Engineering Mathematics I
- Chapter 2. 2nd-Order Linear ODEs

민기복

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1. First-Order ODEs
Summary (1)

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- Separation of variables & Reduction to separable form
  - Separation of variables

$$g(y)y' = f(x)$$

$$g(y)y' = f(x) \quad \Rightarrow \quad \int g(y)\,dy = \int f(x)\,dx + c$$

  - Extended Method: Reduction to separable form

$$y' = f\left(\frac{y}{x}\right)$$

$$y' = f\left(\frac{y}{x}\right) \;\Rightarrow\; u'x + u = f(u) \;\Rightarrow\; \frac{du}{f(u) - u} = \frac{dx}{x}$$

1. First-Order ODEs
Summary (2)

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- Exact differential equation

$$M(x,y)\,dx + N(x,y)\,dy = 0 \quad \leftarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$M(x,y) = \frac{\partial u}{\partial x} \quad \Rightarrow \quad u(x,y) = \int M(x,y)\,dx + k(y) \quad \Rightarrow \quad \frac{\partial u}{\partial y} = N(x,y) \quad \Rightarrow \quad \frac{dk}{dy} \;\;\&\;\; k(y)$$

  - Non exact differential equation (finding integrating factors)

$$Pdx + Qdy = 0 \qquad FPdx + FQdy = 0$$

$$F(x) = \exp\left(\int R(x)\,dx\right), \; where \; R(x) = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$$

1. First-Order ODEs
Summary (3)

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- Linear ODEs $y' + p(x)y = r(x)$
  - Homogeneous ODEs

$$y' + p(x)y = 0 \quad \Rightarrow \quad y = ce^{-\int p(x)dx}$$

  - Nonhomgeneous ODEs

$$y' + p(x)y = r(x) \quad \Rightarrow \quad (py - r)dx + dy = 0$$

$$y = e^{-h}\left[\int e^{h} r dx + c\right], \quad where = h = \int p dx$$

  - Bernoulli Equation (reduction to linear ODEs)

$$y' + p(x)y = g(x)y^{a} \quad (a \neq 0 \ \& \ a \neq 1)$$

$$u' + (1-a)pu = (1-a)g \leftarrow u(x) = [y(x)]^{1-a}$$

1. 2nd-Order Linear ODEs
Introduction

- ODEs
  - Linear ODEs
  - Nonlinear ODEs
- Linear ODEs of the 2nd order
  - The most important ODEs
  - Transition to higher order Equations is immediate

Chapter 2. 2nd-Order Linear ODEs
Introduction

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Homogeneous Linear ODEs of 2nd order

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- Linear ODEs of 2nd order – Standard Form

$$y''+p(x)y'+q(x)y=r(x)$$

  - Homogeneous: $r(x)\equiv 0$

    Ex)

    $$xy''+y'+xy=0 \qquad y''+\frac{1}{x}y'+y=0$$

  - Nonhomogeneous: $r(x)\neq 0$

    Ex)

    $$y''+25y=e^{-x}\cos x$$

    $$y''y+(y')^2=0 \quad ?$$

Homogeneous Linear ODEs of 2nd order
Superposition Principle

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Fundamental Theorem for the Homogeneous Linear ODE (2)

For a homogeneous linear ODE (2), *any linear combination of two solutions on an open interval I is again a solution of* (2) *on I. In particular, for such an equation, sums and constant multiples of solutions are again solutions.*

– If y_1 and y_2 are solutions of Homogeneous Linear ODE

$$y''+p(x)y'+q(x)y=0$$

$$y=c_1y_1 \qquad y=c_2y_2$$

$$y=c_1y_1+c_2y_2$$

– Ex1) $y''+y=0$

– Ex2) $y''+y=1$ $\qquad y_1=1+\cos x \qquad y_2=1+\sin x$

– Ex3) $y''y-xy'=0$ $\qquad y_1=x^2 \qquad y_2=1$

Homogeneous Linear ODEs of 2nd order
Initial Value Problem (IVP)

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$$y'' + p(x)y' + q(x)y = 0$$

– Two Initial Condition for IVP

$$y(x_0) = K_0 \qquad y'(x_0) = K_1$$

– General solution $\quad y = c_1 y_1 + c_2 y_2$

– Two arbitrary constants, C1 and C2 can be obtained from two initial conditions.

Homogeneous Linear ODEs of 2nd order

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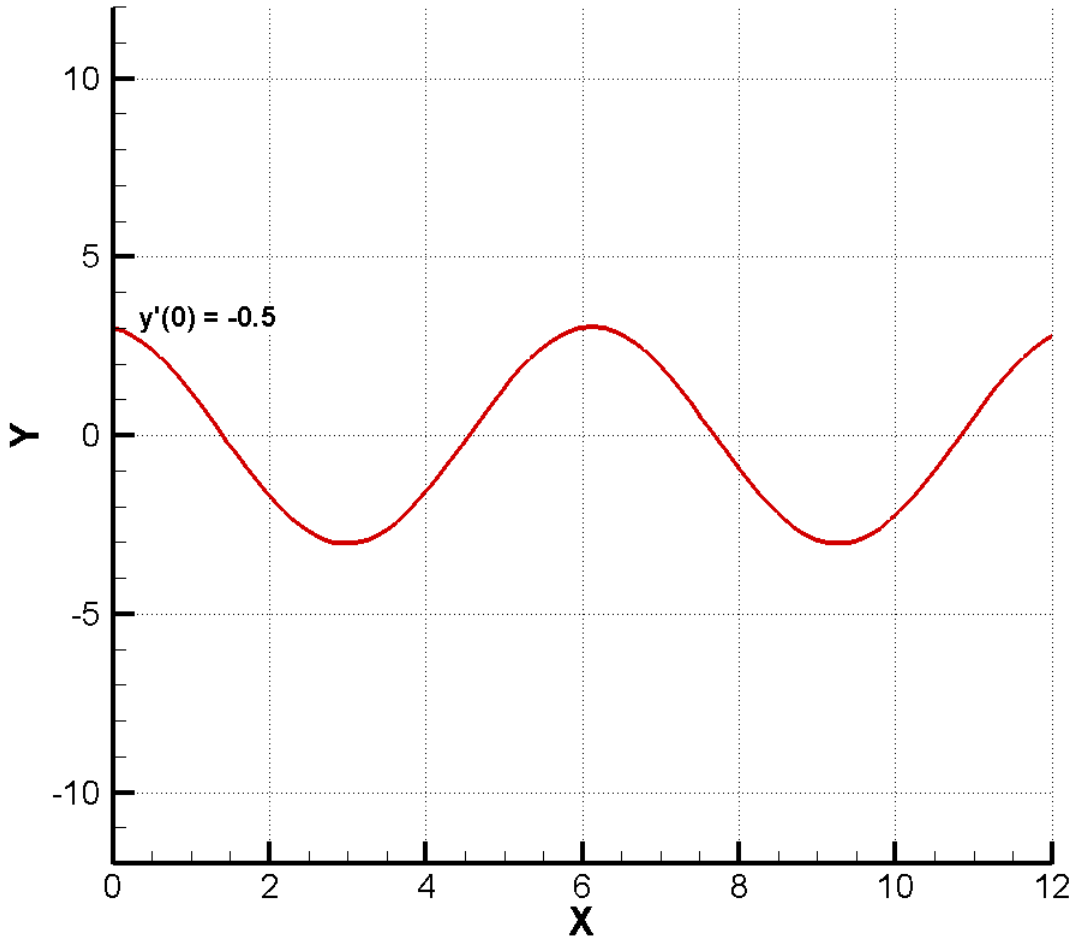
- Example 4

$$y'' + y = 0, \qquad y(0) = 3.0, \qquad y'(0) = -0.5$$

Homogeneous Linear ODEs of 2nd order
Example 4 Initial Value Problem

$$y'' + y = 0, \qquad y(0) = 3.0, \qquad y'(0) = -0.5$$



Homogeneous Linear ODEs of 2nd order
Example 4 Initial Value Problem

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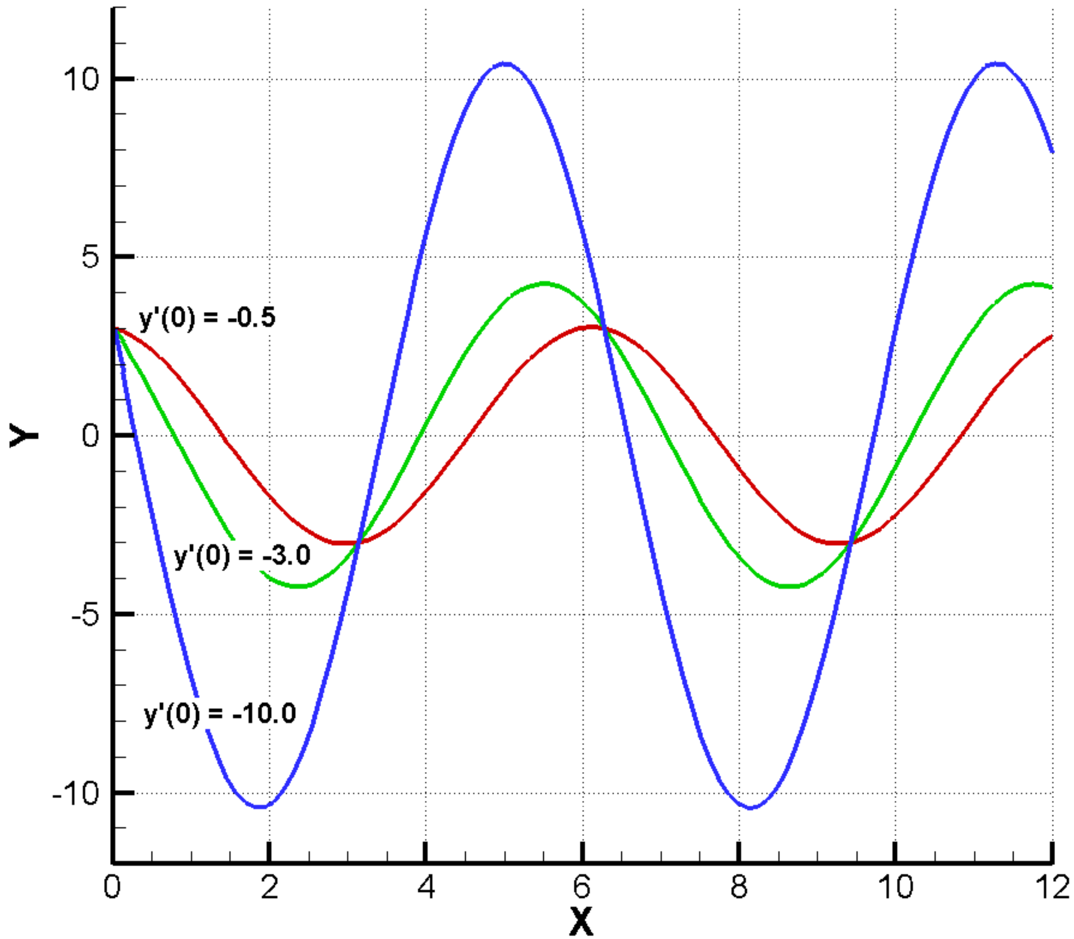
$$y''+y=0, \quad y(0)=3.0, \quad y'(0)=-0.5$$

