ESTIMATION THEORY

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Introduction

Problem: Find the best estimate $\hat{x}$ from the measurements of the form

$z = x + w$

where w is a random process.

(1). The first measurement: $z_1 = x(t_1) + w$

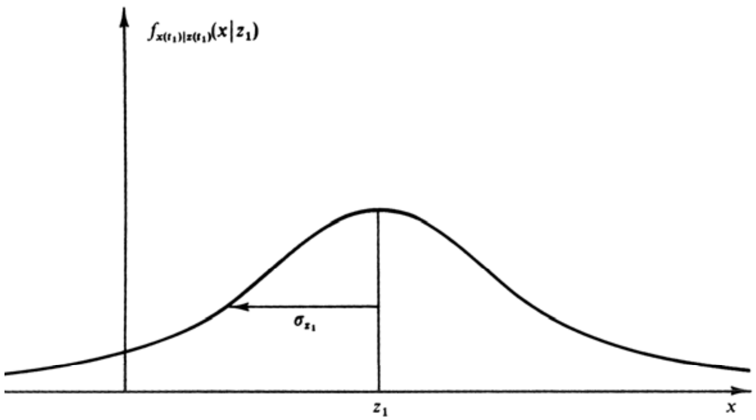


FIG. 1.4 Conditional density of position based on measured value z_1.

$$\hat{x}(t_1) = z_1$$

$$\sigma_x^{\,2}(t_1) = \sigma_{z_1}^{\,2}$$

Introduction (continued)

(2). The second measurement: $z_2,\ t_2 \cong t_1$

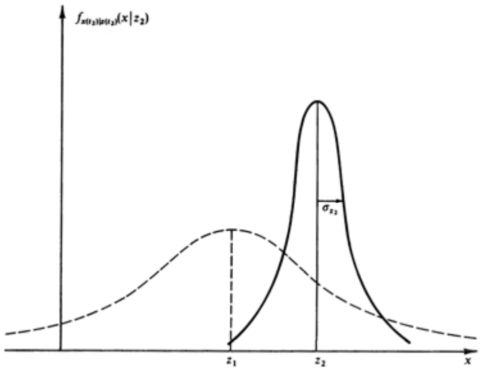


FIG. 1.5 Conditional density of position based on measurement z_2 alone.

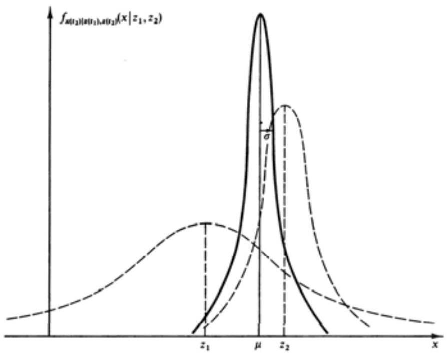


FIG. 1.6 Conditional density of position based on data z_1 and z_2.

$$\mu = \left[\sigma_{z_2}^{\ 2} / \left(\sigma_{z_1}^{\ 2} + \sigma_{z_2}^{\ 2}\right)\right] z_1 + \left[\sigma_{z_1}^{\ 2} / (\sigma_{z_1}^{\ 2} + \sigma_{z_2}^{\ 2})\right] z_2$$

$$1/\sigma^2 = \left(1/\sigma_{z_1}^{\ 2}\right) + \left(1/\sigma_{z_2}^{\ 2}\right)$$

$$\hat{x}(t_2) = \mu = \left[\sigma_{z_2}^{\ 2} / \left(\sigma_{z_1}^{\ 2} + \sigma_{z_2}^{\ 2}\right)\right] z_1 + \left[\sigma_{z_1}^{\ 2} / (\sigma_{z_1}^{\ 2} + \sigma_{z_2}^{\ 2})\right] z_2$$

$$= z_1 + \left[\sigma_{z_1}^{\ 2} / \left(\sigma_{z_1}^{\ 2} + \sigma_{z_2}^{\ 2}\right)\right](z_2 - z_1)$$

$$= \hat{x}(t_1) + K(t_2)\left[z_2 - \hat{x}(t_1)\right] = \text{Predictor} + \text{Corrector}$$

$$\sigma_x^{\ 2}(t_2) = \sigma_x^{\ 2}(t_1) - K(t_2)\sigma_x^{\ 2}(t_1)$$

Introduction (continued)

