Engineering Mathematics I
- Chapter 5. Series Solutions of ODEs. Special Functions
- 5장. 상미분 방정식의 급수해법. 특수함수

민기복

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Ch.4 Systems of ODEs. Phase Plane. Qualitative Methods

- Basics of Matrices and Vectors
- Systems of ODEs as Models
- Basic Theory of Systems of ODEs
- Constant-Coefficient Systems. Phase Plane Method
- Criteria for Critical Points. Stability
- Qualitative Methods for Nonlinear Systems
- Nonhomogeneous Linear Systems of ODEs

Series Solutions of ODEs. Special Functions
상미분 방정식의 급수해법. 특수함수

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- 5.1 Power Series Method (거듭제곱 급수 해법)
- 5.2 Theory of the Power Series Method (거듭제곱 급수 해법의 이론)
- 5.3 Legendre's Equation. Legendre Polynomials $P_n(x)$ Legendre 방정식. Legendre 다항식 $P_n(x)$
- 5.4 Frobenius Method (Frebenius 해법)
- 5.5 Bessel's Equation. Bessel functions $J_v(x)$. Bessel의 방정식. Bessel 함수 $J_v(x)$
- 5.6 Bessel's Functions of the Second Kind $Y_v(x)$. 제2종 Bessel 함수 $Y_v(x)$
- 5.7 Sturm-Liouville Problems. Orthogonal Functions. Sturm-Liouville 문제. 직교함수
- 5.8 Orthogonal Eigenfunction Expansions. 직교 고유함수의 전개

Series Solutions of ODEs. Special Functions.

- 변수계수를 갖는 선형미분 방정식을 풀이하는 표준적인 방법인 power series method (거듭제곱급수, 혹은 멱급수 해법)을 소개한다
- 거듭제곱급수 해법으로 얻을 수 있는 유명한 특수함수:
  - Bessel function (베셀 함수),
  - Legendre function (르장드르 함수),
  - Gauss의 hypergeometric function (초기하함수)

Power Series Method

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- Linear ODEs with variable coefficient ← power series method
- Power series is an infinite series of the form*;

$$\sum_{m=0}^{\infty} a_m (x-x_0)^m = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots$$

  - Coefficients (계수) : $a_0, \ a_1, \ a_2, \ \cdots$
  - Center (중심) : x_0
  - If $x_0 = 0$; $\sum_{m=0}^{\infty} a_m x^m = a_0 + a_1 x + a_2 x^2 + \cdots$

- Maclaurin series (맥클로린 급수) $\frac{1}{1-x} = \sum_{m=0}^{\infty} x^m = 1 + x + x^2 + \cdots \quad (|x|<1)$

$$e^x = \sum_{m=0}^{\infty} \frac{x^m}{m!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$Taylor\ Series: \ f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

$$\cos x = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{(2m)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - + \cdots$$

$$\sin x = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{(2m+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - + \cdots$$

* 통상 음의 거듭제곱이나 분수거듭제곱을 가지는 급수는 포함하지 않음.

Power Series Method

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- Idea of power series
  - For a given ODE $y''+p(x)y'+q(x)y=0$
  - Represent $p(x)$ and $q(x)$ by power series in power of x
  - Assume a solution in the form of power series

$$y=\sum_{m=0}^{\infty}a_m x^m = a_0+a_1x+a_2x^2+a_3x^3+\cdots$$

  - Differentiation of this series, and put into ODE

$$y'=\sum_{m=1}^{\infty}ma_m x^{m-1}=a_1+2a_2x+3a_3x^2+\cdots$$

$$y''=\sum_{m=2}^{\infty}m(m-1)a_m x^{m-1}=2a_2+3\cdot 2a_3x+\cdots$$

  - Determine the unknown coefficients a_m

Power Series Method

- Ex 1. Solve the following ODE by power series

$$y' = 2xy$$

$$y = \sum_{m=0}^{\infty} a_m x^m = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \qquad e^x = \sum_{m=0}^{\infty} \frac{x^m}{m!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$y' = \sum_{m=1}^{\infty} m a_m x^{m-1} = a_1 + 2a_2 x + 3a_3 x^2 + \cdots$$

$$\Rightarrow \quad a_1 + 2a_2 x + 3a_3 x^2 + \cdots = 2x\left(a_0 + a_1 x + a_2 x^2 + \cdots\right)$$