$$y''+y=0, \quad y(0)=3.0, \quad y'(0)=-3.0$$

$$y''+y=0, \quad y(0)=3.0, \quad y'(0)=-10.0$$



Homogeneous Linear ODEs of 2nd order
Example 4 Initial Value Problem

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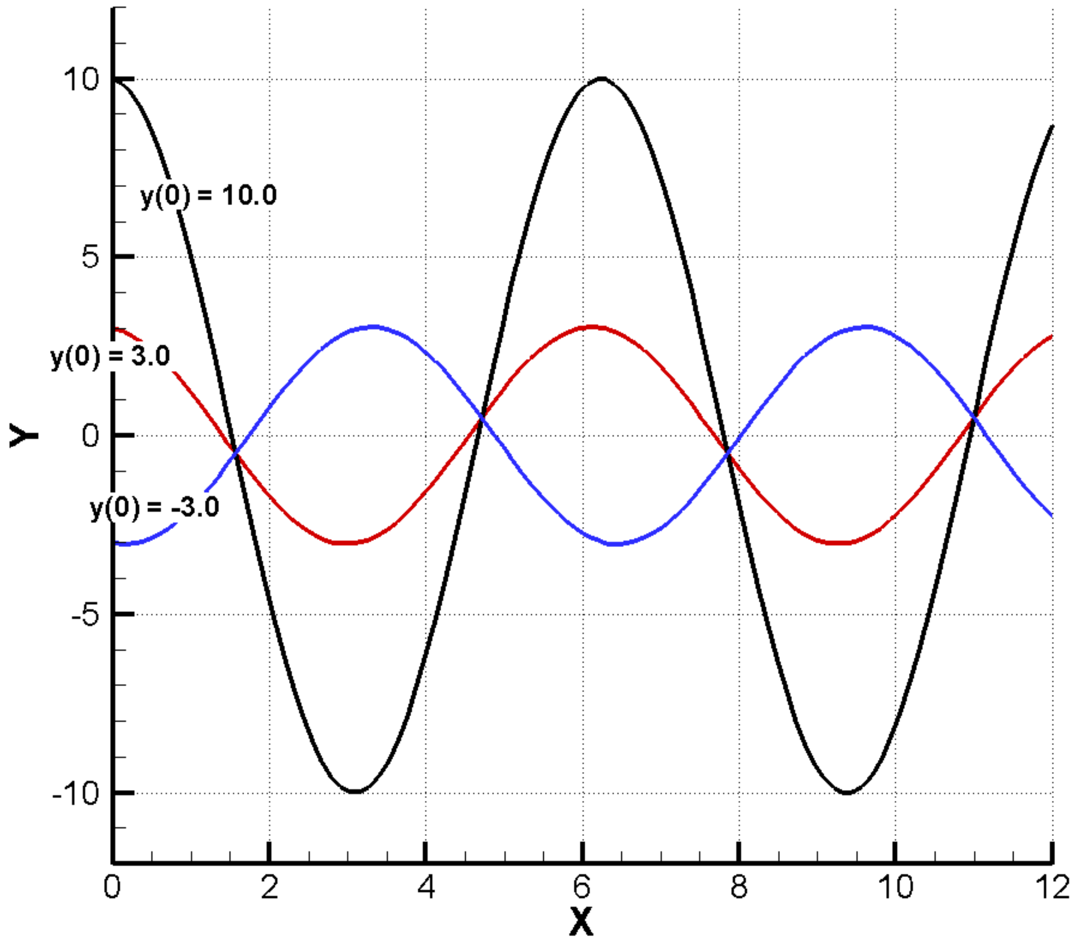
$y''+y=0, \quad y(0)=3.0, \quad y'(0)=-0.5$

$y''+y=0, \quad y(0)=10.0, \quad y'(0)=-0.5$

$y''+y=0, \quad y(0)=-3.0, \quad y'(0)=-0.5$



Homogeneous Linear ODEs of 2nd order

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- General solution (일반해): $y = c_1 y_1 + c_2 y_2$
  - 구간 I에서 비례(proportional) 하지 않는 2계 제차 방정식의 해 y_1, y_2와 임의의 상수 c_1, c_2를 갖는 해

$$k_1 y_1(x) + k_2 y_2(x) = 0$$

Linearly independent → $k_1 = 0 \; and \; k_2 = 0$

Linearly dependent → $y_1 = -\frac{k_2}{k_1} y_2 \; or \; y_2 = -\frac{k_1}{k_2} y_1$

  - Basis of solution (기저): 서로독립인 y_1, y_2를 구간 I에서의 제차방정식의 기저 (basis) 또는 기본계(fundamental system)
  - Particular solution (특수해): 일반해에서 c_1, c_2를 갖는 해

Homogeneous Linear ODEs of 2nd order
Finding a basis if one solution is known

- Method of reduction of order (by Lagrange)

$$\left(x^2 - x\right)y'' - xy' + y = 0$$

$$y_1 = x \longrightarrow y_2 = uy_1$$

Homogeneous Linear ODEs of 2nd order
Finding a basis if one solution is known

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- Method of reduction of order (by Lagrange)

$$y + p(x)y + q(x)y = 0$$

$$y = y_2 = uy_1 \qquad \left(y' = y_2' = u'y_1 + uy_1',\; y'' = y_2'' = u''y_1 + 2u'y_1' + uy_1''\right)$$

$$\Rightarrow \quad u''y_1 + u'\left(2y_1' + py_1\right) + u\left(y_1'' + py_1' + qy_1\right) = 0 \quad \Rightarrow \quad u'' + u'\frac{2y_1' + py_1}{y_1} = 0 \quad \left(\because \; y_1'' + py_1' + qy_1 = 0\right)$$

$$U = u', \quad U' = u'' \quad \Rightarrow \quad U' + \left(2\frac{y_1'}{y_1} + p\right)U = 0$$

$$\Rightarrow \quad \frac{dU}{U} = -\left(2\frac{y_1'}{y_1} + p\right)dx \;\; \& \;\; \ln|U| = -2\ln|y_1| - \int p\,dx \quad \Rightarrow \quad \therefore \; U = \frac{1}{y_1^{\,2}}e^{-\int p\,dx}, \quad y_2 = uy_1 = y_1\int U\,dx$$

Homogeneous Linear ODEs with Constant Coefficients

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- 2nd order homogeneous linear ODEs with constant coefficients (상수계수 2계 제차 선형미분방정식)

$$y'' + ay' + by = 0$$

  - $y = e^{\lambda x}$ can be a solution
  - λ is a solution of the following equation

$$\lambda^2 + a\lambda + b = 0$$

Characteristic equation (특성방정식, or auxiliary equation)

- Roots of quadratic equation

$$\lambda_1 = \frac{1}{2}\left(-a+\sqrt{a^2-4b}\right),\ \lambda_2 = \frac{1}{2}\left(-a-\sqrt{a^2-4b}\right)$$

$$y_1 = e^{\lambda_1 x} \quad and \quad y_2 = e^{\lambda_2 x}$$

- Three kinds of the general solution of the equation
  - (case I) Two real roots $\lambda_1,\ \lambda_2$ if $a^2-4b>0 \quad \Rightarrow \quad y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$
  - (case II) A real double root $\lambda = -a/2$ if $a^2-4b=0 \quad \Rightarrow \quad y = (c_1 + c_2 x)e^{-ax/2}$
  - (case III) Complex conjugate roots $\lambda = -a/2 \pm i\omega$ if

$$a^2-4b<0 \quad \Rightarrow \quad y = e^{-ax/2}\left(A\cos\omega x + B\sin\omega x\right)$$

$$\uparrow$$

$$\omega^2 = b - a^2/4$$

Case I. Two Distinct Real Roots λ_1 and λ_2

$$y'' + ay' + by = 0$$

- Basis of solutions

$$y_1 = e^{\lambda_1 x} \qquad y_2 = e^{\lambda_2 x}$$

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$

- Ex) $y'' + y' - 2y = 0 \qquad y(0) = 4,\ \ y'(0) = -5$

Case II. Real double root λ=-a/2

$$y''+ay'+by=0$$

- (case II) A real double root $\lambda=-a/2$ if $a^2-4b=0$ $\Rightarrow$ $y=(c_1+c_2x)e^{-ax/2}$

$$y_1=e^{-(a/2)x} \xrightarrow{\text{Method of reduction of order}} y_2=xe^{-(a/2)x}$$

$$y=(c_1+c_2x)e^{-ax/2}$$

- Example 4) $y''+y'+0.25y=0,\ \ y(0)=3.0,\ \ y'(0)=-3.5$

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Case III. Complex Conjugate

$$y''+ay'+by=0$$

$$\lambda = \frac{-a \pm \sqrt{a^2-4b}}{2} \qquad \omega^2 = b - a^2/4$$

- (case III) Complex conjugate roots $\lambda = -a/2 \pm i\omega$

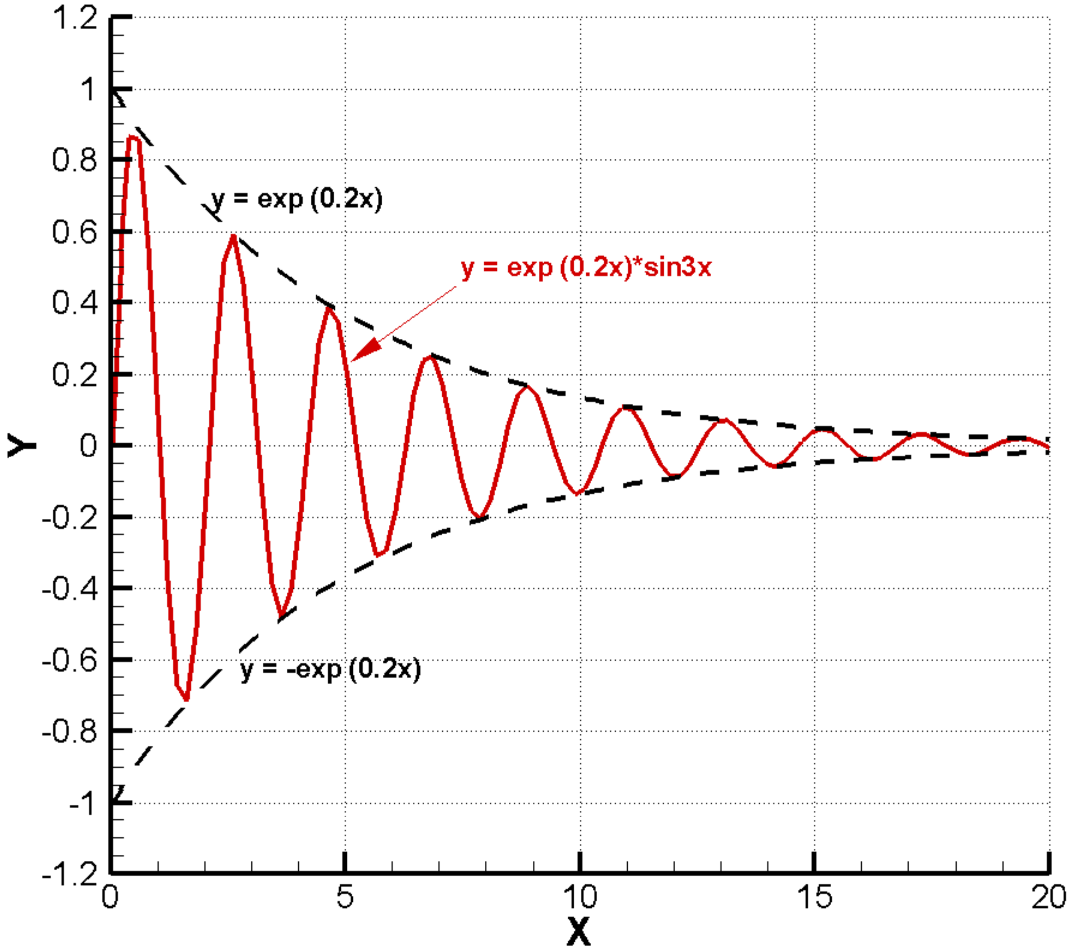
  if $a^2 - 4b < 0 \quad \Rightarrow \quad y = e^{-ax/2}\left(A\cos\omega x + B\sin\omega x\right)$

$$\lambda_1 = -a/2 + i\omega \qquad \lambda_2 = -a/2 - i\omega$$

  - Basis of solutions: $y_1 = e^{-ax/2}\cos\omega x \qquad y_2 = e^{-ax/2}\sin\omega x$
  - General solutions: $y = e^{-ax/2}\left(A\cos\omega x + B\sin\omega x\right)$

- Example 5) $y'' + 0.4y' + 9.04y = 0,\ \ y(0) = 0,\ \ y'(0) = 3$

- Example 5.