(3). The third measurement: $z_3 = x(t_3) + w(t_3); \dfrac{dx}{dt} = u + w$

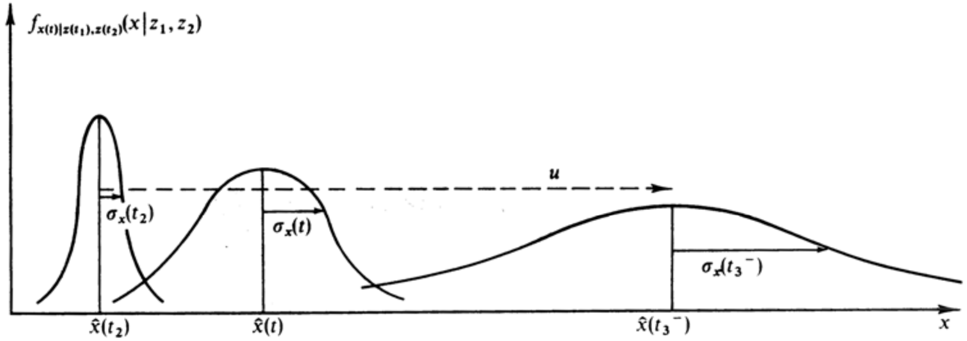


FIG. 1.7 Propagation of conditional probability density.

$$\hat{x}(t_3^-) = \hat{x}(t_2) + u(t_3 - t_2)$$
$$\sigma_x^2(t_3^-) = \sigma_x^2(t_2) + \sigma_w^2(t_3 - t_2)$$

$$\hat{x}(t_3^+) = \hat{x}(t_3^-) + K(t_3)[z_3 - \hat{x}(t_3^-)]$$
$$\sigma_x^2(t_3^+) = \sigma_x^2(t_3^-) - K(t_3)\sigma_x^2(t_3^-)$$
$$K(t_3) = \sigma_x^2(t_3^-)/[\sigma_x^2(t_3^-) + \sigma_{z_3}^2]$$

As $\sigma_{z_3}^2 \to \infty, \quad K(t_3) = 0.$

As $\sigma_w^2 \to \infty, \sigma_x^2(t_3^-) \to \infty$ and $K(t_3) = 1.$

Chapter 1

Linear Systems Theory

Matrix Algebra

(1). Matrix Multiplication

Suppose that A is an $n \times r$ matrix and B is an $r \times p$ matrix. Then the product of A and B is written as $C = AB$. Each element in the matrix product C is computed as

$$C_{ij} = \sum_{k=1}^{r} A_{ik} B_{kj}; \quad i = 1, \cdots, n; \quad j = 1, \cdots, p. \tag{1.13}$$

In general, $AB \neq BA$. (no commutability)

(2). Vector Products

Inner Product: $\underline{x}^T \underline{x} = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1^2 + \cdots + x_n^2.$

Outer Product: $$\underline{xx}^T = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} = \begin{bmatrix} x_1^2 & \cdots & x_1 x_n \\ \vdots & \ddots & \vdots \\ x_n x_1 & \cdots & x_1^2 \end{bmatrix}. \tag{1.14}$$

Matrix Algebra (continued)

(3). Rank and Nonsingularity of $A = (n \times n)$

- A is nonsingular.
- A^{-1} exists.
- The rank of A is equal to n.
- The rows of A are linearly independent.
- The columns of A are linearly independent.
- $|A| \neq 0$.
- $A\underline{x} = \underline{b}$ has a unique solution $\underline{x}$ for all $\underline{b}$.
- 0 is not an eigenvalue of A.

(4). Trace of a square matrix: $Tr(A) = \sum_i A_{ii}$

Note that $Tr(AB) = Tr(BA)$,

$$(AB)^T = B^T A^T,$$

$$(AB)^{-1} = B^{-1} A^{-1}.$$

Matrix Algebra (continued)

(5). Definiteness of a Symmetric $n \times n$ matrix A

A is:

- Positive definite if $\underline{x}^T A \underline{x} > 0$ for all nonzero $n \times 1$ vectors $\underline{x}$. This is equivalent to saying that all of the eigenvalues of A are oisutuve reak numbers. If A is positive definite, then is also positive definite.
- Positive semidefinite if $\underline{x}^T A \underline{x} \geq 0$.
- Negative definite if $\underline{x}^T A \underline{x} < 0$.
- Negative semidefinite if $\underline{x}^T A \underline{x} \leq 0$.
- Indefinite if it does not fit into any of the above four caregories.

(6). Matrix Inversion Lemma

Suppose we have the partitioned matrix $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ where A and D are invertible square matrices, and the B and C matrices may or may not be square. Then,

$$\left(A - BD^{-1}C\right)^{-1} = A^{-1} + A^{-1}B\left(D - CA^{-1}B\right)^{-1} CA^{-1}. \tag{1.38}$$

