5. Properties of Kalman Filtering

Orthogonality Principle

The Kalman filter is a special case of the mean-square estimator, that is, the Kalman estimates are the conditional mean estimates, viz.,

$$\underline{\hat{x}}_{KF} = E\{\underline{x} \mid \underline{z}\}. \tag{5-1}$$

Now, we want to show that the Kalman filter satisfies the orthogonality principle. Let RVs x and z be jointly distributed. Then for any function $g(\cdot)$,

$$E\{g(z)(x - E\{x \mid z\})^T\} = 0.$$

That is, any function of z is orthogonal to x once the conditional mean has been subtracted out. To show this, write

$$\begin{aligned} E\{g(z)(x - E\{x \mid z\})^T\} &= E\{g(z)x^T\} - E\{g(z)E\{x^T \mid z\}\} \\ &= E\{g(z)x^T\} - E\{E\{g(z)x^T \mid z\}\} \\ &= E\{g(z)x^T\} - E\{g(z)x^T\} = 0 \end{aligned}$$

We used the fact that

$$g(z)E\{x^T \mid z\} = E(g(z)x^T \mid z\}$$

for any function $g(\cdot)$ since $g(z)$ is deterministic if z is fixed. The orthogonality principle says that the RV

$$\tilde{x} = x - E\{x \mid z\},$$

which is the estimation error, is orthogonal to all other RVs g(z). It has the following implication for estimation theory. If $g(\cdot)$ is any function (or filter), then

$$E\left\|x - E\{x \mid z\}\right\| \leq E\left\|x - g(z)\right\|,$$

$$\left(\begin{aligned} \text{Proof: } E\left(x - g(z)\right)^2 &= E\left(x - E\{x \mid z\} + E\{x \mid z\} - g(z)\right)^2 \\ &= E\left(x - E\{x \mid z\}\right)^2 + E\left(E\{x \mid z\} - g(z)\right)^2 \text{ (orthogonality principle applied)} \\ &\geq E\left(x - E\{x \mid z\}\right)^2 \end{aligned}\right)$$

where $\|\cdot\|$ denotes the Euclidean norm. Thus no other function of z provides a “closer approximation to x” in a probabilistic sense than does the Kalman filter which estimates x as $E\{x \mid z\}$.

White Gaussian Residual

The residual is defined by

$$r_k = z_k - H_k \hat{x}_k(-), \tag{5-2}$$

which can be expressed

$$\begin{aligned} r_k &= H_k x_k + v_k - H_k \hat{x}_k(-) \\ &= H_k [\Phi_{k-1} x_{k-1} + \int_{t_{k-1}}^{t_k} \Phi(t_{k-1}, \tau) G(\tau)\, d\beta(\tau)] + v_k - H_k \Phi_{k-1} \hat{x}_{k-1}(+) \\ &= H_k \Phi_{k-1} \tilde{x}_{k-1}(+) + H_k \int_{t_{k-1}}^{t_k} \Phi(t_{k-1}, \tau) G(\tau)\, d\beta(\tau) + v_k \end{aligned} \tag{5-3}$$

where $\beta(\tau)$ is the Brownian motion, i.e., $d\beta(\tau) = w(\tau)\, d\tau$.

Let's investigate the right-hand side terms in Eq. (5-3). Notice that the unconditional density

$$
\begin{aligned}
f_{\tilde{x}_{k-1}(+)}(\xi) &= \int_{-\infty}^{\infty} f_{\tilde{x}_{k-1}(+)z_{k-1}}(\xi, z_{k-1})\, dz_{k-1} \\
&= \int_{-\infty}^{\infty} f_{\tilde{x}_{k-1}(+)|z_{k-1}}(\xi) f_{z_{k-1}}(z_{k-1})\, dz_{k-1}
\end{aligned}
$$

$$
\text{(Note that } f_{\tilde{x}_{k-1}(+)|z_{k-1}}(\xi \mid z_{k-1}) = \left[(2\pi)^{n/2} \left|P_{k-1}(+)\right|^{1/2}\right]^{-1} \exp\left\{-\frac{1}{2}\xi^T P_{k-1}^{-1}(+)\xi\right\}
$$

$$
= f_{\tilde{x}_{k-1}(+)|z_{k-1}}(\xi) \text{ since } P_{k-1}(+) \text{ is independent of } z_{k-1}.)
$$

$$
\begin{aligned}
&= f_{\tilde{x}_{k-1}(+)|z_{k-1}}(\xi) \int_{-\infty}^{\infty} f_{z_{k-1}}(z_{k-1})\, dz_{k-1} \\
&= f_{\tilde{x}_{k-1}(+)|z_{k-1}}(\xi)
\end{aligned}
$$