1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1.2
Y
0
5
10
15
20
X
y = exp (0.2x)
y = exp (0.2x)*sin3x
y = -exp (0.2x)

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summary

$$y'' + ay' + by = 0 \xrightarrow{y = e^{\lambda x}} \lambda^2 + a\lambda + b = 0$$

Summary of Cases I–III

Case	Roots of (2)	Basis of (1)	General Solution of (1)
I	Distinct real λ_1, λ_2	$e^{\lambda_1 x}, e^{\lambda_2 x}$	$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$
II	Real double root $\lambda = -\frac{1}{2}a$	$e^{-ax/2}, xe^{-ax/2}$	$y = (c_1 + c_2 x)e^{-ax/2}$
III	Complex conjugate $\lambda_1 = -\frac{1}{2}a + i\omega$, $\lambda_2 = -\frac{1}{2}a - i\omega$	$e^{-ax/2}\cos\omega x$ $e^{-ax/2}\sin\omega x$	$y = e^{-ax/2}(A\cos\omega x + B\sin\omega x)$

Differential Operators (미분연산자).

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- Operator (연산자): A transformation that transforms a function into another function

- Differential Operator *D* (미분연산자)
  - An operator which transforms a (differentiable) function into its derivative $$Dy = y' = \frac{dy}{dx}$$
  - Identity Operator *I* : $Iy = y$
  - Second-order differential operator (2계 미분연산자)
  $$L = P(D) = D^2 + aD + bI \quad \Rightarrow \quad Ly = P(D)y = y'' + ay' + by$$

- Example1. Factor $L = P(D) = D^2 - 3D - 40I$ and solve $P(D)y = 0$

Modeling: Free Oscillations (자유진동). (Mass-Spring System)
Introduction

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- A basic mechanical system
  - a mass on an elastic spring, which moves up and down*.

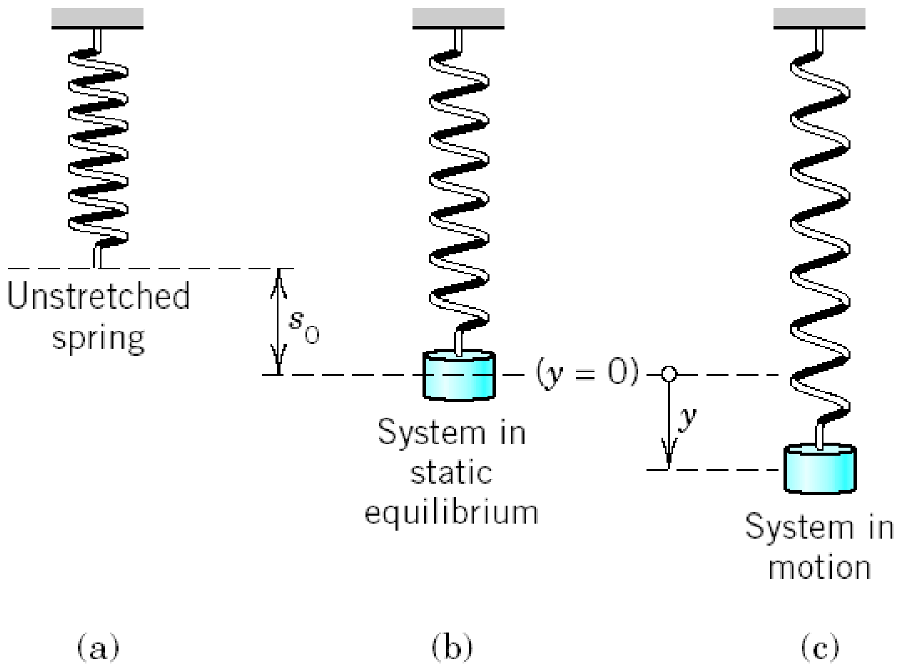
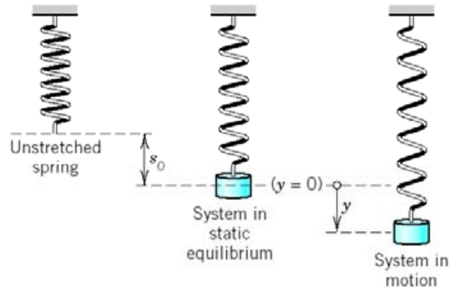


Fig. 32. Mechanical mass–spring system

*We choose the downward direction as the (+) direction.

Modeling: Free Oscillations. (Mass-Spring System)

Modeling



- Physical Information
  - Newton's second law : Mass x Acceleration = Force
  - Hooke's law: restoring force is directly proportional to the distance
- Modeling
  - System in static equilibrium

$$F_0 = -ks_0 \quad (k : \text{spring constant})$$

Weight of body : $W = mg$

$$F_0 + W = -ks_0 + mg = 0$$

  - System in motion

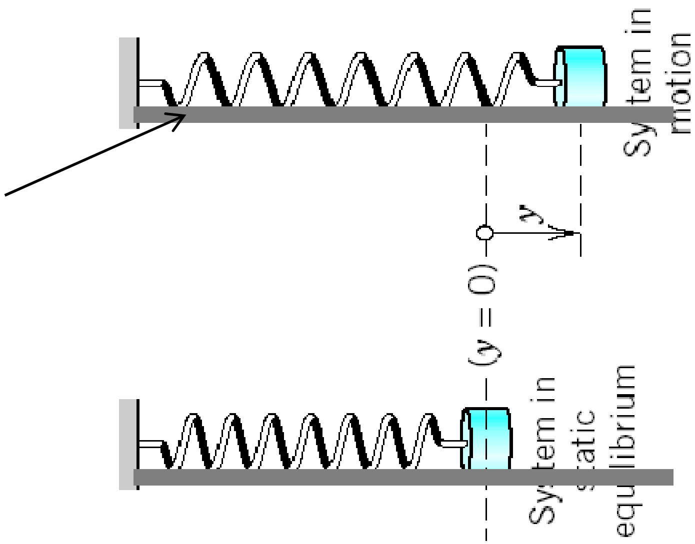
Restoring force : $F_1 = -ky$ (Hooke's law)

(Newton's second law) $\sum F = F_1 = my''$

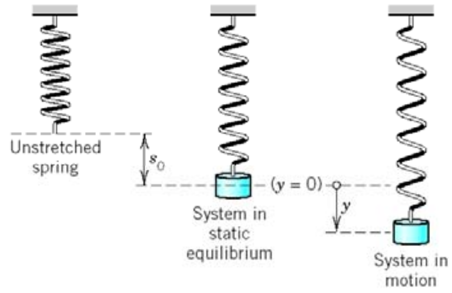
$$my'' + ky = 0$$

- <u>Alternative way of thinking</u>
  - a mass on an elastic spring, which moves horizontally.

plate



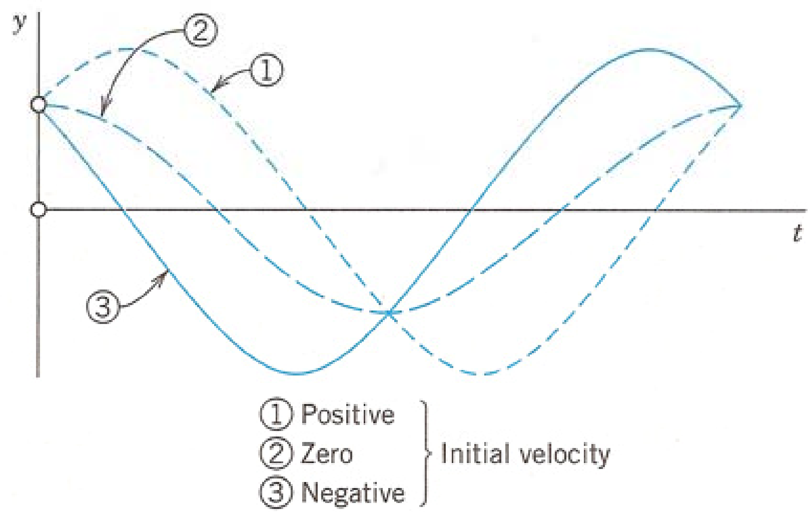
Modeling: Free Oscillations. (Mass-Spring System)



Undamped system

- Undamped System : ODE and Solution
  - ODE : $my'' + ky = 0$
  - Harmonic oscillation (조화진동)

$$y(t) = A\cos\omega_0 t + B\sin\omega_0 t = C\cos(\omega_0 t - \delta), \qquad \omega_0^{\,2} = \frac{k}{m}$$



Modeling: Free Oscillations. (Mass-Spring System)

Undamped system

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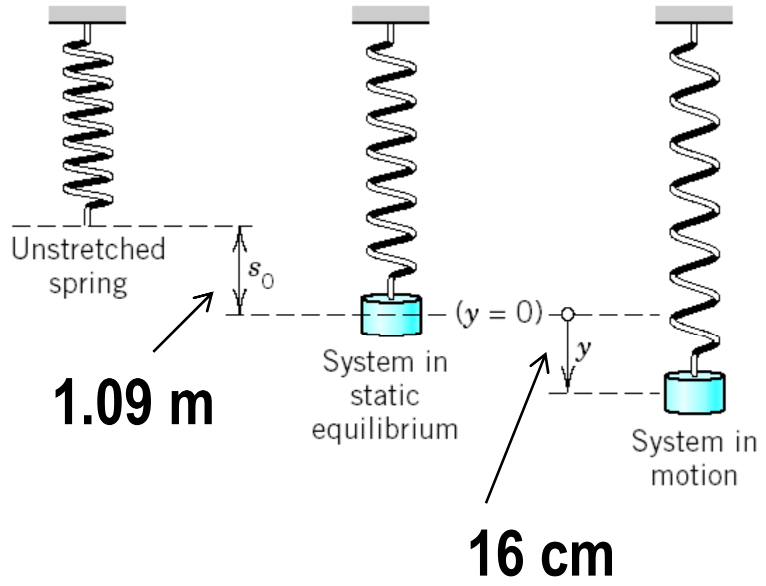
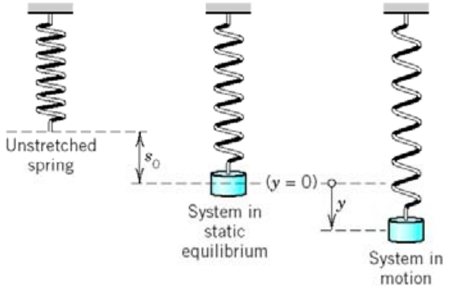
- Example 1
  - Iron ball weight = 98 N
  - Stretched 1.09 m
  - 1) How many cycles per minutes? And its motion?
  - 2) Pull down 16 cm and let it start with zero velocity. its motion?

Not to scale!