Matrix Calculus

$$(1).\ \frac{\partial f(\underline{x})}{\partial \underline{x}} = \left[\frac{\partial f}{\partial x_1} \cdots \frac{\partial f}{\partial x_n}\right].$$

$$(2).\ \frac{\partial f(A)}{\partial A} = \begin{bmatrix} \partial f / \partial A_{11} \cdots \partial f / \partial A_{1n} \\ \vdots \qquad \ddots \qquad \vdots \\ \partial f / \partial A_{m1} \cdots \partial f / \partial A_{mn} \end{bmatrix}; \quad A = \left(A_{ij}\right),\ i = 1, \cdots, m,\ j = 1, \cdots, n.$$

$$(3).\ \frac{\partial\left(\underline{x}^T \underline{y}\right)}{\partial \underline{x}} = \left[\partial\left(\underline{x}^T \underline{y}\right) / \partial x_1 \cdots \partial\left(\underline{x}^T \underline{y}\right) / \partial x_n\right] = \left[y_1 \cdots y_n\right] = \underline{y}^T;\ \frac{\partial\left(\underline{x}^T \underline{y}\right)}{\partial \underline{y}} = \underline{x}^T.$$

$$(4).\ \frac{\partial\left(\underline{x}^T A \underline{x}\right)}{\partial \underline{x}} = \underline{x}^T A^T + \underline{x}^T A;\ \frac{\partial\left(\underline{x}^T A \underline{x}\right)}{\partial \underline{x}} = 2\underline{x}^T A \text{ if } A = A^T.$$

$$(5).\ \frac{\partial(A\underline{x})}{\partial \underline{x}} = A;\ \frac{\partial\left(\underline{x}^T A\right)}{\partial \underline{x}} = A.$$

$$(6).\ \frac{\partial Tr\left(ABA^T\right)}{\partial A} = AB^T + AB;\ \frac{\partial Tr\left(ABA^T\right)}{\partial A} = 2AB,\ \text{if } B = B^T.$$

Continuous, Deterministic Linear Systems

Models

$$\underline{\dot{x}} = A\underline{x} + B\underline{u}$$

$$\underline{y} = C\underline{x}$$

Solution

$$\underline{x}(t) = e^{A(t-t_0)}\underline{x}(t_0) + \int_{t_0}^{t} e^{A(t-\tau)} B\underline{u}(\tau)d\tau$$

$$e^{At} = \sum_{j=0}^{\infty} \frac{(At)^j}{j!}$$

$$= \mathfrak{L}^{-1}\left[(sI - A)^{-1}\right].$$

Nonlinear Systems

Models

$$\underline{\dot{x}} = f\left(\underline{x}, \underline{u}, \underline{w}\right)$$

$$\underline{y} = h\left(\underline{x}, \underline{v}\right) \tag{1.83}$$

Linearized Models Employing the Taylor Series Expansion

$$f\left(\underline{x}\right) = f\left(\underline{\overline{x}}\right) + \left.\frac{\partial f}{\partial \underline{x}}\right|_{\underline{\overline{x}}} \underline{\tilde{x}} + \frac{1}{2!}\left.\frac{\partial^2 f}{\partial \underline{x}^2}\right|_{\underline{\overline{x}}} \underline{\tilde{x}}^2 + \frac{1}{3!}\left.\frac{\partial^3 f}{\partial \underline{x}^3}\right|_{\underline{\overline{x}}} \underline{\tilde{x}}^3 + \cdots$$

$$= f\left(\underline{\overline{x}}\right) + \left.\left(\tilde{x}_1 \frac{\partial}{\partial x_1} + \cdots + \tilde{x}_n \frac{\partial}{\partial x_n}\right) f\left(\underline{x}\right)\right|_{\underline{\overline{x}}} +$$

$$\frac{1}{2!}\left.\left(\tilde{x}_1 \frac{\partial}{\partial x_1} + \cdots + \tilde{x}_n \frac{\partial}{\partial x_n}\right)^2 f\left(\underline{x}\right)\right|_{\underline{\overline{x}}} +$$

$$\frac{1}{3!}\left.\left(\tilde{x}_1 \frac{\partial}{\partial x_1} + \cdots + \tilde{x}_n \frac{\partial}{\partial x_n}\right)^3 f\left(\underline{x}\right)\right|_{\underline{\overline{x}}} + \cdots$$

<u>Nonlinear Systems</u> (continued)

$$f(\underline{x}) = f(\overline{\underline{x}}) + D_{\tilde{\underline{x}}} f + \frac{1}{2!} D_{\tilde{\underline{x}}}^2 f + \frac{1}{3!} D_{\tilde{\underline{x}}}^3 f + \cdots \qquad \left(D_{\tilde{\underline{x}}}^k f = \left(\sum_{i=1}^{k} \tilde{x}_i \frac{\partial}{\partial x_i} \right)^k f(\underline{x}) \bigg|_{\overline{\underline{x}}} \right)$$

$$\approx f(\overline{\underline{x}}) + D_{\tilde{\underline{x}}} f = f(\overline{\underline{x}}) + \frac{\partial f}{\partial \underline{x}}\bigg|_{\overline{\underline{x}}} \tilde{\underline{x}} = f(\overline{\underline{x}}) + A\tilde{\underline{x}} \tag{1.90}$$