$$\Rightarrow \quad a_1 + 2a_2 x + 3a_3 x^2 + \cdots = 2a_0 x + 2a_1 x^2 + 2a_2 x^3 + \cdots$$

$$\Rightarrow \quad a_1 = 0,\ 2a_2 = 2a_0,\ 3a_3 = 2a_1,\ 4a_4 = 2a_2,\ 5a_5 = 2a_3,\ 6a_6 = 2a_4,\ \cdots$$

$$\Rightarrow \quad a_2 = a_0,\ a_4 = \frac{a_2}{2} = \frac{a_0}{2!},\ a_6 = \frac{a_4}{3} = \frac{a_0}{3!},\ \cdots$$

$$\therefore\ y = a_0\left(1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!} + \cdots\right) = a_0 e^{x^2}$$

Power Series Method

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- Ex. 2 Solve the following ODE by power series.

$$y''+y=0$$

$$y=\sum_{m=0}^{\infty} a_m x^m \qquad y''=\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2}$$

$$\Rightarrow \sum_{m=2}^{\infty} m(m-1)a_m x^{m-2}+\sum_{m=0}^{\infty} a_m x^m=0$$

$$\Rightarrow \sum_{s=0}^{\infty}(s+2)(s+1)a_{s+2}x^s=-\sum_{s=0}^{\infty} a_s x^s$$

(첫 번째 항은 $m=s+2$, 두 번째 항은 $m=s$)

$$\cos x=\sum_{m=0}^{\infty}\frac{(-1)^m x^{2m}}{(2m)!}=1-\frac{x^2}{2!}+\frac{x^4}{4!}-+\cdots$$

$$\sin x=\sum_{m=0}^{\infty}\frac{(-1)^m x^{2m+1}}{(2m+1)!}=x-\frac{x^3}{3!}+\frac{x^5}{5!}-+\cdots$$

Recursion Formula (순환공식): $a_{s+2}=-\dfrac{a_s}{(s+2)(s+1)} \qquad (s=0,\ 1,\ \cdots)$

$$a_2=-\frac{a_0}{2\cdot 1}=-\frac{a_0}{2!}, \qquad a_3=-\frac{a_1}{3\cdot 2}=-\frac{a_1}{3!}$$

$$a_4=-\frac{a_2}{4\cdot 3}=\frac{a_0}{4!}, \qquad a_5=-\frac{a_3}{5\cdot 4}=\frac{a_1}{5!}$$

$$\therefore\ y=a_0+a_1x-\frac{a_0}{2!}x^2-\frac{a_1}{3!}x^3+\frac{a_0}{4!}x^4+\frac{a_1}{5!}x^5+\cdots=a_0\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-+\cdots\right)+a_1\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-+\cdots\right)$$

$$=a_0\cos x+a_1\sin x$$

Theory of the Power Series Method
Basic Concepts

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- Basic Concepts

$$\sum_{m=0}^{\infty} a_m (x-x_0)^m = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots + a_n(x-x_0)^n + a_{n+1}(x-x_0)^{n+1} + a_{n+2}(x-x_0)^{n+2} + \cdots$$

$s_n(x)$ **: n-th partial sum (부분합)** $R_n(x)$ **: remainder (나머지)**

- Series is convergent at $x=x_1$ → $\lim_{n\to\infty} s_n(x_1) = s(x_1)$

  Value (수렴값) or sum (합)

- $s(x_1) = \sum_{m=0}^{\infty} a_m (x_1 - x_0)^m$
- For every n, $s(x_1) = s_n(x_1) + R_n(x_1)$
- If this sequence diverges at $x=x_1$, series (1) is called divergent at $x=x_1$
- In case of convergence, for any positive ε, there is an N such that

$$|R_n(x_1)| = |s(x_1) - s_n(x_1)| < \varepsilon \quad for\ all\ n > N$$

Theory of the Power Series Method
Convergence Interval (수렴구간), Radius of Convergence (수렴반지름)

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- 수렴구간: 급수가 수렴하는 값들의 구간 ($|x - x_0| < R$ 의 형태로 나타남)

- 수렴반지름 (R):
  급수는 $|x - x_0| < R$ 인 모든 x 에 대하여 수렴하고,
  인 모든 x 에 대하여 발산할 때 $|x - x_0| > R$

$$R = \frac{1}{\lim_{m\to\infty} \sqrt[m]{|a_m|}} \qquad R = \frac{1}{\lim_{m\to\infty} \left| a_{m+1} / a_m \right|}$$