This equation indicates that $\tilde{x}_{k-1}$ is independent of z_{k-1}. The second term of the right-hand side of Eq. (5-3) is also independent of z_{k-1} since it involves only transition matrix and process noise. Finally, v_k is independent of z_{k-1} because it is white. Therefore, we can say that r_k is independent of z_{k-1}. On the other hand, we can notice that r_k is a linear function of z_k. Therefore, we can conclude that r_k is a white sequence.

Eq. (5-3) also tells us that since z_k is Gaussian, r_k is Gaussian with the following mean and variance.

$$E\{r_k\} = H_k \Phi_{k-1} E\{\tilde{x}_{k-1}(+)\} + H_k E\{I(\cdot,\cdot)\} + E\{v_k\} = 0 \tag{5-4}$$

$$\begin{aligned} E\{r_k r_k{}^T\} &= E\{(H_k[x_k - \hat{x}_k(-)] + v_k)(H_k[x_k - \hat{x}_k(-)] + v_k)^T\} \\ &= H_k P_k(-) H_k{}^T + R_k \end{aligned} \tag{5-5}$$

This verifies that the residual r_k is a white Gaussian sequence. This property is useful:

(1) to verify a design of Kalman filter,

(2) to detect a sensor failure or bad data.

Stability

Consider the following time-invariant system and the deterministic asymptotic estimation

$$\underline{x}_{k+1} = A\underline{x}_k + B\underline{u}_k \tag{5-6a}$$
$$\underline{z}_k = H\underline{x}_k \tag{5-6b}$$

where state $\underline{x}_k \in R^n$, control input $\underline{u}_k \in R^m$, output $\underline{z}_k \in R^p$; and *A*, *B*, and *H* are known constant matrices of appropriate dimension. All variables are deterministic, so that if initial state $\underline{x}_0$ is known then Eq. (5-6) can be solved exactly for $\underline{x}_k$, $\underline{z}_k$ for $k \geq 0$.

Deterministic asymptotic estimation problem: Design an estimator whose output $\underline{\hat{x}}_k$ converges with *k* to the actual state $\underline{x}_k$ of Eq. (5-6) when the initial state $\underline{x}_0$ is unknown, but $\underline{u}_k$ and $\underline{z}_k$ are given exactly.

An estimator of observer which solves this problem has the form

$$\underline{\hat{x}}_{k+1} = A\underline{\hat{x}}_k + L\left(\underline{z}_k - H\underline{\hat{x}}_k\right) + B\underline{u}_k, \tag{5-7}$$

as shown in Fig. 5-1.