Applying Eq. (1.90) into Eq. (1.83)

$$\dot{\underline{x}} = f(\underline{x}, \underline{u}, \underline{w})$$

$$\approx f(\overline{\underline{x}}, \overline{\underline{u}}, \overline{\underline{w}}) + \frac{\partial f}{\partial \underline{x}}\bigg|_{\overline{\underline{x}}} (\underline{x} - \overline{\underline{x}}) + \frac{\partial f}{\partial \underline{u}}\bigg|_{\overline{\underline{u}}} (\underline{u} - \overline{\underline{u}}) + \frac{\partial f}{\partial \underline{w}}\bigg|_{\overline{\underline{w}}} (\underline{w} - \overline{\underline{w}})$$

$$= \dot{\overline{\underline{x}}} + A\tilde{\underline{x}} + B\tilde{\underline{u}} + L\tilde{\underline{w}}.$$

$$\dot{\tilde{\underline{x}}} = A\tilde{\underline{x}} + B\tilde{\underline{u}} + L\underline{w}. \quad (\text{Say, } \overline{\underline{w}} = \underline{0}) \tag{1.93}$$

Similarly,

$$\tilde{\underline{y}} = \frac{\partial h}{\partial \underline{x}}\bigg|_{\overline{\underline{x}}} \tilde{\underline{x}} + \frac{\partial h}{\partial \underline{v}}\bigg|_{\overline{\underline{v}}} \tilde{\underline{v}}$$

$$= C\tilde{\underline{x}} + D\underline{v}. \tag{1.94}$$

Simulation/Trapezoidal Integration

We want to numerically solve the state equation, $\dot{x} = f(x,u,t)$.

$$x(t_f) = x(t_0) + \int_{t_0}^{t_f} f[x(t),u(t),t]\,dt$$

$$= x(t_0) + \sum_{k=0}^{L} \int_{t_k}^{t_{k+1}} f[x(t),u(t),t]\,dt \text{ where } t_k = \mathrm{k}T \text{ for } k = 0,\cdots,L \text{ and } T = t_f/L.$$

For some $n \in [0, L-1]$, we can write $x(t_n)$ and $x(t_{n+1})$ as

$$x(t_n) = x(t_0) + \sum_{k=0}^{n} \int_{t_k}^{t_{k+1}} f[x(t),u(t),t]\,dt$$

$$x(t_{n+1}) = x(t_0) + \sum_{k=0}^{n+1} \int_{t_k}^{t_{k+1}} f[x(t),u(t),t]\,dt$$

$$= x(t_n) + \int_{t_n}^{t_{n+1}} f[x(t),u(t),t]\,dt. \qquad (1.110)$$

Simulation/Trapezoidal Integration (continued)

Approximate the integral in Eq. (1.110) as a trapezoid

$$f(x,u,t) \approx f\left(x(t_n),u(t_n),t_n\right) + \left(\frac{f\left(x(t_{n+1}),u(t_{n+1}),t_{n+1}\right) - f\left(x(t_n),u(t_n),t_n\right)}{T}\right)(t-t_n) \ \text{ for } t \in [t_n, t_{n+1}]$$

$$\begin{aligned} x(t_{n+1}) &\approx x(t_n) + \int_{t_n}^{t_{n+1}} \left\{ f\left(x(t_n),u(t_n),t_n\right) + \left(\frac{f\left(x(t_{n+1}),u(t_{n+1}),t_{n+1}\right) - f\left(x(t_n),u(t_n),t_n\right)}{T}\right)(t-t_n) \right\} dt \\ &= x(t_n) + \left(\frac{f\left(x(t_n),u(t_n),t_n\right) + f\left(x(t_{n+1}),u(t_{n+1}),t_{n+1}\right)}{2}\right)T \\ &= x(t_n) + \frac{1}{2}\left(f\left(x(t_n),u(t_n),t_n\right)T + f\left(x(t_{n+1}),u(t_{n+1}),t_{n+1}\right)T\right) \qquad (1.114) \end{aligned}$$

Defining

$$\begin{aligned} \triangle x_1 &= f\left(x(t_n),u(t_n),t_n\right)T \\ \triangle x_2 &= f\left(x(t_{n+1}),u(t_{n+1}),t_{n+1}\right)T \\ &\approx f\left(x(t_n)+\triangle x_1 +, u(t_{n+1}),t_{n+1}\right)T, \end{aligned}$$

Eq. (1.114) may be expressed by

$$x(t_{n+1}) \approx x(t_n) + \frac{1}{2}(\triangle x_1 + \triangle x_2). \qquad (1.115)$$

Simulation/Trapezoidal Integration (continued)

Trapezoidal Integration Algorithm

Assume that $x(t_0)$ is given

for $t = t_0 : T : t_f - T$

$$\triangle x_1 = f\left(x(t), u(t), t\right)T$$

$$\triangle x_2 = f\left(x(t) + \triangle x_1 +, u(t+T), t+T\right)T,$$

$$x(t+T) = x(t) + \frac{1}{2}\left(\triangle x_1 + \triangle x_2\right)$$

end

Observability and Controllability

Consider the following time-invariant system and the deterministic asymptotic estimation

$$\underline{x}_{k+1} = A\underline{x}_k + B\underline{u}_k$$

$$\underline{z}_k = H\underline{x}_k \qquad (1.157)$$

where state $\underline{x}_k \in R^n$, control input $\underline{u}_k \in R^m$, output $\underline{z}_k \in R^p$;

and A, B, and H are known constant matrices of appropriate dimension.

