# Functions

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# Functions

- A function can be understood as a machine that takes an input and produces an output
  - consistency: every time we put a specific input, the same answer emerges.
- A function can be understood as a "rule" or "mechanism" that transforms one quantity into another.
  - $f(x) = x^2+4$
  - $g(x) = |x|$

# Function Definition and Notation

- A function is a relation with special property.
- Definition 23.1 (Function): A relation *f* is called a function provided $(a,b) \in f$ and $(a,c) \in f$ imply $b=c$.
- Examples
  - $f=\{(1,2), (2,3), (3,1), (4,7)\}$
  - $g=\{(1,2), (1,3), (4,7)\}$
- Definition 23.3 (Function notation): Let *f* be a function and let *a* be an object. The notation $f(a)$ is defined provided there exists an object *b* such that $(a,b) \in f$ . In this case, $f(a)$ equals *b*. Otherwise [there is no ordered pair of the form $(a,\_) \in f$ ], the notation $f(a)$ is undefined. The symbol $f(a)$ are pronounced "*f* of *a*."
- Example 23.4: Express the integer function $f(x)=x^2$ as a set of ordered pairs
  - $f = \{..., (-3,9), (-2,4), (-1,1), (0,0), (1,1), (2,4), (3,9), ...\}$

    $$f = \{(x,y) : x, y \in Z, y = x^2\}$$

# Domain and Image

- Definition 23.5 (Domain, image) Let *f* be a function. The set of all possible first elements of the ordered pairs in *f* is called the *domain* of *f* and is denoted dom *f*. The set of all possible second elements of the ordered pairs in *f* is called the *image* of *f* and is denoted im *f*.

$$\text{dom } f = \{a : \exists b, (a,b) \in f\} \text{ and } \text{im } f = \{b : \exists a, (a,b) \in f\}$$

- Example 23.6: Let *f*={(1,2),(2,3),(3,1),(4,7)}.
  - dom *f* = {1,2,3,4} and im *f* ={1,2,3,7}
- Example 23.7: Let *f* be the function

$$f = \{(x,y) : x, y \in Z, y = x^2\}$$

  - the domain of *f* is the set of all integers, and the image of *f* is the set of all perfect squares

# $f: A \rightarrow B$

- Definition 23.8 ($f: A \rightarrow B$) Let $f$ be a function and let $A$ and $B$ be sets. We say that $f$ is a function from $A$ to $B$ provided dom $f = A$ and $\operatorname{im} f \subseteq B$. In this case, we write $f: A \rightarrow B$. We also say that $f$ is a *mapping from A to B.*

- Example 23.9: Consider the sine function

$$\sin : R \rightarrow R$$
$$\sin : R \rightarrow [-1,1]$$

# Counting Functions

- How many functions from *A* to *B* are there?
- Proposition 23.10: Let *A* and *B* be finite sets with $|A|=a$ and $|B|=b$. The number of functions from *A* to *B* is $b^a$.
- Example: Let A={1,2,3} and B={4,5}. Find all functions f:A→B.

# Inverse Functions

- A function is a special type of relation.
- We defined the inverse of a relation *R*, denoted by $R^{-1}$, to be the relation formed from *R* by reversing all its ordered pairs.
- Since a function *f* is a relation, we may also consider $f^{-1}$.
- If *f* is a function from *A* to *B*, is $f^{-1}$ a function from *B* to *A*?
- What are the conditions that make the above true?

# One-to-one

- Definition 23.13 (One-to-one) A function f is called one-to-one provided that, whenever $(x,b),(y,b) \in f$, we must have x=y. In other words, if

$$\text{if } x \neq y, \text{ then } f(x) \neq f(y).$$

- M-to-N
  - One-to-many: f is not a function
  - Many-to-one: $f^{-1}$ is not a function
  - One-to-one: both f and $f^{-1}$ are functions
- Proposition 23.14: Let $f$ be a function. The inverse relation $f^{-1}$ is a function if and only if $f$ is one-to-one.
  - Proof?
- Proposition 23.15: Let $f$ be a function and suppose $f^{-1}$ is also a function. Then dom $f$ = im $f^{-1}$ and im $f$ = dom $f^{-1}$.
  - Proof?