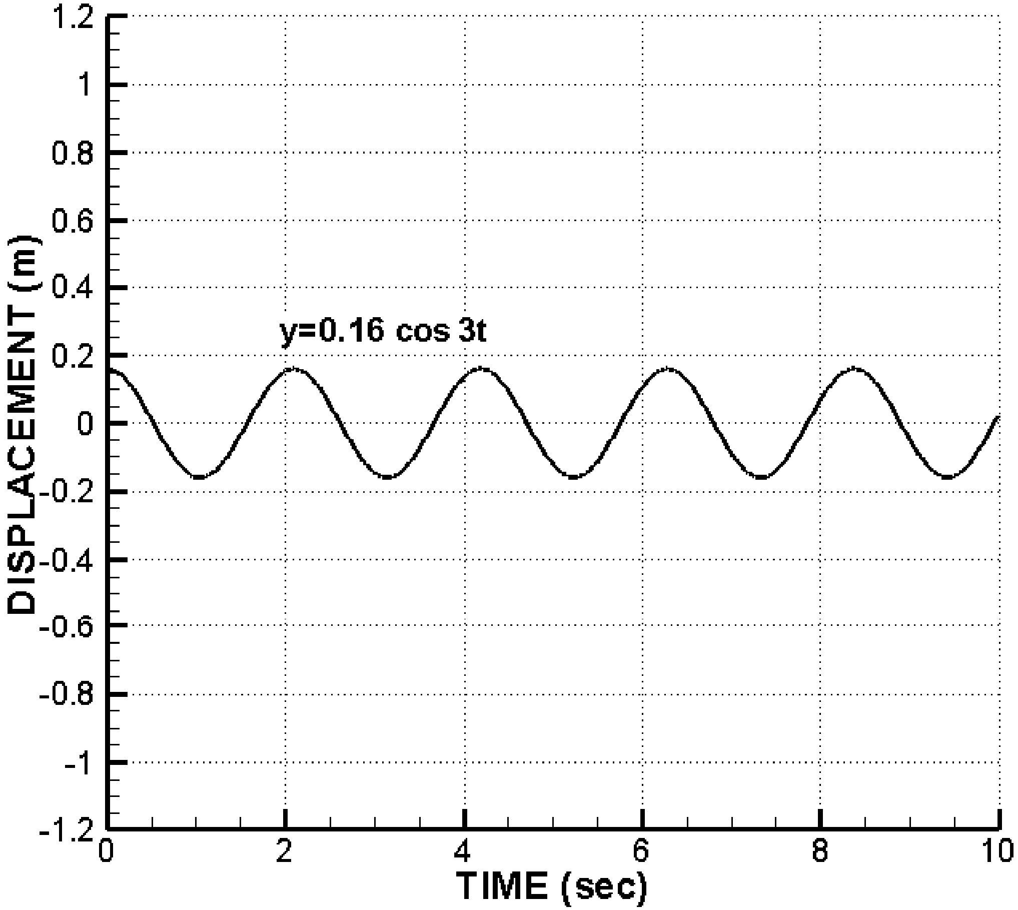


Modeling: Free Oscillations. (Mass-Spring System)

Undamped system

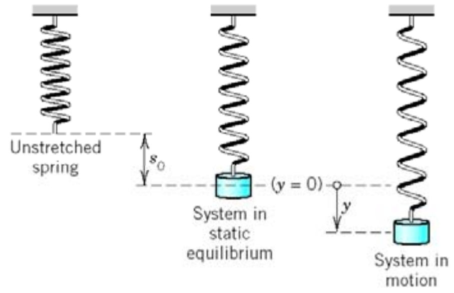


- Example 1.

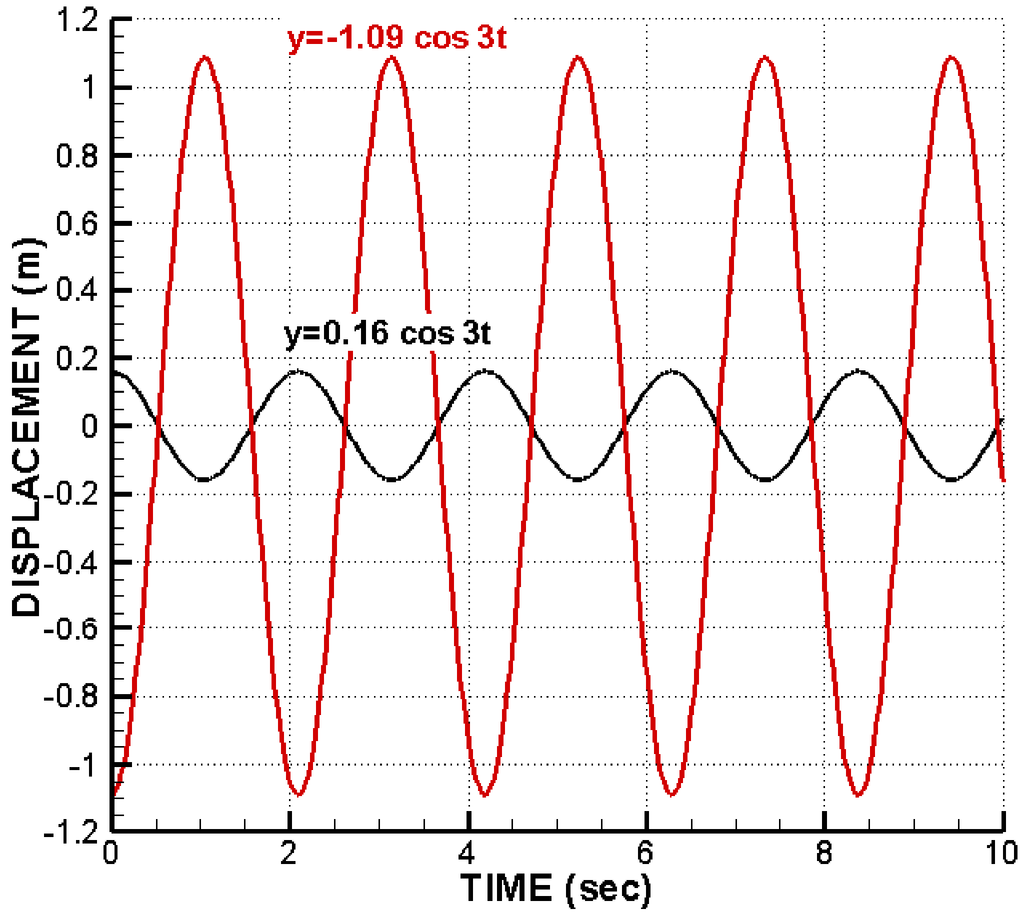


Modeling: Free Oscillations. (Mass-Spring System)

Undamped system



- Example 1.



Damped system

Damping Force

$$F_2 = -cy'$$

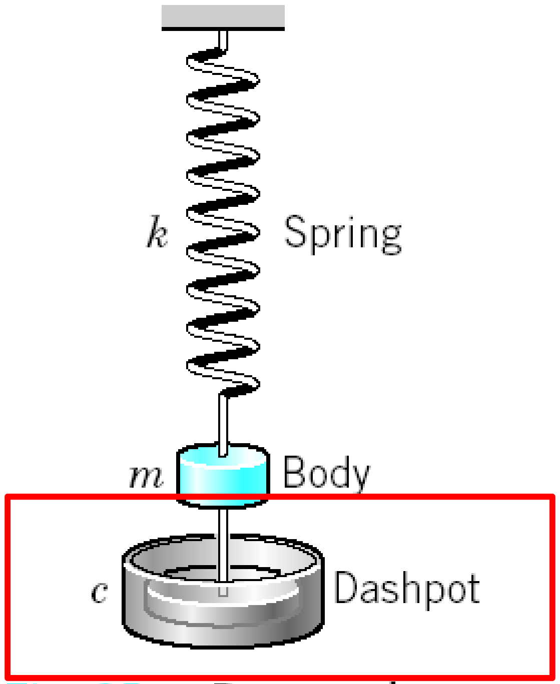
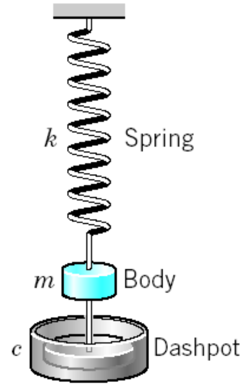


Fig. 35. Damped system

Modeling: Free Oscillations. (Mass-Spring System)

Damped system



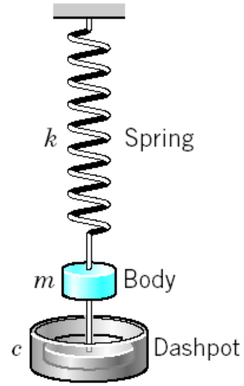
- Damped System : ODE and Solutions
  - Damping force : $F_2 = -cy'$ (c: the damping constant)
  - (Newton's second law) $\sum F = F_1 + F_2 = my''$

$$my'' + cy' + ky = 0$$

- Three types of motion
  - **Case 1 (Overdamping)** $c^2 > 4mk$ Distinct real roots λ_1 , λ_2
  - **Case 2 (Critical damping)** $c^2 = 4mk$. A real double root.
  - **Case 3 (Underdamping)** $c^2 < 4mk$ Complex conjugate roots.

Modeling: Free Oscillations. (Mass-Spring System)

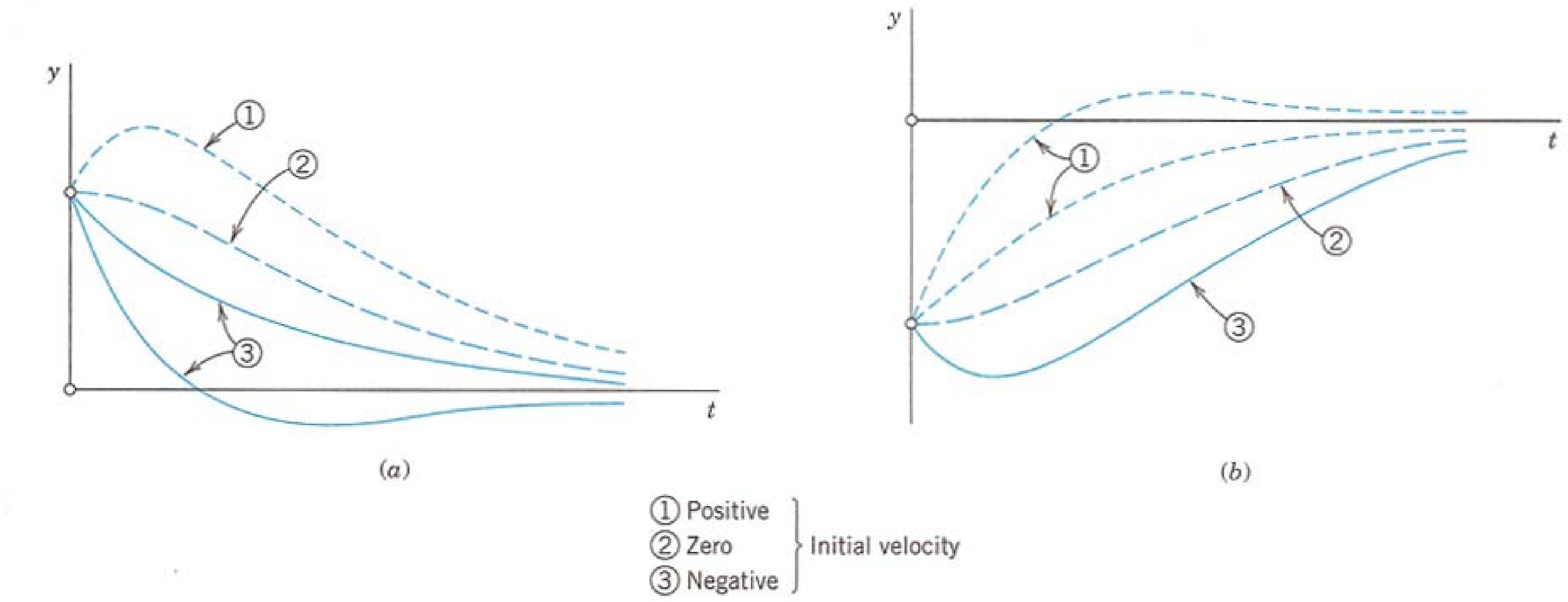
Damped system - overdamping



$$my'' + cy' + ky = 0$$

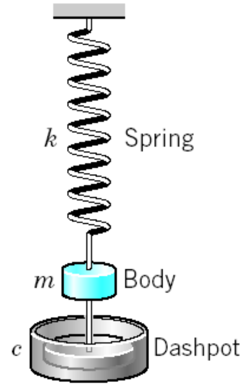
- Overdamping $c^2 > 4mk$

$$y(t) = c_1 e^{-(\alpha-\beta)t} + c_2 e^{-(\alpha+\beta)t} \qquad \alpha = \frac{c}{2m}, \quad \beta = \frac{\sqrt{c^2 - 4mk}}{2m}$$