All variables are deterministic, so that if initial state $\underline{x}_0$ is known then Eq. (1.157) can be solved exactly for $\underline{x}_k$, $\underline{z}_k$ for $k \geq 0$.

Deterministic asymptotic estimation problem: Design an estimator whose output $\hat{\underline{x}}_k$ converges with k to the actual state $\underline{x}_k$ of Eq. (1.157) when the initial state $\underline{x}_0$ is unknown, but $\underline{u}_k$ and $\underline{z}_k$ are given exactly.

An estimator of observer which solves this problem has the form

$$\hat{\underline{x}}_{k+1} = A\hat{\underline{x}}_k + L\left(\underline{z}_k - H\hat{\underline{x}}_k\right) + B\underline{u}_k$$

as shown in Fig. 2.1-1.

Observability and Controllability (continued)

DETERMINISTIC STATE OBSERVER

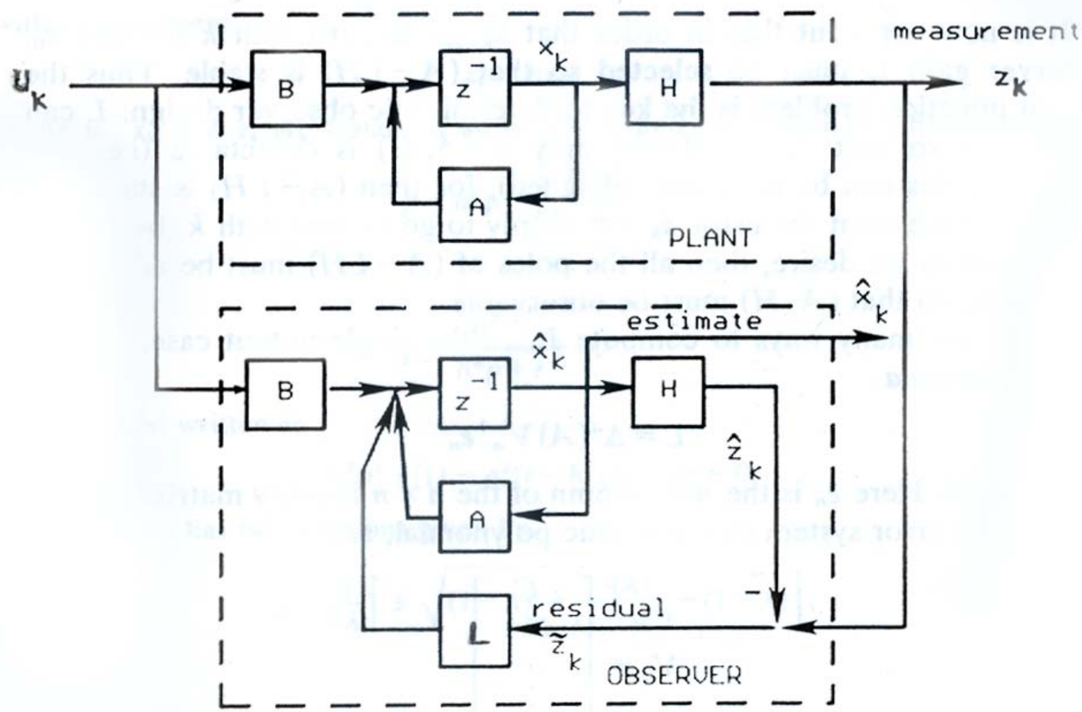


FIGURE 2.1-1 State observer.

Observability and Controllability (continued)

To Choose L in Eq. (1.157) so that the estimation error $\underline{\tilde{x}}_k = \underline{x}_k - \underline{\hat{x}}_k$ goes to zero with k for all $\underline{x}_0$, it is necessary to examine the dynamics of $\underline{\tilde{x}}_k$. Write

$$\begin{aligned}\underline{\tilde{x}}_{k+1} &= \underline{x}_{k+1} - \underline{\hat{x}}_{k+1} \\ &= A\underline{x}_k + B\underline{u}_k - \left[A\underline{\hat{x}}_k + L\left(\underline{z}_k - H\underline{\hat{x}}_k\right) + B\underline{u}_k\right] \\ &= A\left(\underline{x}_k - \underline{\hat{x}}_k\right) - L\left(H\underline{x}_k - H\underline{\hat{x}}_k\right) \\ &= (A - LH)\underline{\tilde{x}}_k\end{aligned}$$

It is now apparent that in order that $\underline{\tilde{x}}_k$ go to zero with k for any $\underline{\tilde{x}}_0$, observer gain L must be selected so that $(A - LH)$ is stable. L can be chosen so that $\underline{\tilde{x}}_k \to 0$ if and only if (A, H) is detectable which is defined in the sequel.