# Proof Template 20 (One-to-one)

- Proving a function is one-to-one
  - Direct method: Suppose $f(x)=f(y)$..... Therefore $x=y$. Therefore $f$ is one-to-one.
  - Contrapositive method: Suppose $x \neq y$ .... Therefore $f(x) \neq f(y)$ . Therefore $f$ is one-to-one.
  - Contradiction method: Suppose $f(x) = f(y)$ but $x \neq y$ .... "contradiction". Therefore $f$ is one-to-one.
- Example 23.16: Let $f{:}Z \rightarrow Z$ by $f(x)=3x+4$. Prove that $f$ is one-to-one.
  - Proof: Suppose f(x) = f(y). Then 3x+4=3y+4. Subtracting 4 from both sides gives 3x=3y. Dividing both sides by 3 gives x=y. Therefore f is one-to-one.
- Example 23.17: Let $f{:}Z \rightarrow Z$ by $f(x)=x^2$. Prove that $f$ is not one-to-one.
  - Proof: Notice that $f(3)=f(-3)=9$, but 3 is not -3. Therefore $f$ is not one-to-one.

# Onto

- For the inverse of a function also to be a function, it is necessary and sufficient that the function be one-to-one.
- What is the condition that makes the inverse of $f{:}A \rightarrow B$ is a function from $B$ to $A$?
- Definition 23.18 (Onto) Let $f{:}A \rightarrow B$. We say that $f$ is onto $B$ provided that for every $b \in B$ there is an $a \in A$ so that $f(a)=b$. In other words, im $f = B$.
- Examples: Let A={1,2,3,4,5,6} and B={7,8,9,10}
  - f={(1,7),(2,7),(3,8),(4,9),(5,9),(6,10)}
  - g={(1,7),(2,7),(3,7),(4,9),(5,9),(6,10)}

# Proof Template 21 (Onto)

- Proving a function is onto
  - Direct method: Let *b* be an arbitrary element of *B*. Explain how to find/construct an element $a \in A$ such that *f*(*a*)=*b*. Therefore *f* is onto B.
  - Set method: Show that the sets *B* and im *f* are equal.
- Example 23.20: Let $f: Q \rightarrow Q$ by $f(x)=3x+4$. Prove that *f* is onto *Q*.
  - Proof. ...

# One-to-one and Onto

- Theorem 23.21: Let $A$ and $B$ be sets and let $f:A \rightarrow B$. The inverse relation $f^{-1}$ is a function from $B$ to $A$ if and only if $f$ is one-to-one and onto $B$
  - Proof: Let $f:A \rightarrow B$.
  - => Suppose $f$ is one-to-one and onto $B$.
    - We need to prove that $f^{-1}:B \rightarrow A$.
    - Since $f$ is one-to-one, $f^{-1}$ is a function
    - Since $f$ is onto $B$, im $f = B$. Thus, dom $f^{-1} = B$
    - Since the domain of $f$ is $A$, im $f^{-1} = A$.
    - Therefore $f^{-1}: B \rightarrow A$
  - <= Suppose $f^{-1}:B \rightarrow A$. Since $f^{-1}$ is a function, $f$ is one-to-one. Since im $f$ = dom $f^{-1}$, we see that $f$ is onto $B$.
- Definition 23.22: Let $f:A \rightarrow B$. We call f a bijection provided it is both one-to-one and onto B.
- Example: Let A be the set of even integers and let B be the set of odd integers. The function $f:A \rightarrow B$ defined by $f(x)=x+1$ is a bijection.
  - Proof???

# Counting functions

- How many functions f:A→B are one-to-one? How many are onto?
- Proposition 23.24 (Pigeonhole Principle) Let $A$ and $B$ be finite sets and let $f:A\to B$. If $|A|>|B|$, then $f$ is not one-to-one. If $|A|<|B|$, then $f$ is not onto.
- Proposition 23.25 Let $A$ and $B$ be finite sets and let $f:A\to B$. If $f$ is a bijection, then $|A|=|B|$
- Let $A$ and $B$ be finite sets with $|A|=a$ and $|B|=b$.
  - The number of functions from $A$ to $B$: # of length-$a$ lists made from $b$ elements allowing repetition=$b^a$
  - If $a <= b$, the number of one-to-one functions: # of length-$a$ lists made from $b$ elements without allowing repetition=$(b)_a$
  - If $a>=b$, the number of onto functions: # of length-$a$ lists made from $b$ elements allowing repetition and at least once usage of all $b$ elements=

$$\sum_{j=0}^{b} (-1)^j \binom{b}{j} (b-j)^a$$

  - If $a=b$, the number of bijections: $a!$

# Pigeonhole Principle

- Proposition 23.24 (Pigeonhole Principle) Let $A$ and $B$ be finite sets and let $f: A \rightarrow B$. If $|A|>|B|$, then $f$ is not one-to-one. If $|A|<|B|$, then $f$ is not onto.
- What does it have to do with pigeons?
- How this principle can be used?