Modeling: Free Oscillations. (Mass-Spring System)

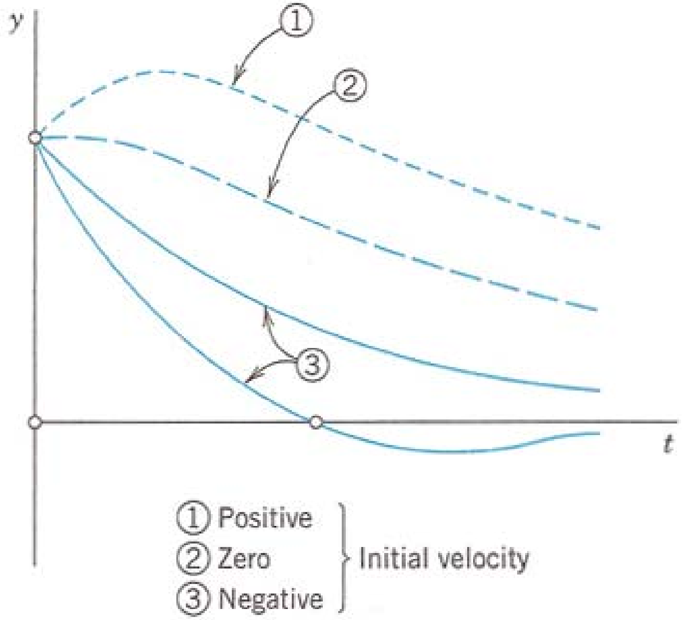
Damped system – critical damping



$$my'' + cy' + ky = 0$$

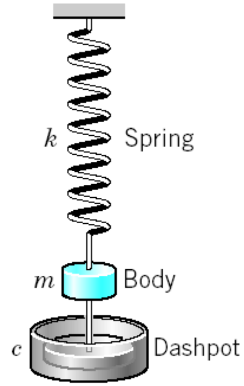
- Critical damping $\qquad c^2 = 4mk$

$$y(t) = (c_1 + c_2 t)e^{-\alpha t}, \qquad \alpha = \frac{c}{2m}$$



Modeling: Free Oscillations. (Mass-Spring System)

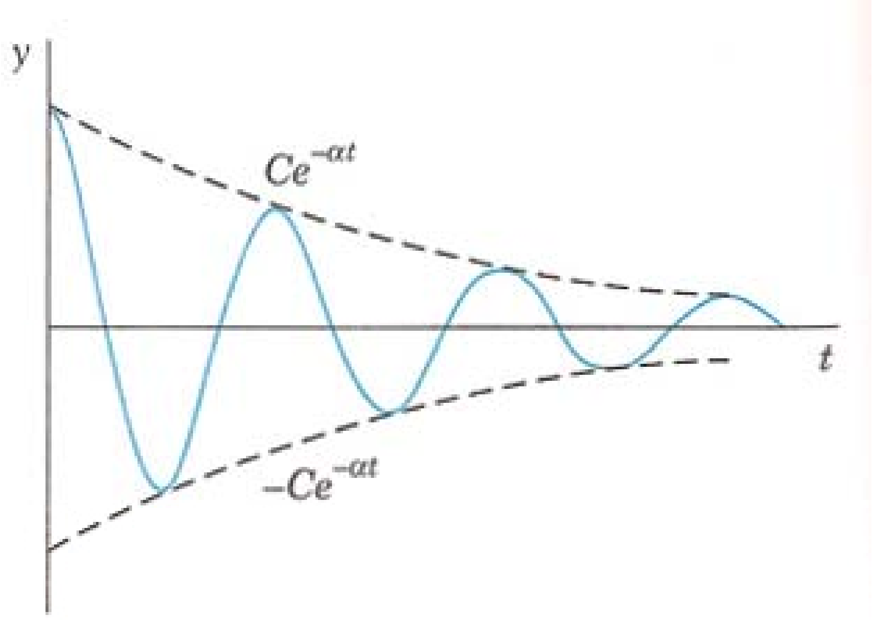
Damped system - underdamping



$$my'' + cy' + ky = 0$$

- Underdamping $\left(c^2 < 4mk\right)$

$$y(t) = e^{-\alpha t}\left(A\cos\omega^* t + B\sin\omega^* t\right) = Ce^{-\alpha t}\cos\left(\omega^* t - \delta\right)$$



$$\omega^* = \frac{1}{2m}\sqrt{4mk - c^2}$$

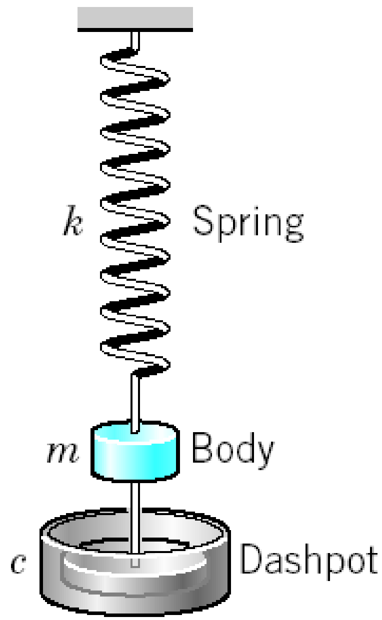
Modeling: Free Oscillations. (Mass-Spring System)

Damped system

- Example2. $my'' + cy' + ky = 0$

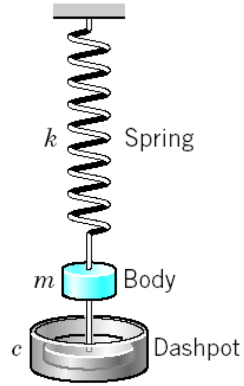
$$m = 10,\ k = 90,\ c = 100,\ 60,\ 10$$

$$y(0) = 0.16,\ y'(0) = 0$$

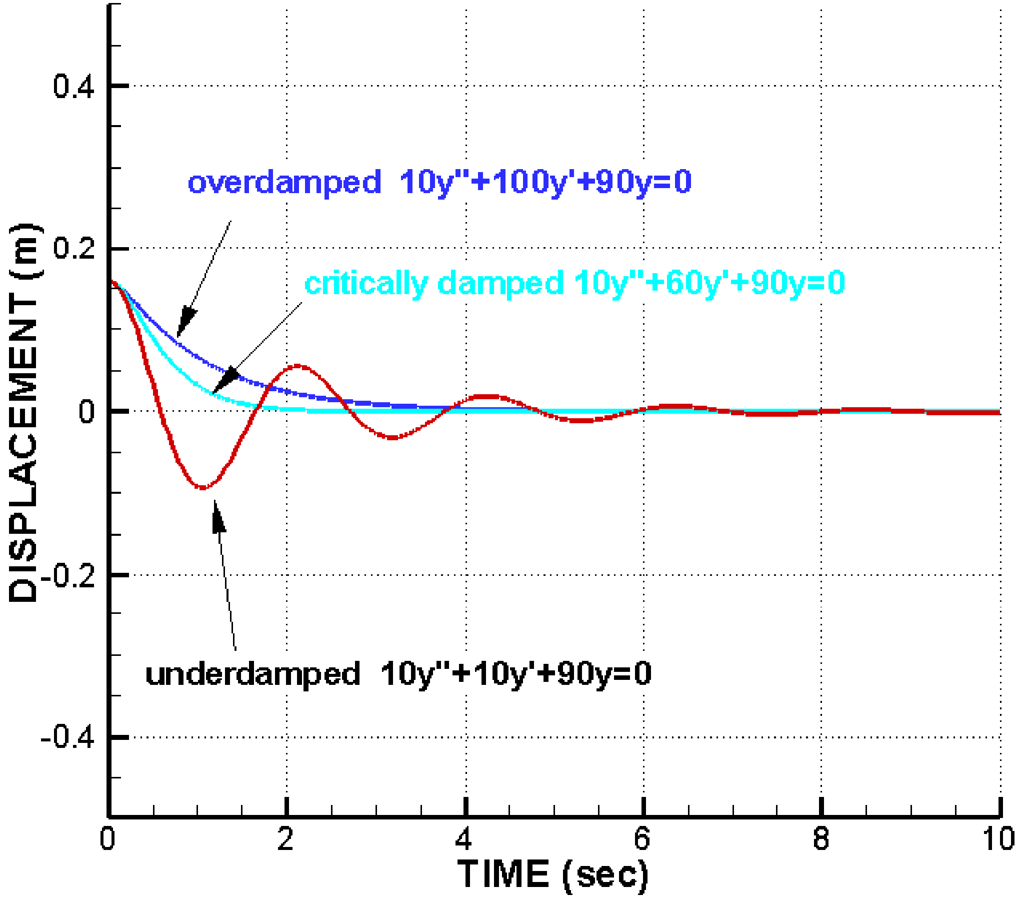


Modeling: Free Oscillations. (Mass-Spring System)

Damped system

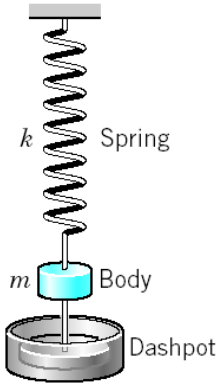


- Example2

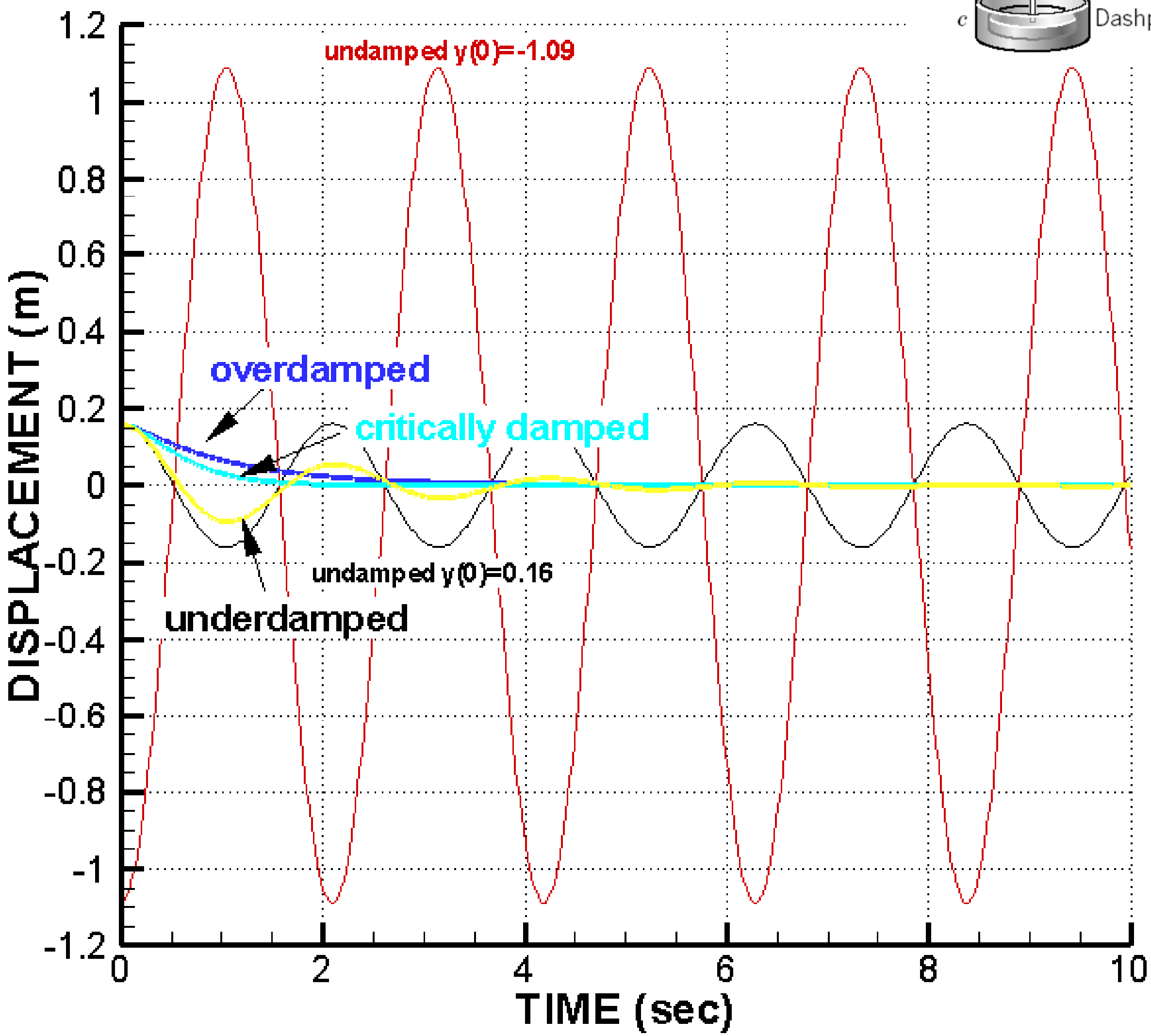


Modeling: Free Oscillations. (Mass-Spring System)