(1). (A, H) is observable if the poles of $(A - LH)$ can be arbitrarily assigned by appropriate choice of the output injection matrix L.

(2). (A, H) is detectable if $(A - LH)$ can be made asymptotically stable by some matrix L. (If (A, H) is observable, then the pair is detectable; but the reverse is not necessarily true.)

(3). (A, B) is controllable (reachable) if the poles of (A-BK) can be arbitrarily assigned by appropriate choice of the feedback matrix K.

(4). (A, B) is stabilizable if (A-BK) can be made asymptotically stable by some matrix K. (If (A, B) is controllable, then (A, B) is stabilizable; but the reverse is not necessarily true.)

Observability and Controllability (continued)

Theorem (Observability): The n-state discrete linear time-invariant system

$$\underline{x}_k = A\underline{x}_{k-1} + B\underline{u}_{k-1}$$

$$\underline{y}_k = H\underline{x}_k$$

has the observability matrix Q defined by

$$Q = \begin{bmatrix} H \\ HA \\ \vdots \\ HA^{n-1} \end{bmatrix}.$$

The system is observable if and only if $\rho(Q) = n$.

Theorem (Controllability): The n-state discrete linear time-invariant system $\underline{x}_k = A\underline{x}_{k-1} + B\underline{u}_{k-1}$ has the controllability matrix P defined by

$$P = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}.$$

The system is controllable if and only if $\rho(P) = n$.

Chapter 2

Probability Theory

Probability

Probability Space

$\mathfrak{P} = \{S, A, P\}$

$S =$ sample space, e.g., $S = \{f_1, f_2, f_3, f_4, f_5, f_6\}$

$A =$ event space, $A \subset S$, $A = \{\phi, \{odd\}, \{even\}, S\}$

$P =$ probability assigned to events, e.g., $P[\phi] = 0$, $P[\{odd\}] = P[\{even\}] = 1/2$, $P[S] = 1$

Probability Axioms

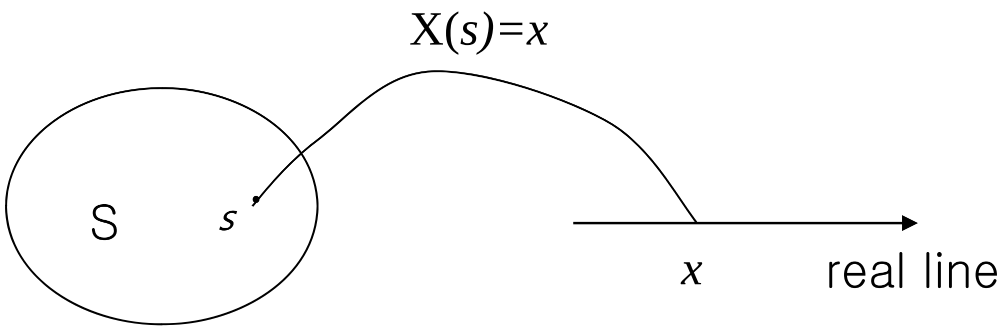
Axiom 1 For any event A, $P[A] \geq 0$.

Axiom 2 $P[S] = 1$.

Axiom 3 For any countable collection $A_1, A_2, \cdots$ of mutually exclusive events

$$P[A_1 \cup A_2 \cup \cdots] = P[A_1] + P[A_2] + \cdots.$$

Random Variables



Probability Distribution Function (PDF)

$$F_X(x) = P(X \le x)$$

$$F_X(x) \in [0,1]$$

$$F_X(-\infty) = 0$$

$$F_X(\infty) = 1$$

$$F_X(a) \le F_X(b) \text{ if } a \le b$$

$$P(a < X \le b) = F_X(b) - F_X(a)$$

$$F_X(x \mid A) = P\big((X \le x) \mid A\big) = \frac{P(X \le x, A)}{P(A)}$$

Random Variables (continued)