# Pigeonhole Principle (Examples)

- Proposition 24.1: Let $n \in N$. Then there exist positive integers a and b, with $a \neq b$, such that $n^a$ - $n^b$ is divisible by 10.
  - Proof: Consider the 11 natural numbers $n^1$, $n^2$, $n^3$, ... $n^{11}$. The ones digits of these numbers take on values in the set {0, 1, ,2, ..., 9}. Since there are only 10 possible ones digits, and we have 11 different numbers, two of these numbers (say $n^a$ and $n^b$) must have the same ones digit. Therefore, $n^a$ - $n^b$ is divisible by 10.
- Definition: A point (x,y) whose coordinates are both integers is called a lattice point.
- Proposition 24.2: Given five distinct lattice points in the plane, at least one of the line segments determined by these points has a lattice point as its midpoint.
  - Proof Hints: Four possible parity types (even, even), (even, odd), (odd, even), and (odd, odd) but five lattice points → by Pigeonhole Principle
  - Mid point of (a,b), (c,d) is ((a+c)/2, (b+d)/2)

# Cantor's Theorem

- Is it possible to find bijections between infinite sets?
- Example: a bijection from N to Z

$$f : N \to Z,\ f(n) = \begin{cases} -n/2 & \text{if } n \text{ is even and} \\ (n+1)/2 & \text{if } n \text{ is odd} \end{cases}$$

- Two infinite sets have the same size provided there is a bijection between them
- Do all infinite sets have the same size?
- Theorem 24.4 (Cantor) Let $A$ be a set. If $f{:}A \to 2^A$, then $f$ is not onto.
  - Easy for finite set A
  - How about infinite set A?
- Implication of Cantor's Theorem: $|Z| \neq |2^Z|$

# Composition

- Just like +, * operations for combining integers, there is a natural operation for combining functions
- Definition 25.1 (Composition of functions) Let $A$, $B$, and $C$ be sets and let $f:A \rightarrow B$ and $g:B \rightarrow C$. The function $g \circ f$ is a function from $A$ to $C$ defined by

$$(g \circ f)(a) = g[f(a)]$$

  where $a \in A$. The function $g \circ f$ is called the composition of $g$ and $f$.
- dom $g \circ f$ = dom $f$
- In order for $g \circ f$ to make sense, every output of f must be an acceptable input to g. This holds. Why?
- What about $f \circ g$ ?

# Properties of Composition

- It is possible that $g \circ f$ and $f \circ g$ both make sense.
- However, $g \circ f$ and $f \circ g$ are generally not equal.
- The function composition does not satisfy the commutative property.
- It satisfies the associative property (Proof?)

# Proof Template 22
# (Equality of Functions)

- To prove *f* = *g*, do the following
  - Prove that dom *f* = dom *g*.
  - Prove that for every *x* in their common domain, *f*(*x*)=*g*(*x*).
- Proposition 25.6: Let A, B, C, and D be sets and let f:A→B, g:B→C, and h:C→D. Then

$$h \circ (g \circ f) = (h \circ g) \circ f$$

- Proof: ???
  - domains of both sides are the same.
  - For any a, both side functions produce the same value.

# Identity Function

- Definition 25.7 (Identity function) Let *A* be a set. The *identity function on A* is the function $\mathrm{id}_A$ whose domain is A, and for all $a \in A$, $\mathrm{id}_A(a)=a$. In other words,

$$\mathrm{id}_A = \{(a,a) : a \in A\}$$

- Proposition 25.8: Let A and B be sets, Let f:A→B. Then

$$f \circ \mathrm{id}_A = \mathrm{id}_B \circ f = f.$$

  - Proof???
- Proposition 25.9: Let A and B be sets and suppose f:A→B is one-to-one and onto. Then

$$f \circ f^{-1} = \mathrm{id}_B \text{ and } f^{-1} \circ f = \mathrm{id}_A.$$

  - Proof???

# Homework

- 23.1, 23.4, 23.7, 23.9
- 24.1, 24.2
- 25.1, 25.2, 25.7