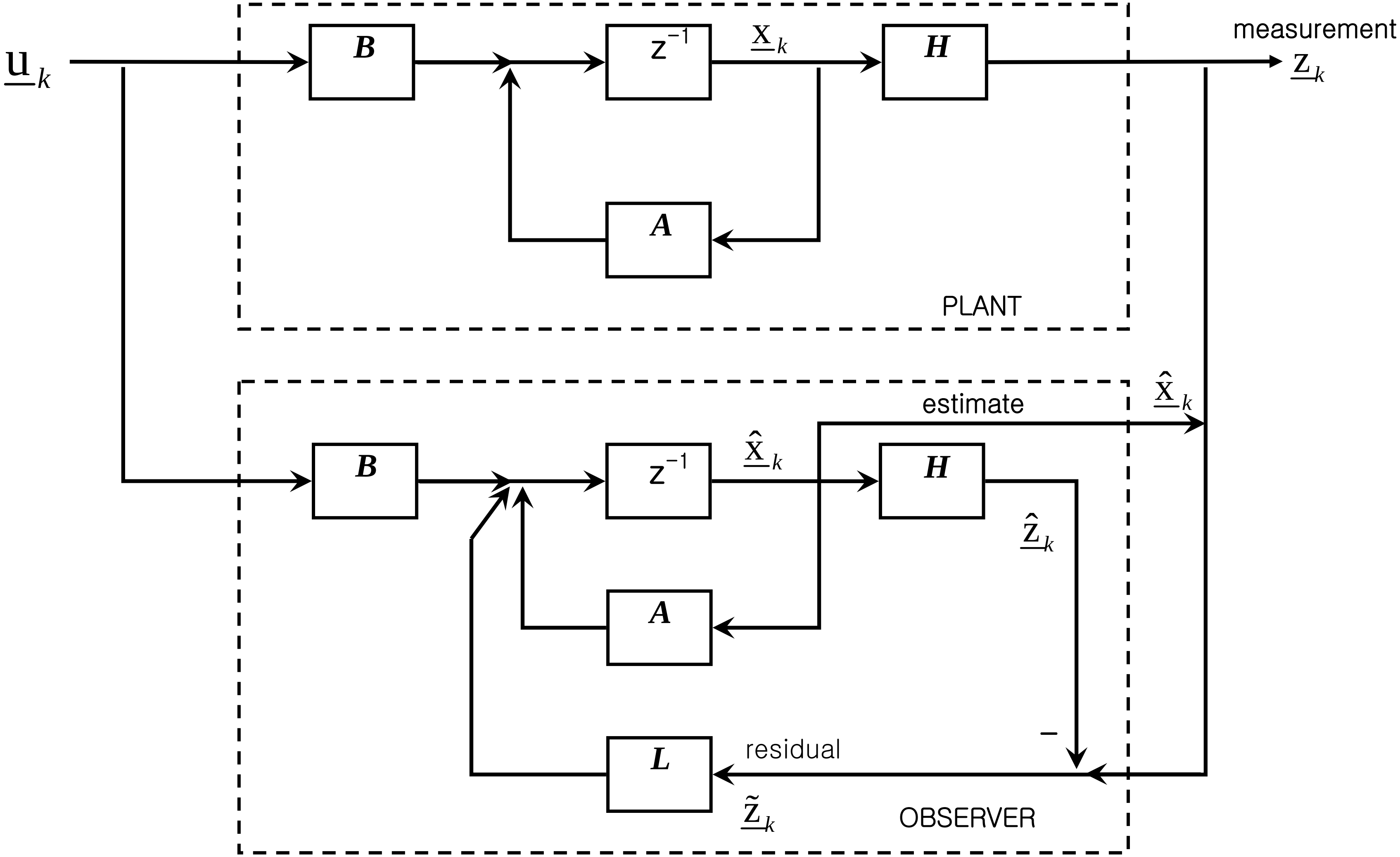

Figure 5-1 State Observer

To Choose L in Eq. (5-7) so that the estimation error $\underline{\tilde{x}}_k = \underline{x}_k - \underline{\hat{x}}_k$ goes to zero with k for all $\underline{x}_0$, it is necessary to examine the dynamics of $\underline{\tilde{x}}_k$. Write

$$
\begin{aligned}
\underline{\tilde{x}}_{k+1} &= \underline{x}_{k+1} - \underline{\hat{x}}_{k+1} \\
&= A\underline{x}_k + B\underline{u}_k - \left[A\underline{\hat{x}}_k + L\left(\underline{z}_k - H\underline{\hat{x}}_k\right) + B\underline{u}_k \right] \\
&= A\left(\underline{x}_k - \underline{\hat{x}}_k\right) - L\left(H\underline{x}_k - H\underline{\hat{x}}_k\right) \\
&= (A - LH)\underline{\tilde{x}}_k
\end{aligned}
$$

It is now apparent that in order that $\underline{\tilde{x}}_k$ go to zero with k for any $\underline{\tilde{x}}_0$, observer gain L must be selected so that $(A - LH)$ is stable. L can be chosen so that $\underline{\tilde{x}}_k \to 0$ if and only if (A, H) is detectable which is defined in the sequel.

(1). *(A, H)* is observable if the poles of *(A-LH)* can be arbitrarily assigned by appropriate choice of the output injection matrix *L*.