Damped system



- Example2



Euler-Cauchy Equations

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- Euler-Cauchy Equations

$$x^2 y'' + axy' + by = 0$$

Euler-Cauchy Equations

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- Euler-Cauchy Equations : $x^2 y'' + axy' + by = 0$

$$\downarrow \; y = x^m$$

  - Auxiliary Equation $m^2 + (a-1)m + b = 0$

- Three kinds of the general solution of the equation
  - Case 1 Two real roots $m_1, \; m_2 \quad \Rightarrow \quad y = c_1 x^{m_1} + c_2 x^{m_2}$
  - Case 2 A real double root $m = {(1-a)}/{2} \quad \Rightarrow \quad y = (c_1 + c_2 \ln x) x^m$
  - Case 3 Complex conjugate roots

$$m = \mu \pm i\upsilon \quad \Rightarrow \quad y = x^{\mu} \left[A\cos(\upsilon \ln x) + B\sin(\upsilon \ln x) \right]$$

Euler-Cauchy Equations

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- Example 1

$$x^2 y'' + 1.5xy' - 0.5y = 0$$

- Example 2

$$x^2 y'' - 5xy' + 9y = 0$$

- Example 3

$$x^2 y'' + 0.6xy' + 16.04y = 0$$

Existence and Uniqueness of Solutions. Wronskian

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- Theorem 1. Existence and Uniqueness Theorem for Initial Value Problems

$$y''+p(x)y'+q(x)y=0 \qquad (1)$$

$$y(x_0)=K_0, \quad y'(x_0)=K_1 \qquad (2)$$

  - 위의 초기값 문제에서 $p(x)$와 $q(x)$가 열린구간 I에서 연속함수이고, x_0 구간 I내에 있다면, 초기값 문제는 구간 I에서 유일한 해를 갖는다.

Existence and Uniqueness Theorem for Initial Value Problems

If $p(x)$ and $q(x)$ are continuous functions on some open interval I (see Sec. 1.1) *and x_0 is in I, then the initial value problem consisting of* (1) *and* (2) *has a unique solution $y(x)$ on the interval I.*

Existence and Uniqueness of Solutions. Wronskian

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- Theorem 2. Linear Dependence and Independence of Solutions.
- 상미방이 열린 구간 *I* 에서 연속인 계수 $p(x)$와 $q(x)$를 갖는다고 가정. 이 때 구간 *I* 에서 제차 선형상미방의 두 개의 해 y_1, y_2가 구간 *I* 에서 일차종속이 되기 위한 필요충분조건은 그들의 Wronskian이 구간 *I* 내의 어떤 x_0 에서 0이 되는 것임. $x = x_0$ 에서 $W = 0$ 이라면, 구간 *I* 에서 $W \equiv 0$. 만약 W가 0이 아닌 x_1 이 구간 *I* 내에 존재하면, 구간 *I* 에서 y_1, y_2는 일차독립.

Linear Dependence and Independence of Solutions

Let the ODE (1) have continuous coefficients $p(x)$ and $q(x)$ on an open interval I. Then two solutions y_1 and y_2 of (1) on I are linearly dependent on I if and only if their **"Wronskian"**

(6)
$$W(y_1, y_2) = y_1 y_2' - y_2 y_1'$$

is 0 at some x_0 in I. Furthermore, if $W = 0$ at an $x = x_0$ in I, then $W \equiv 0$ on I; hence if there is an x_1 in I at which W is not 0, then y_1, y_2 are linearly independent on I.

Existence and Uniqueness of Solutions. Wronskian

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- Wronskian or Wronskian Determinant (론스키안 행렬식)

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

- Example 1.

$$y'' + \omega^2 y = 0$$

$$y_1 = \cos \omega x \qquad y_2 = \sin \omega x$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos \omega x & \sin \omega x \\ -\omega \sin \omega x & \omega \cos \omega x \end{vmatrix} = \omega \cos^2 \omega x + \omega \sin^2 \omega x = \omega$$

- Example 2

$$y'' - 2y' + y = 0$$

- Theorem 3. Existence of a General Solution
  - If *p*(*x*) and *q*(*x*) are continuous on an open interval *I*, then has a general solution on *I*.
- Theorem 4. A general solution includes All solutions
  - If the ODE $y''+p(x)y'+q(x)y=0$ has continuous coefficients *p*(*x*) and *q*(*x*) on some open interval *I*, then every solution *y* = y(*x*) of the equation on *I* is of the form $Y(x)=C_1y_1(x)+C_2y_2(x)$ where y_1, y_2 is any basis of solutions of the equation on *I* and c_1, c_2 are suitable constants. Hence the equation does not have **singular solutions** (that is, solutions not obtainable from a general solution).

Nonhomogeneous ODEs

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- Nonhomogeneous Linear ODEs

$$y''+p(x)y'+q(x)y=r(x), \quad r(x)\neq 0 \qquad (1)$$

$$y''+p(x)y'+q(x)y=0 \qquad (2)$$

General Solution, Particular Solution

A **general solution** of the nonhomogeneous ODE (1) on an open interval I is a solution of the form

(3) $$y(x) = y_h(x) + y_p(x);$$

here, $y_h = c_1y_1 + c_2y_2$ is a general solution of the homogeneous ODE (2) on I and y_p is any solution of (1) on I containing no arbitrary constants.

A **particular solution** of (1) on I is a solution obtained from (3) by assigning specific values to the arbitrary constants c_1 and c_2 in y_h.

Nonhomogeneous ODEs

$$y'' + p(x)y' + q(x)y = r(x), \quad r(x) \neq 0 \qquad (1)$$

$$y = y_h + y_p \qquad (3)$$

A General Solution of a Nonhomogeneous ODE Includes All Solutions

If the coefficients $p(x)$, $q(x)$, and the function $r(x)$ in (1) *are continuous on some open interval I, then every solution of* (1) *on I is obtained by assigning suitable values to the arbitrary constants c_1 and c_2 in a general solution* (3) *of* (1) *on I.*

Nonhomogeneous ODEs
Steps in solving nonhomogeneous ODE

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$$y''+p(x)y'+q(x)y=r(x)$$

- Steps in solving nonhomogeneous ODE
  - Step1. solve homogeneous ODE → find y_h
  - Step2. find any solution of nonhomogeneous ODE → find y_p
    → $y = y_h + y_p$
  - Step 3. Solution of initial value problem

- How to find y_p?

Nonhomogeneous ODEs
Method of undetermined coefficients

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$$y''+ay'+by=r(x)$$

- Method of Undetermined Coefficients (미정계수법)
  - Method of finding y_p
  - Suitable for constant coefficient a, b (상수계수에 적합)
  - When r(x)'s derivatives are similar to itself
    - Exponential, power of x, cosine, sine, or sum of these

Nonhomogeneous ODEs
Method of undetermined coefficients

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$$y'' + ay' + by = r(x)$$

Table 2.1 Method of Undetermined Coefficients

Term in $r(x)$	Choice for $y_p(x)$
$ke^{\gamma x}$	$Ce^{\gamma x}$
$kx^n \ (n = 0, 1, \cdots)$	$K_n x^n + K_{n-1}x^{n-1} + \cdots + K_1 x + K_0$
$k \cos \omega x$	$K \cos \omega x + M \sin \omega x$
$k \sin \omega x$	
$ke^{\alpha x} \cos \omega x$	$e^{\alpha x}(K \cos \omega x + M \sin \omega x)$
$ke^{\alpha x} \sin \omega x$	

Nonhomogeneous ODEs
Method of undetermined Coefficients

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- Choice Rules
  - Basic Rule (기본규칙):
    If r(x) is one of the functions in the table → choose the function in the table.
  - Modification Rule (변형규칙):
    choice of y_p = solution of homogeneous ODE → multiply by x (or x^2 if solution is double root)
  - Sum Rule (합규칙):
    If r(x) is a sum of functions in the table → sum of the functions in the table

Nonhomogeneous ODEs

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- Example 1.

$$y''+y=0.001x^2, \quad y(0)=0, \quad y'(0)=1.5$$

- Example 2.

$$y''+3y'+2.25y=-10e^{-1.5x}, \quad y(0)=1, \quad y'(0)=0$$

- Example 3.

$$y''+2y'+5y=e^{0.5x}+40\cos 10x-190\sin 10x,$$
$$y(0)=0.16, \quad y'(0)=40.08$$

Nonhomogeneous ODEs

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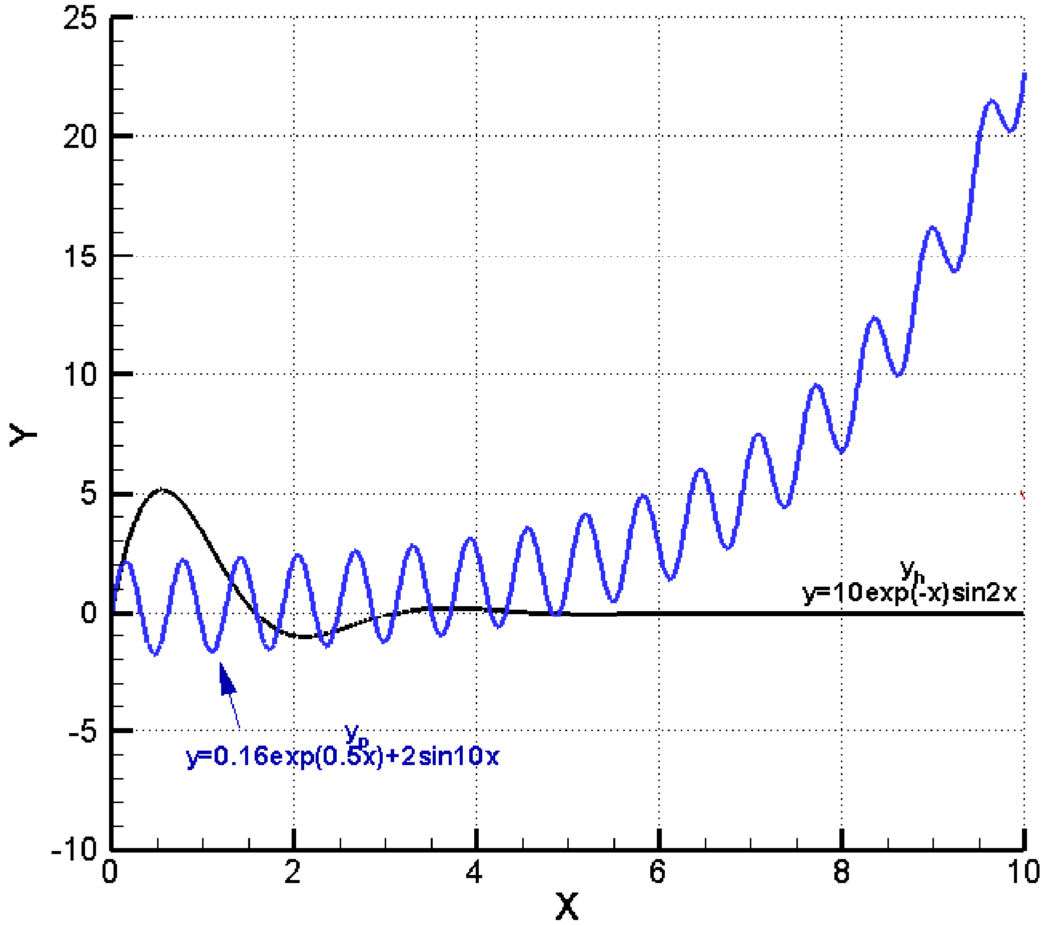
- Example 3



