EI}. \tag{7.13.27}$$

Since the bending moment is not a function of y_3, Eq. (7.13.27) yields the important result that the second derivative of lateral displacement for every longitudinal fiber of the beam is a constant. (A longitudinal fiber is an element of the beam with the same y_3-coordinate.) If we denote the lateral deflection of the axis of the beam by $w(y_1)$ instead of u_3, we have

$$\frac{d^2 w}{dy_1^2} = \frac{M}{EI}. \tag{7.13.28}$$

We recall from calculus that the curvature of a given curve, $y = f(x)$, is given by

$$\frac{1}{R} = \frac{d^2f/dx^2}{[1 + (df/dx)^2]^{3/2}}, \tag{7.13.29}$$

where R is the radius of curvature. For a very flat curve for which df/dx is very small in comparison to unity, the curvature may be represented simply by the second derivative of the curve, that is,

$$\frac{1}{R} = \frac{d^2f}{dx^2}. \tag{7.13.30}$$

The deformed shapes of beams are such that the curvatures are small; thus d^2w/dy_1^2 is the curvature of the deformed beam. We also find that the normal strain ϵ_{11} is related to the curvature by

$$\epsilon_{11} = -y_3 \frac{d^2w}{dy_1^2}. \tag{7.13.31}$$

We consider next the strain energy dU for an element dy_1 of the beam. Since the only stress component is σ_{11}, we have

$$dU = \left(\frac{1}{2} \iint \frac{\sigma_{11}^2}{E}\, dy_2\, dy_3\right) dy_1, \tag{7.13.32}$$

or

$$dU = \left(\frac{1}{2} \iint E\epsilon_{11}^2\, dy_2\, dy_3\right) dy_1. \tag{7.13.33}$$

Substituting Eq. (7.13.15) into Eq. (7.13.32), and Eq. (7.13.31) into Eq. (7.13.33), we obtain respectively

$$dU = \frac{M^2}{2EI}\, dy_1, \tag{7.13.34}$$

and

$$dU = \frac{EI}{2}\left(\frac{d^2w}{dy_1^2}\right)^2 dy_1. \tag{7.13.35}$$

7.14 ENGINEERING BEAM THEORY

At this point it is worth while to mention the so-called engineering beam theory which covers the nonuniform beam under general lateral loading conditions. In such a case, the bounding surface of the beam may not be free of stress, and/or the body force F_3 may not be zero. Also, in general, the shear force in each section is not zero, and hence the bending moment M is not constant along the beam. The engineering beam theory, however, also neglects the normal stresses σ_{22} and σ_{33}, because it can be shown that these components are of a much smaller order of magnitude than σ_{11}. The theory also assumes that for a beam, the deformation due to shear strain is negligible in comparison to that due to the normal strain ϵ_{11}. As a result, the normal stresses and strains may still be calculated using Eqs. (7.13.15) and (7.13.31), although the bending moment, curvature, and moment of inertia are no longer constant along y_1. This engineering beam theory also assumes that the moment-curvature relation, Eq. (7.13.28), will still hold, and that the strain energy of a complete beam can be calculated by integrating Eq. (7.13.35).

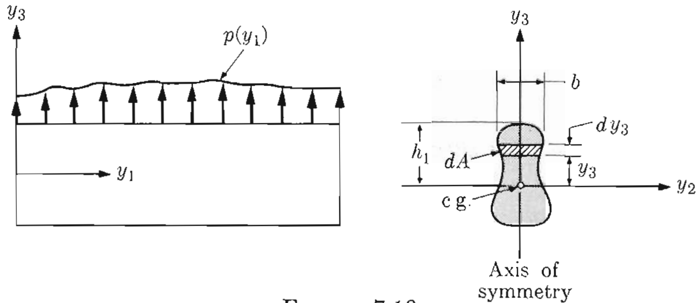

FIGURE 7.10

7.15 SUMMARY: ENGINEERING BEAM THEORY

The equations of engineering beam theory (see Figure 7.10 for coordinates and other nomenclature) are as follows:

Definition of equipollent transverse shear and bending moment

$$S = -\int_A \sigma_{13}\, dA, \tag{7.15.1}$$

$$M = -\int_A \sigma_{11} y_3\, dA. \tag{7.15.2}$$

Displacements

$$u_3(y_1, y_2, y_3) = w(y_1), \tag{7.15.3}$$

$$u_1(y_1, y_2, y_3) = -y_3 \frac{dw}{dy_1}. \tag{7.15.4}$$

Strain-displacement relation

$$\epsilon_{11} = -y_3 \frac{d^2w}{dy_1^2}. \tag{7.15.5}$$

Moment-curvature relation

$$\frac{d^2w}{dy_1^2} = \frac{M}{EI}. \tag{7.15.6}$$

Force equilibrium

$$\frac{dS}{dy_1} = p. \tag{7.15.7}$$

Moment equilibrium

$$\frac{dM}{dy_1} = S. \tag{7.15.8}$$

Stress-strain relation

$$\sigma_{11} = E\epsilon_{11}. \tag{7.15.9}$$

Bending stress versus moment relation

$$\sigma_{11} = -\frac{My_3}{I}. \tag{7.15.10}$$

Shear stress versus transverse shear relation

$$\sigma_{13} = \frac{SQ}{bI}. \tag{7.15.11}$$

Cross-sectional moment of inertia

$$I = \iint y_3^2\, dA. \tag{7.15.12}$$

Static moment

$$Q = \iint_{y_3}^{h_1} y_3\, dA. \tag{7.15.13}$$

PROBLEMS

7.1 (a) Thin flat panels (Fig. P.7.1) with external forces acting in the y_1y_2-plane of the panel are said to be in a *state of plane stress*, that is, $\sigma_{33} = \sigma_{13} = \sigma_{23} = 0$ (see Problem 6.4). For an isotropic material verify the following expressions for γ_{11}, γ_{22}, and γ_{12} in terms of σ_{11}, σ_{22}, and σ_{12} and the elastic constants E and ν.

$$\gamma_{11} = \frac{1}{E}(\sigma_{11} - \nu\sigma_{22}), \qquad \gamma_{22} = \frac{1}{E}(\sigma_{22} - \nu\sigma_{11}), \qquad \gamma_{12} = \frac{1+\nu}{E}\sigma_{12}.$$

(b) Determine the following inverse stress-strain relations, i.e., the expression of σ_{11}, σ_{22}, and σ_{12} in terms of γ_{11}, γ_{22}, and γ_{12}:

$$\sigma_{11} = \frac{E}{1-\nu^2}(\gamma_{11} + \nu\gamma_{22}), \qquad \sigma_{22} = \frac{E}{1-\nu^2}(\gamma_{22} + \nu\gamma_{11}), \qquad \sigma_{12} = \frac{E}{1+\nu}\gamma_{12}.$$

How are the elastic constants related to the elasticity tensor E_{mnpr}?

7.2 The fibers of a reinforced plastic panel are evenly distributed along three different preferred directions, AA, BB, and CC, which are 60° apart. All are parallel to the face of the panel, as shown in Fig. P.7.2. One of these directions is along the