(2). *(A, H)* is detectable if *(A-LH)* can be made asymptotically stable by some matrix *L*. (If *(A, H)* is observable, then the pair is detectable; but the reverse is not necessarily true.)

(3). *(A, B)* is reachable if the poles of *(A-BK)* can be arbitrarily assigned by appropriate choice of the feedback matrix *K*.

(4). *(A, B)* is stabilizable if *(A-BK)* can be made asymptotically stable by some matrix *K*. (If *(A, B)* is reachable, then *(A, B)* is stabilizable; but the reverse is not necessarily true.)

Now, consider the following stochastic system

$$\underline{\mathrm{x}}_{k+1} = A\underline{\mathrm{x}}_k + B\underline{\mathrm{u}}_k + G\underline{\mathrm{w}}_k,\ \underline{\mathrm{x}}_0 \sim N(\underline{\overline{\mathrm{x}}}_0, P_0),\ \underline{\mathrm{w}}_k \sim N(0, Q)$$
$$\underline{\mathrm{z}}_k = H\underline{\mathrm{x}}_k + \underline{\mathrm{v}}_k,\ \underline{\mathrm{v}}_k \sim N(0, R)\,.$$

The stability of the Kalman filter designed for this system is presented by two theorems.

Theorem 5-1. (Sufficient condition for a Kalman filter to be stable)

Let *(A, H)* be detectable. Then for every choice of P_0 there is a bounded limiting solution *P* to

$$P_{k+1}(-) = A[P_k(-) - P_k(-)H^T(HP_k(-)H^T + R)^{-1}HP_k(-)]A^T + GQG^T. \qquad (5\text{-}8)$$

Furthermore, *P* is a positive semidefinite solution to the algebraic Riccati equation

$$P = A[P - PH^T(HPH^T + R)^{-1}HP]A^T + GQG^T. \qquad (5\text{-}9)$$

(Proof)

(A, H) detectable means that there exists an *L* that makes *(A-LH)* asymptotically stable.

Construct a suboptimal $\hat{x}_k$ using the *L*

$\widehat{x}_{k+1}^L = (A - LH)\hat{x}_k^L + Bu_k + Lz_k$. (This is a state observer.)

The estimation error in this case is given by

$$
\begin{aligned}
\tilde{x}_{k+1}^{L} &= x_{k+1} - \hat{x}_{k+1}^{L} \\
&= Ax_k + Bu_k + Gw_k - (A - LH)\hat{x}_{k+1}^{L} - Bu_k - Lz_k \\
&= Ax_k + Bu_k + Gw_k - (A - LH)\hat{x}_{k+1}^{L} - Bu_k - L\left[Hx_k + v_k\right] \\
&= (A - LH)\tilde{x}_k + Gw_k - Lv_k
\end{aligned}
$$

The estimation error covariance is

$$S_{k+1} = (A - LH)S_k(A - LH)^T + LRL^T + GQG^T .$$

If *(A-LH)* is asymptotically stable, S_k has a finite steady-state value according to the Lyapunov theorem.

Since the Kalman is optimal, $P_k(-) < S_k$, where $P_k(-)$ satisfies Eq. (5-8).

From the above rationale, if *(A, H)* is detectable,

$$\lim_{k\to\infty} P_{k+1}(-) = \lim_{k\to\infty} P_k(-) = P$$

and P is the solution of the algebraic Riccati equation, Eq. (5-9).

Theorem 5-2: (Necessary and sufficient condition for a Kalman filter to be stable)

Let $\sqrt{Q}$ be a square root of the process noise covariance so that $Q = \sqrt{Q}\sqrt{Q}^T \geq 0$, and let the measurement noise have $R>0$. Suppose $(A, G\sqrt{Q})$ is reachable. Then (A, H) is detectable if and only if

a. There is a unique positive definite limiting solution P to Eq. (5-8) which is independent of P_0. Furthermore, P is the unique positive definite solution to the algebraic Riccati equation.

b. The steady-state error system defined by Kalman filter, viz.,

$$\tilde{x}_{k+1}(-) = A(I - K_k H)\tilde{x}_k(-) + Gw_k - AK_k v_k \tag{5-10}$$

with steady-state Kalman gain

$$K = PH^T(HPH^T + R)^{-1} \tag{5-11}$$

is asymptotically stable.