Nonhomogeneous ODEs

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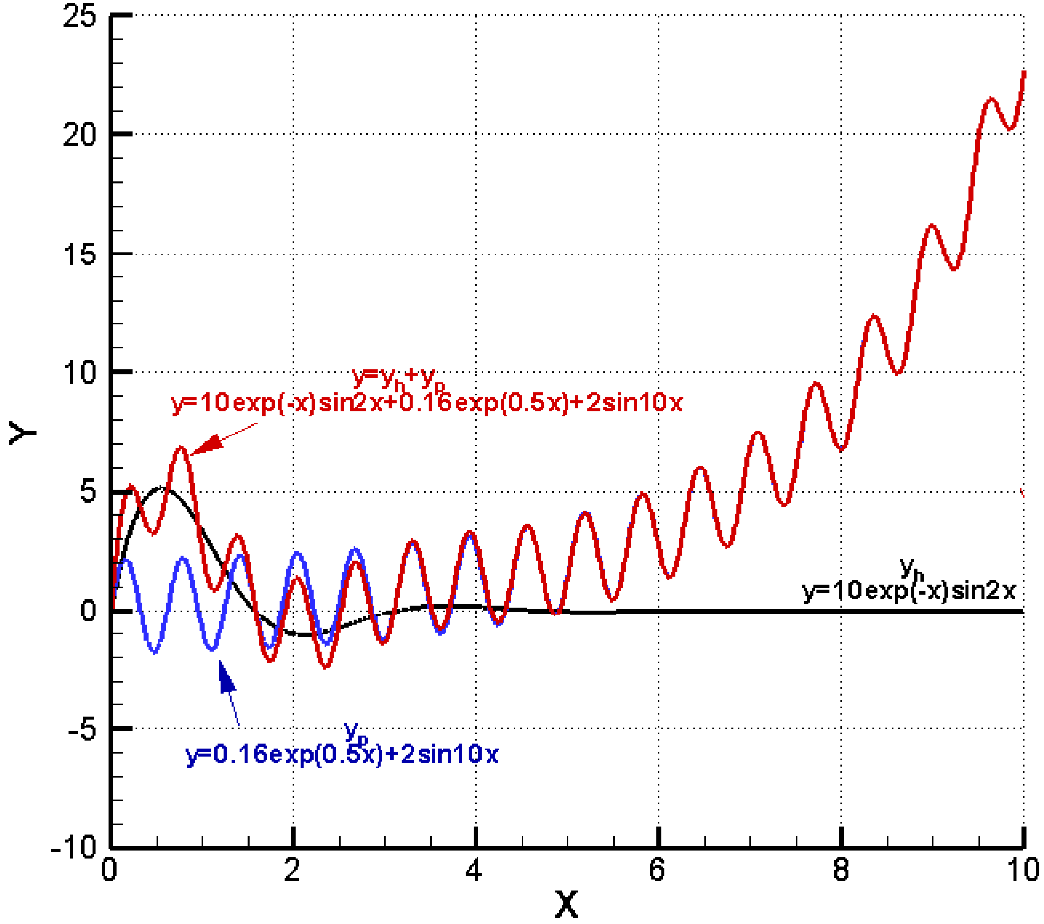
- Example 3



Nonhomogeneous ODEs
Example 3

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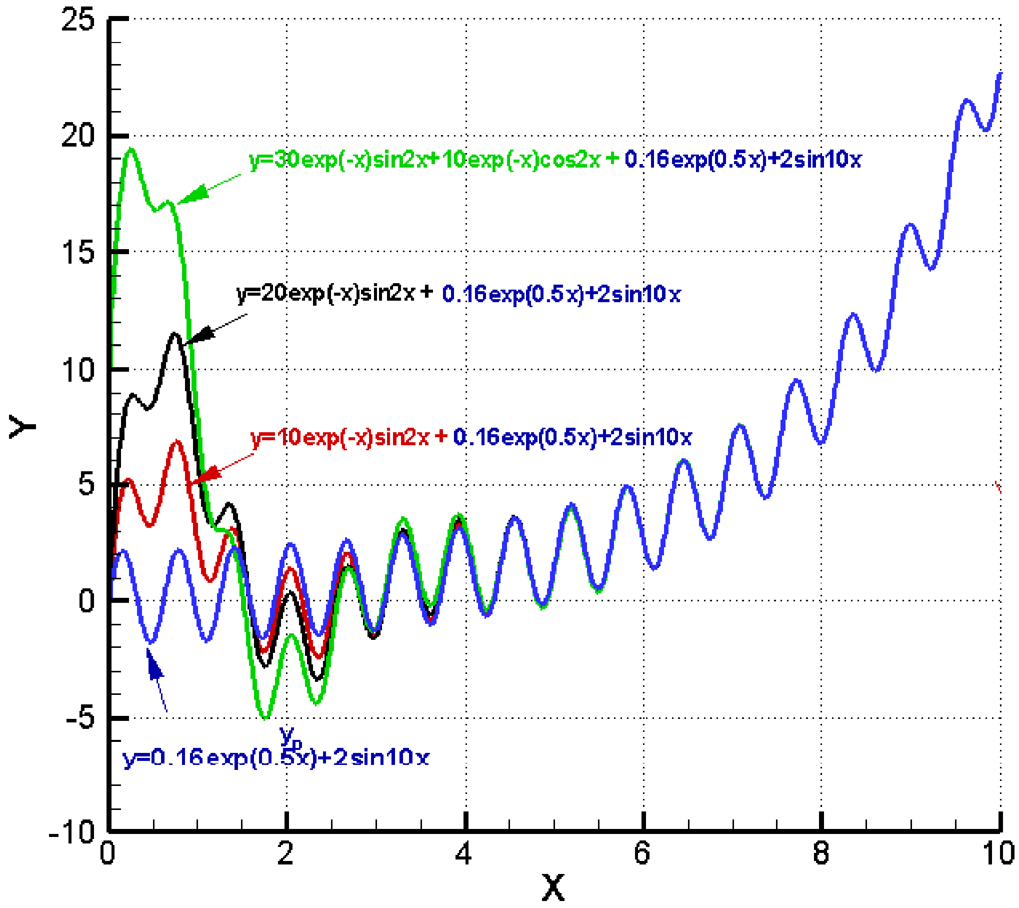

Stable → nonhomogeneous ODE approaches Steady State solution, y_p

Otherwise → unstable

Modeling: Forced Oscillations. Resonance
Free Motion vs. Forced Motion

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- Free Motion : Motions in the absence of external forces caused solely by internal forces.

$$my'' + cy' + ky = 0$$

- Forced Motion : Model by including an external force.

$$my'' + cy' + ky = r(t)$$

Input or driving force

y(t) : Output or response

Modeling: Forced Oscillations. Resonance

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- Motion with Periodic external forces
  - Nonhomogeneous ODE : $my'' + cy' + ky = F_0 \cos \omega t$
  - Through the method of undetermined coefficients

$$y_p = a \cos \omega t + b \sin \omega t$$

↓ **By putting y_p into the homogeneous ODE**

$$a = F_0 \frac{m\left(\omega_0^{\ 2} - \omega^2\right)}{m^2\left(\omega_0^{\ 2} - \omega^2\right)^2 + \omega^2 c^2}, \quad b = F_0 \frac{\omega c}{m^2\left(\omega_0^{\ 2} - \omega^2\right)^2 + \omega^2 c^2}$$

Modeling: Forced Oscillations. Resonance
Undamped system

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- Undamped Forced Oscillations

$$c=0 \quad \Rightarrow \quad y_p = \frac{F_0}{m\left(\omega_0^{\ 2}-\omega^2\right)}\cos\omega t \quad \Rightarrow \quad y = \underbrace{C\cos\left(\omega_0 t-\delta\right)}_{y_h} + \underbrace{\frac{F_0}{m\left(\omega_0^{\ 2}-\omega^2\right)}\cos\omega t}_{y_p}$$

  - Natural frequency: $\dfrac{\omega_0}{2\pi}$
  - Frequency of the driving force $\dfrac{\omega}{2\pi}$
  - What will happen if $\omega = \omega_0$?

Modeling: Forced Oscillations.
Undamped Forced System: Resonance

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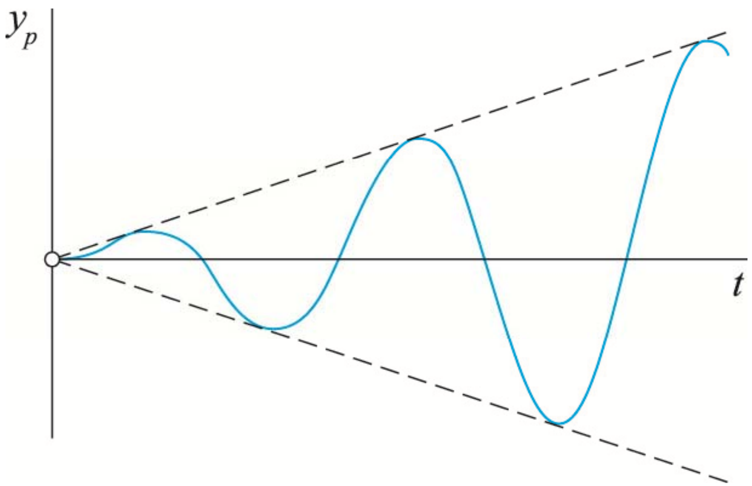
- Resonance (공진) : Excitation of large oscillations by matching input and natural frequencies. $\omega = \omega_0$
  - y_p is no longer in the form of $y_p = a\cos\omega t + b\sin\omega t$
  - Why?
  - From modification rule → $y_p = t(a\cos\omega_0 t + b\sin\omega_0 t)$



$$y_p = \frac{F_0}{2m\omega_0} t \sin\omega_0 t$$

What will happen if input and natural frequency are very close?