Probability Density Function (pdf):

$$f_X(x) = \frac{dF_X(x)}{dx}$$

$$F_X(x) = \int_{-\infty}^{x} f_X(z)\,dz$$

$$f_X(x) \geq 0$$

$$\int_{-\infty}^{\infty} f_X(x)\,dx = 1$$

$$P(a < X \leq b) = \int_{a}^{b} f_X(x)\,dx$$

$$f_X(x \mid A) = \frac{dF_X(x \mid A)}{dx}$$

Expected Value:

$$E(X) = \int_{-\infty}^{\infty} x f_X(x)\,dx$$

Variance:

$$\sigma_X^2 = E\left[\left(X - \bar{X}\right)^2\right] = \int_{-\infty}^{\infty} \left(x - \bar{X}\right)^2 f_X(x)\,dx$$

Random Variables (continued)

Uniform Random Variable:

$$f_X(x) = \frac{1}{b-a} \quad a \le x \le b$$

$$E[X] = \frac{a+b}{2}; \quad VAR[X] = \frac{(b-a)^2}{12}$$

Gaussian (Normal) Random Variable:

$$f_X(x) = \frac{e^{-\left(x-\bar{X}\right)^2/2\sigma_X^2}}{\sqrt{2\pi}\sigma_X} \quad -\infty < x < \infty, \sigma_X > 0$$

$$E[X] = \bar{X}; \quad VAR[X] = \sigma_X^2$$

Multiple Random Variables

Joint Probability Distribution Function:

$$F_{XY}(x,y) = P(X \le x, Y \le y); \quad F(x,y) \in [0,1]$$

$$F(x,-\infty) = F(-\infty,y) = 0; \quad F(\infty,\infty) = 1$$

$$F(a,c) \le F(b,d) \text{ if } a \le b \text{ and } c \le d$$

$$P(a < x \le b, c < y \le d) = F(b,d) + F(a,c) - F(a,d) - F(b,c)$$

$$F(x,\infty) = F(x); \; F(\infty,y) = F(y)$$

Joint Probability Density Function:

$$f_{XY}(x,y) = \frac{\partial^2 F_{XY}(x,y)}{\partial x \partial y}$$

$$F(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f(z_1,z_2)\, dz_1 dz_2$$

$$f(x,y) \ge 0; \quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, dx dy = 1$$

$$P(a < x \le b, c < y \le d) = \int_{c}^{d}\int_{a}^{b} f(x,y)\, dx dy$$

$$f(x) = \int_{-\infty}^{\infty} f(x,y)\, dy; \quad f(y) = \int_{-\infty}^{\infty} f(x,y)\, dx$$

Multiple Random Variables (continued)

Correlation Matrix:

$$R_{\underline{X}\,\underline{Y}}(\underline{x},\underline{y}) = E\left(\underline{X}\,\underline{Y}^T\right) = \begin{bmatrix} E\left(X_1Y_1\right) & \cdots & E\left(X_1Y_m\right) \\ \vdots & & \vdots \\ E\left(X_1Y_1\right) & \cdots & E\left(X_nY_m\right) \end{bmatrix}$$

Covariance Matrix:

$$C_{\underline{X}\,\underline{Y}}(\underline{x},\underline{y}) = E\left[\left(\underline{X}-\bar{\underline{X}}\right)\left(\underline{Y}-\bar{\underline{Y}}\right)^T\right] = E\left(\underline{X}\,\underline{Y}^T\right) - \bar{\underline{X}}\,\bar{\underline{Y}}^T$$

Gaussian Random Vector:

$$f_{\underline{X}}(\underline{x}) = \frac{1}{(2\pi)^{n/2}\left|C_{\underline{X}}\right|^{1/2}}\exp\left[-\frac{1}{2}\left(\underline{X}-\bar{\underline{X}}\right)^T C_{\underline{X}}^{-1}\left(\underline{X}-\bar{\underline{X}}\right)\right]$$

Statistical Independence:

$$P\left(X \le x, Y \le y\right) = P\left(X \le x\right)P\left(Y \le y\right)$$

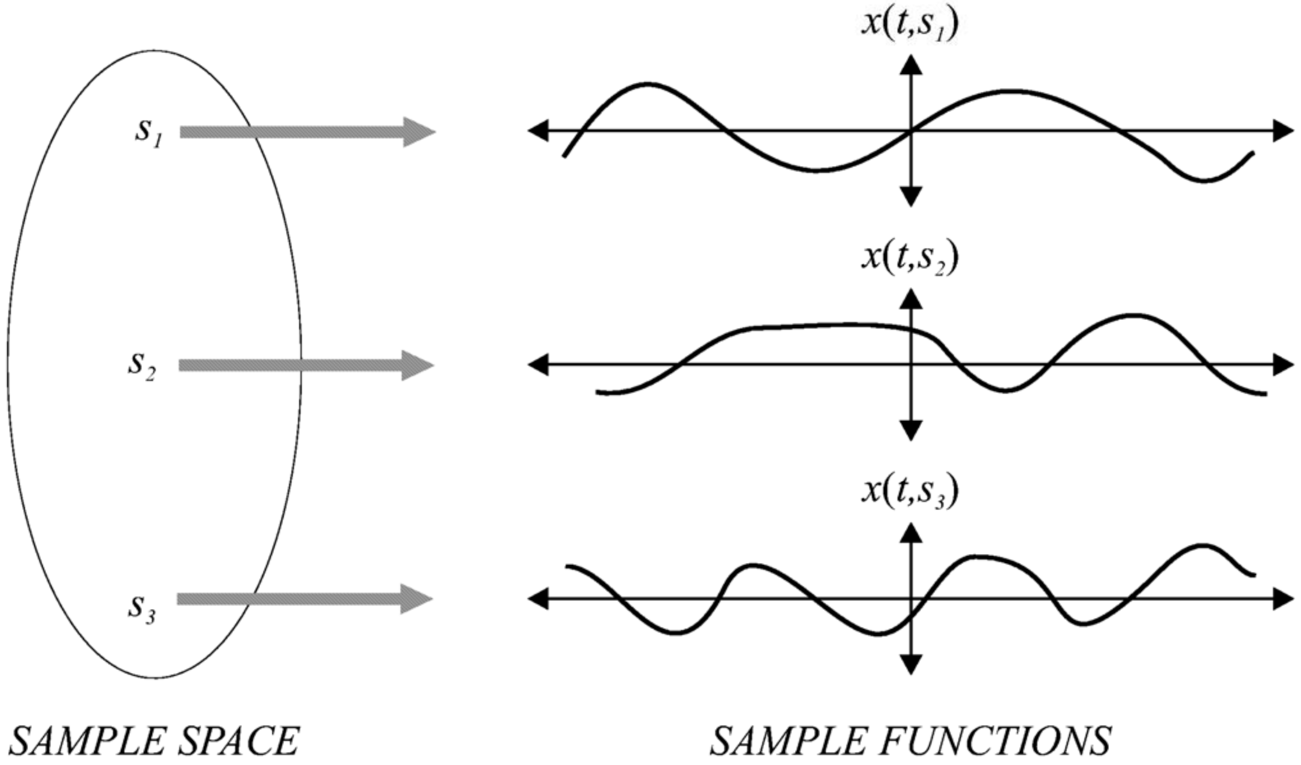
$$F_{XY}(x,y) = F_X(x)F_Y(y)$$

$$f_{XY}(x,y) = f_X(x)f_Y(y)$$

$$R_{XY} = E\left(XY\right) = E(X)E(Y) \quad \text{(uncorrelatedness)}$$

Stochastic Processes

Conceptual Representation of Stochastic Process



(1) Ensemble Average
(2) Time Average

Stochastic Processes (continued)

$X(t, s), t \in R^1, s \in S$

(1) t, s = fixed $\qquad X =$ a single number (an outcome of an experiment)

(2) t = variable $\qquad X =$ a time function

s = fixed

(3) t = fixed $\qquad X =$ a random variable

s = variable

(4) t = variable $\qquad X =$ a random process (a family of time functions)

s = variable

Stochastic Processes (continued)

Stationary Process:

A stochastic process $X(t)$ is stationary *if and only if for all sets of time instants* $t_1, \ldots, t_m$*, and any time difference* τ,

$$f_{X(t_1),\ldots,X(t_m)}(x_1, \ldots, x_m) = f_{X(t_1+\tau),\ldots,X(t_m+\tau)}(x_1, \ldots, x_m).$$

. . . .

A random sequence X_n is stationary *if and only if for any set of integer time instants* $n_1, \ldots, n_m$*, and integer time difference* k,

$$f_{X_{n_1},\ldots,X_{n_m}}(x_1, \ldots, x_m) = f_{X_{n_1+k},\ldots,X_{n_m+k}}(x_1, \ldots, x_m).$$

Stochastic Processes (continued)

Wide-Sense Stationary (WSS):

$X(t)$ *is a* wide sense stationary stochastic process *if and only if for all* t,

$$E[X(t)] = \mu_X, \quad \textit{and} \quad R_X(t,\tau) = R_X(0,\tau) = R_X(\tau).$$

. . . .

X_n *is a* wide sense stationary random sequence *if and only if for all* n,

$$E[X_n] = \mu_X, \quad \textit{and} \quad R_X[n,k] = R_X[0,k] = R_X[k].$$

Properties of WSS:

$$R_X(0) = E\left[X^2(t)\right]$$

$$R_X(-\tau) = R_X(\tau)$$

$$\left|R_X(\tau)\right| \le R_X(0)$$

White Noise and Colored Noise

Power Spectrum:

$$S_X(\omega) = \int_{-\infty}^{\infty} R_X(\tau) e^{-j\omega\tau} \, d\tau$$

$$R_X(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_X(\omega) e^{j\omega\tau} \, d\omega$$

White Noise:

$$S_X(\omega) = R_X(0) \quad \text{for all } \omega$$

$$R_X(\tau) = R_X(0)\delta(\tau)$$