(Proof)

Define D by $R = DD^T$.

$R>0$ implies that $|D| \neq 0$. Then, there exists an M that holds $H=DM$.

$(A, G\sqrt{Q})$ reachable implies

$$rank[zI - A \quad G\sqrt{Q}] = n. \tag{5-12}$$

(Suppose that $rank[zI - A \quad G\sqrt{Q}] \neq n$, then there exists an n-dimensional non-zero vector q such that $q[zI - A \quad G\sqrt{Q}] = 0$, that is, $qA=zq$ and $qG\sqrt{Q} = 0$. These equations may be expanded to

$$qAG\sqrt{Q} = zqG\sqrt{Q} = 0$$

$$qA^2G\sqrt{Q} = zqAG\sqrt{Q} = z^2qG\sqrt{Q} = 0$$

.

.

.

resulting in

$$qC(A, G\sqrt{Q}) = q[G\sqrt{Q},\ AG\sqrt{Q}, \cdots, A^{n-1}G\sqrt{Q}] = 0.$$

This contradicts to the assumption that $(A, G\sqrt{Q})$ is reachable.)

Eq. (5-12) may be expanded to

$$\begin{aligned} n &= rank[zI - A, G\sqrt{Q}, LD] \\ &= rank[zI - A, G\sqrt{Q}, LD]\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ M & 0 & I \end{bmatrix} \\ &= rank[zI - A + LH, G\sqrt{Q}, LD]. \end{aligned}$$

This implies that $((A - LH), [G\sqrt{Q}, LD])$ is reachable.

To prove necessity, let gain L define a suboptimal estimate $\hat{x}_k^L$ as in the proof of Theorem 5-1, with *(A-LH)* asymptotically stable. By Theorem 5-1, *(A, H)* detectable implies $P_k(-) \to P$ with P bounded and at least positive semidefinite. But $E\|\tilde{x}_k(-)\|^2 \leq E\|\tilde{x}_k^L\|^2$ for all L because of the optimality of the Kalman filter. Hence, system *A(I-KH)* is also asymptotically stable with

$$K = PH^T (HPH^T + R)^{-1}.$$

Define $L' = AK$. Then we can write the algebraic Riccati equation as

$$\begin{aligned} P &= (A - L'H)P(A - L'H)^T + L'DD^T (L')^T + G\sqrt{Q}\sqrt{Q}^T G^T \\ &= (A - L'H)P(A - L'H)^T + [G\sqrt{Q} \quad L'D][G\sqrt{Q} \quad L'D]^T. \end{aligned} \tag{5-13}$$

We know that P is an at least positive semidefinite solution of Eq. (5-13). But Eq. (5-13) is also a Lyapunov equation, for which we know that $(A - L'H)$ is stable (according to the Lyapunov Theorem) and also that the pair $((A - L'H), [G\sqrt{Q}, L'D])$ is reachable (shown above). So the solution P is also unique and positive definite, and the gain $K = PH^T (HPH^T + R)^{-1}$ is uniquely defined.

(**Lyapunov Theorem**: (Ref. Panos J. Antsaklis and Anthony N. Michel, *Linear Systems*)
If there is a positive definite and symmetric matrix X and a positive definite and symmetric matrix Q satisfying ,

$A^T XA - X + Q = 0$, (Lyapunov Equation)

then the matrix A is stable. Conversely, if A is stable, then, given any symmetric matrix Q, the above Lyapunov equation has a unique solution, and if Q is positive definite then X is positive definite.)

To show sufficiency, note that if *A(I-KH)* is asymptotically stable, there is an L=AK for which the system *(A-LH)* is stable and hence the pair *(A, H)* is detectable. Q.E.D.

In summary, Theorem 5-2 says that if the state is *reachable by the process noise*, so that every mode is excited by w_k, then the Kalman filter is asymptotically stable if *(A, H) is detectable*. Thus, we can *guarantee a stable filter* by selecting the measurement matrix *H* correctly and ensuring that the process model is sufficiently corrupted by noise!