Modeling: Forced Oscillations.
Undamped Forced System : Beat

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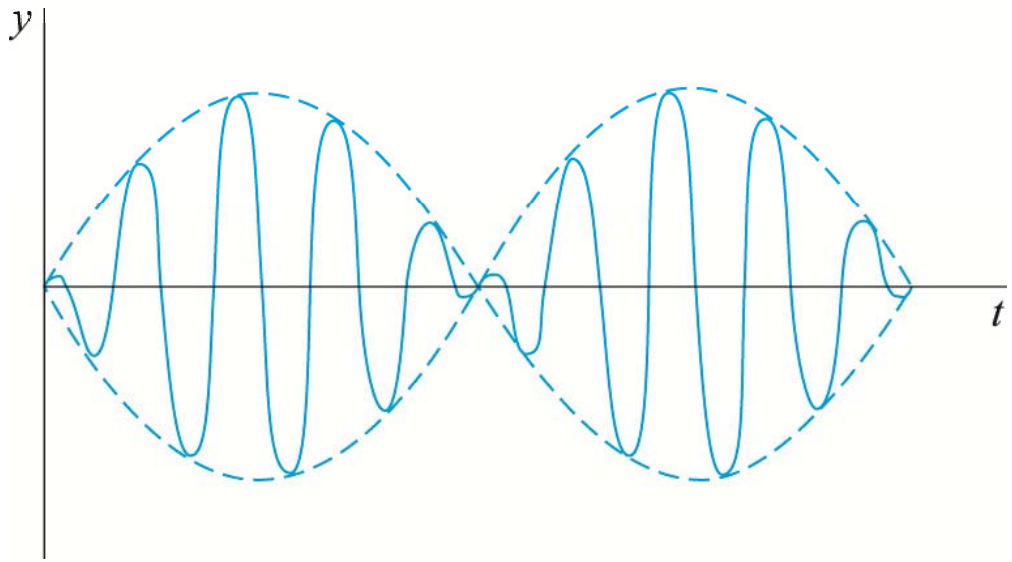
- Beats (맥놀이):
  - Forced undamped oscillation when the difference of the input and natural frequencies is small.

$$y = \frac{F_0}{m\left(\omega_0{}^2 - \omega^2\right)}\left(\cos\omega t - \cos\omega_0 t\right) = \frac{2F_0}{m\left(\omega_0{}^2 - \omega^2\right)}\sin\left(\frac{\omega_0 + \omega}{2}t\right)\sin\left(\frac{\omega_0 - \omega}{2}t\right)$$



Modeling: Forced Oscillations.
Damped Forced System

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- Damped Forced Oscillation
  - Transient Solution : The general solution $y = y_h + y_p$ of the nonhomogeneous ODE
  - Steady-State Solution : The particular solution y_p

$$y_p = a\cos\omega t + b\sin\omega t \longrightarrow y = C^*\cos(\omega t - \eta)$$

$$C^* = \frac{F_0}{\sqrt{m^2(\omega_0{}^2 - \omega^2)^2 + \omega^2 c^2}}$$

  - Amplitude C* is a function of ω and we can find ω when amplitude is the maximum

$$C^*(\omega_{\max}) = \frac{2mF_0}{c\sqrt{4m^2\omega_0{}^2 - c^2}}$$

Solution by Variation of Parameters

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- Method of Variation of Parameter (매개변수 변환)
  - More general method when r(x) is not in a form that we want

$$y'' + p(x)y' + q(x)y = r(x)$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$y_p(x) = -y_1 \int \frac{y_2 r}{W} dx + y_2 \int \frac{y_1 r}{W} dx$$

Solution by Variation of Parameters

- Example 1. Solve the nonhomogeneous ODE

$$y'' + y = \sec x$$

Solution by Variation of Parameters

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- Derivation of $y_p(x) = -y_1 \int \frac{y_2 r}{W} dx + y_2 \int \frac{y_1 r}{W} dx$

- Put $y_p(x) = u(x) y_1(x) + v(x) y_2(x)$

2^{nd}-Order Linear ODEs
Summary (1)

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- Linear ODEs of 2^{nd} order – Standard Form

$$y'' + p(x)y' + q(x)y = r(x)$$

  - Homogeneous: $r(x) \equiv 0$
  - Nonhomogeneous: $r(x) \neq 0$

2^{nd}-Order Linear ODEs
Summary (2)

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- Linear ODEs of constant coefficients

$$y'' + ay' + by = 0 \xrightarrow{y = e^{\lambda x}} \lambda^2 + a\lambda + b = 0$$

Summary of Cases I–III

Case	Roots of (2)	Basis of (1)	General Solution of (1)
I	Distinct real λ_1, λ_2	$e^{\lambda_1 x}, e^{\lambda_2 x}$	$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$
II	Real double root $\lambda = -\frac{1}{2}a$	$e^{-ax/2}, xe^{-ax/2}$	$y = (c_1 + c_2 x)e^{-ax/2}$
III	Complex conjugate $\lambda_1 = -\frac{1}{2}a + i\omega$, $\lambda_2 = -\frac{1}{2}a - i\omega$	$e^{-ax/2}\cos \omega x$ $e^{-ax/2}\sin \omega x$	$y = e^{-ax/2}(A\cos \omega x + B\sin \omega x)$

2^{nd}-Order Linear ODEs
Summary (3)

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- Free Motion : No external forces

$$my'' + cy' + ky = 0$$

- Forced Motion : Model by including an external force.

$$my'' + cy' + ky = r(t)$$

Input or driving force

y(t) : Output or response

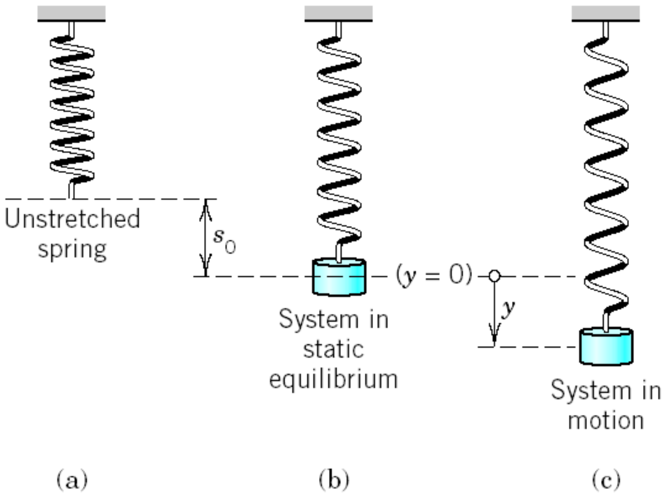


Fig. 32. Mechanical mass–spring system

2^{nd}-Order Linear ODEs
Summary (4)

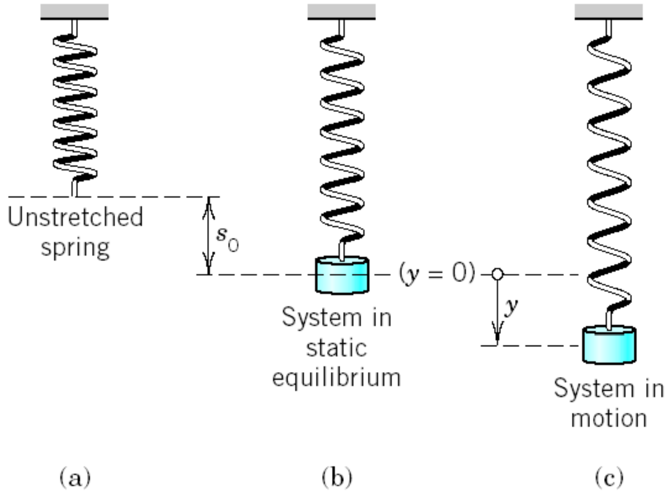


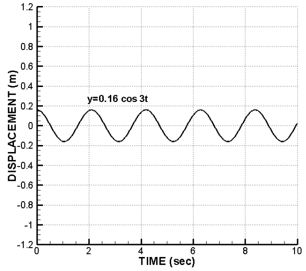
Fig. 32. Mechanical mass–spring system

- Mass-Spring system

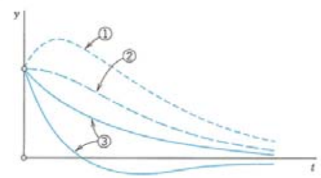
$$my''+cy'+ky=0$$

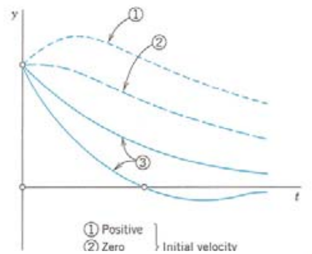
  - Undamped system (c = 0)

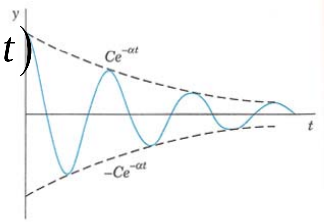
$$y(t)=A\cos\omega_0 t+B\sin\omega_0 t$$



  - Damped system
    - Overdamped $c^2>4mk$ $y(t)=c_1e^{-(\alpha-\beta)t}+c_2e^{-(\alpha+\beta)t}$
    - Critically damped $c^2=4mk$ $y(t)=(c_1+c_2t)e^{-\alpha t}$
    - underdamped $c^2<4mk$ $y(t)=e^{-\alpha t}(A\cos\omega^* t+B\sin\omega^* t)$







2^{nd}-Order Linear ODEs
Summary (5)

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- Euler-Cauchy Equations : $x^2 y'' + axy' + by = 0$

$$\downarrow \; y = x^m$$

  - Auxiliary Equation $m^2 + (a-1)m + b = 0$

- Three kinds of the general solution of the equation
  - Case 1 Two real roots $m_1, \; m_2 \quad \Rightarrow \quad y = c_1 x^{m_1} + c_2 x^{m_2}$
  - Case 2 A real double root $m = {(1-a)}/{2} \quad \Rightarrow \quad y = (c_1 + c_2 \ln x) x^m$
  - Case 3 Complex conjugate roots

$$m = \mu \pm i\upsilon \quad \Rightarrow \quad y = x^{\mu}\left[A\cos(\upsilon \ln x) + B\sin(\upsilon \ln x)\right]$$

2^{nd}-Order Linear ODEs
Summary (6)

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$$y'' + ay' + by = r(x)$$

$$y = y_h + y_p$$

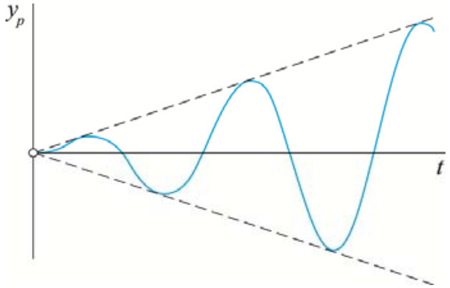
Table 2.1 Method of Undetermined Coefficients

Term in $r(x)$	Choice for $y_p(x)$
$ke^{\gamma x}$	$Ce^{\gamma x}$
$kx^n \ (n = 0, 1, \cdots)$	$K_n x^n + K_{n-1}x^{n-1} + \cdots + K_1 x + K_0$
$k \cos \omega x$ $k \sin \omega x$	$K \cos \omega x + M \sin \omega x$
$ke^{\alpha x} \cos \omega x$ $ke^{\alpha x} \sin \omega x$	$e^{\alpha x}(K \cos \omega x + M \sin \omega x)$

2^{nd}-Order Linear ODEs
Summary (7)

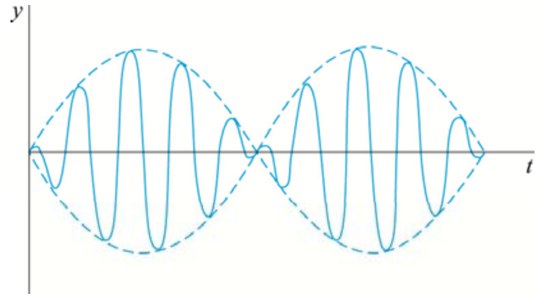
- Undamped forced oscillation
  - Resonance : Excitation of large oscillations by $\omega = \omega_0$

$$y_p = \frac{F_0}{2m\omega_0} t \sin \omega_0 t$$



  - Beat: When $\omega \approx \omega_0$

$$y = \frac{2F_0}{m\left(\omega_0^{\,2} - \omega^2\right)} \sin\left(\frac{\omega_0 + \omega}{2} t\right) \sin\left(\frac{\omega_0 - \omega}{2} t\right)$$



- Damped Forced Oscillation

$$y_p = a \cos \omega t + b \sin \omega t$$

$$C^* = \frac{F_0}{\sqrt{m^2\left(\omega_0^{\,2} - \omega^{\,2}\right)^2 + \omega^2 c^2}}$$

Announcement

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- Home Assignment: Chapter 2 — 28 March 2011
- 1st Exam: Ch.1, Ch.2, Ch.3 — 30 March 2011