# 2 Gauss Elimination

## 2.1 Inverse matrices

A is invertible if there exists a matrix $A^{-1}$ such that

$$A^{-1}A = I \qquad \text{and} \qquad AA^{-1} = I$$

(1) inverse exists $\Leftrightarrow$ elimination produces n pivots (row exchanges allowed)

(2) A cannot have two different inverses.

$$BA = I$$
$$AC = I$$

i.e. A left-inverse and right-inverse are the same.

(3) If A is invertible, the one and only solution to $A\mathrm{x} = \mathrm{b}$ is $\mathrm{x} = A^{-1}\mathrm{b}$ .

(4) If there is a nonzero vector $x$ such that $A\mathrm{x} = 0$ , then $A$ cannot have an inverse.
If $A$ is invertible, $x = 0$ is the only solution to $A\mathrm{x} = 0$

(5) A is invertible $\Leftrightarrow$ determinant is not zero.

ex) $$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{cd - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

(6)

$$\begin{bmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{bmatrix}^{-1} = \begin{bmatrix} 1/d_1 & & 0 \\ & \ddots & \\ 0 & & 1/d_n \end{bmatrix}$$

- If A and B are invertible, then so is AB.

$$(AB)^{-1} = B^{-1}\,A^{-1}$$

## 2.2 Gauss Elimination

• Calculating $A^{-1}$ by Gauss - Jordan Elimination

$$AA^{-1} = I$$

$$A\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} = \begin{bmatrix} e_1 & e_2 & e_3 \end{bmatrix}$$

Gauss - Jordan method solves three systems of equations($Ax_i = e_i$) together.

• Augmented matrix

$$\begin{bmatrix} A & e_1 & e_2 & e_3 \end{bmatrix} = \left[\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$\frac{1}{2}$ row 1 + row 2

$\Longrightarrow$

$$\left[\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & \frac{1}{2} & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$\frac{2}{3}$ row 2 + row 3

$\Longrightarrow$

$$\left[\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & \frac{1}{2} & 1 & 0 \\ 0 & 0 & \frac{4}{3} & \frac{1}{3} & \frac{2}{3} & 1 \end{array}\right]$$

Continue!!

$\frac{3}{4}$ row 3 + row 2

$\Longrightarrow$

$$\left[\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & \frac{3}{2} & 0 & \frac{3}{4} & \frac{3}{2} & \frac{3}{4} \\ 0 & 0 & \frac{4}{3} & \frac{1}{3} & \frac{2}{3} & 1 \end{array}\right]$$

$\frac{2}{3}$ row 2 + row 1

$\Longrightarrow$

$$\left[\begin{array}{ccc|ccc} 2 & 0 & 0 & \frac{3}{2} & 1 & \frac{1}{2} \\ 0 & \frac{3}{2} & 0 & \frac{3}{4} & \frac{3}{2} & \frac{3}{4} \\ 0 & 0 & \frac{4}{3} & \frac{1}{3} & \frac{2}{3} & 1 \end{array}\right]$$

divide each row by its pivots

$\Longrightarrow$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 1 & 0 & \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{array}\right] = \left[\begin{array}{c|ccc} I & x_1 & x_2 & x_3 \end{array}\right]$$

i.e

$$\boxed{\begin{array}{l} \text{G-J process multiplies,} \\ \qquad A^{-1}\,[\,A \quad I\,] \\ \text{to get } [\,I \quad A^{-1}\,] \end{array}}$$

• Let $A$ be a square matirx.
$A^{-1}$ exists (and Gauss-Jordan finds it) exactly when A has $n$ pivots.

$$\text{If } AC = I, \text{ then } CA = I \text{ and } C = A^{-1}.$$

**Example 1.**

If $L$ is lower triangular with 1's on the diagonal, so is $L^{-1}$.

$$[\,L \quad I\,] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ * & 1 & 0 & 0 & 1 & 0 \\ * & * & 1 & 0 & 0 & 1 \end{array}\right]$$

$\Longrightarrow$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & * & 1 & 0 \\ 0 & * & 1 & * & 0 & 1 \end{array}\right]$$

$\Longrightarrow$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & * & 1 & 0 \\ 0 & 0 & 1 & * & * & 1 \end{array}\right] = [\,I \mid L^{-1}\,]$$

• Elimination = LU Factorization (Strang P883)

**Example 2.**

$$A = \begin{bmatrix} 2 & 1 \\ 6 & 8 \end{bmatrix}$$

$$E_{21}A = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 6 & 8 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 5 \end{bmatrix} = U$$

$$\underbrace{E_{21}^{-1}U}_{LU=A} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 6 & 8 \end{bmatrix} = A$$

**Example 3.**

A is 3x3, no row exchange.
Then,

$$E_{21}A$$

$$E_{31}(E_{21}A)$$

$$E_{32}(E_{31}E_{21}A)$$

$$\Longrightarrow A = \underbrace{(E_{21}^{-1} \quad E_{31}^{-1} \quad E_{32}^{-1})}_{\text{lower triangular, 1's on the diagonal}} \quad U = LU$$

Each multiplier $l_{ij}$ goes directly into $(i, j)$ position.

**Example 4.**

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & \frac{2}{3} & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 2 & 1 & 0 \\ 0 & \frac{2}{3} & 1 \\ 0 & 0 & \frac{4}{3} \end{bmatrix}}_{U}$$

Observation :

- If a row of A starts with 0, then so does that row of $L$
- If a column of A starts with 0,then so does that column of $U$

In the above example,

Row 3 of $U$ = Row of $A$ - $l_{31}$ (Row1 of $U$) -$l_{32}$(Row2 of $U$)

$\Longrightarrow$ Row3 of $A$

$= l_{31}$(Row1 of $U$) + $l_{32}$(Row2 of $U$) + 1 (Row3 of $U$)

= Row3 of $LU$

This shows why A = LU.

## 2.3 LDU Factorization

Divide $U$ by diagonal elements so that $U$ has 1's on the diagonal.

In the previous example,

$$\begin{aligned} A &= \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & \frac{2}{3} & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & \frac{3}{2} & 0 \\ 0 & 0 & \frac{4}{3} \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 0 & 1 & \frac{2}{3} \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ * & 1 & 0 \\ * & * & 1 \end{bmatrix} \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} \begin{bmatrix} 1 & \frac{U_{12}}{d_1} & \frac{U_{13}}{d_1} \\ 0 & 1 & \frac{U_{23}}{d_2} \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

## 2.4 Transposes

$$A^T = \text{transpose of } A$$
$$\text{column of } A^T = \text{row of } A$$
$$(A^T)_{ij} = A_{ji}$$

- $(A + B)^T = A^T + B^T$
- $(AB)^T = B^T A^T$

- $(A^{-1})^T = (A^T)^{-1}$
- For any vectors x and y,

$$(Ax)^T y = x^T A^T y = x^T (A^T y)$$

- Symmetric matrix : $AT = A$. $(a_{ji} = a_{ij})$
- Inverse of an invertible, symmetric matrix is also symmetric.

$$\therefore (A^{-1})^T = (A^T)^{-1} = A^{-1}$$

- Choose any matrix $R$, then $R^T R$ is square, and

$$(R^T R)^T = R^T (R^T)^T = R^T R$$

$$\therefore R^T R \text{ is a symmetric matrix.}$$

$$(RR^T \text{ is also a symmetric matrix.})$$

In the previous example,

$$\underbrace{A}_{symmetric} = LDU = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & \frac{2}{3} & 1 \end{bmatrix} \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 0 & 1 & \frac{2}{3} \\ 0 & 0 & 1 \end{bmatrix}$$

$$= LDL^T$$

$\Longrightarrow$ If $A = A^T$, $A = LDU$ (with no row exchange), then
$U = L^T$, and $A = LDL^T$