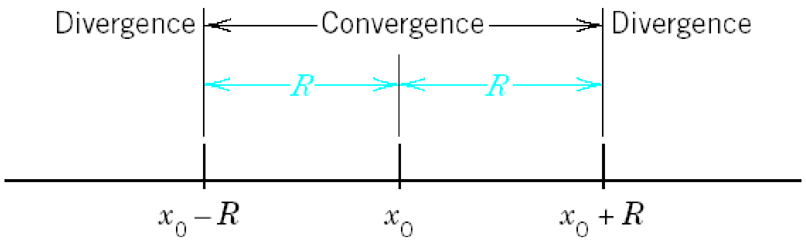

Fig. 103. Convergence interval (6) of a power series with center x_0

Theory of Power Series Method

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- Case 1: (useless) The series always converges at the center.
- Case 2. (usual) If there are further values of x for which the series converges, these values form an interval, called the convergence interval.
- Case 3. (best) The convergence interval may sometimes be infinite, that is, the series converges for all x.

Theory of Power Series Method

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- Example 1.

$$\sum_{m=0}^{\infty} m!x^m = 1 + x + 2x^2 + 6x^3 + \cdots$$

$$\frac{a_{m+1}}{a_m} = ?$$

R = 0, converges only at the center x

- Example 2.

$$\frac{1}{1-x} = \sum_{m=0}^{\infty} x^m = 1 + x + x^2 + x^3 + \cdots$$

R = 1, converges when $|x| < 1$

- Example 3.

$$e^x = \sum_{m=0}^{\infty} \frac{x^m}{m!} = 1 + x + \frac{x^2}{2!} + \cdots$$

$R = \infty$, converges for all x

Theory of Power Series Method

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- Example 4.

$$\sum_{m=0}^{\infty} \frac{(-1)^m}{8^m} x^{3m} = 1 - \frac{x^3}{8} + \frac{x^6}{64} - \frac{x^9}{512} + - \cdots$$

$$\left|\frac{a_{m+1}}{a_m}\right| = \frac{8^m}{8^{m+1}} = \frac{1}{8} \quad \Rightarrow \quad R = 8 \qquad |t| = \left|x^3\right| < 8$$

converges when $|x| < 2$

Theory of Power Series Method
Operations on Power Series

– Termwise Differentiation:

$$y = \sum_{m=0}^{\infty} a_m (x - x_0)^m \quad (|x - x_0| < R) \longrightarrow y' = \sum_{m=1}^{\infty} m a_m (x - x_0)^{m-1} \quad (|x - x_0| < R)$$

– Termwise Addition: $\sum_{m=0}^{\infty} a_m (x - x_0)^m + \sum_{m=0}^{\infty} b_m (x - x_0)^m = \sum_{m=0}^{\infty} (a_m + b_m)(x - x_0)^m$

– Termwise Multiplication:

$$\sum_{m=0}^{\infty} (a_0 b_m + a_1 b_{m-1} + \dots + a_m b_0)(x - x_0)^m$$
$$= a_0 b_0 + (a_0 b_1 + a_1 b_0)(x - x_0) + (a_0 b_2 + a_1 b_1 + a_2 b_0)(x - x_0)^2 + \dots$$

– If a power series has a positive radius of convergence and a sum that is identically zero throughout its interval of convergence, then each coefficient of the series must be zero.(Vanishing of All Coefficients)

Theory of Power Series Method

Existence of Power Series Solutions of ODEs. Real Analytic Functions.

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- Definition: Real Analytic Function
  - A real function f(x) is called analytic at a point $x=x_0$ if it can be represented by a power series in powers of $x-x_0$ with radius of convergence R>0
- Theorem1. Existence of Power Series Solutions

Existence of Power Series Solutions

If p, q, and r in (9) are analytic at $x = x_0$, then every solution of (9) is analytic at $x = x_0$ and can thus be represented by a power series in powers of $x - x_0$ with radius of convergence $R > 0$. Hence the same is true if $\tilde{h}$, $\tilde{p}$, $\tilde{q}$, and $\tilde{r}$ in (10) are analytic at $x = x_0$ and $\tilde{h}(x_0) \neq 0$.

$$y'' + p(x)y' + q(x)y = r(x) \qquad (9)$$

$$\tilde{h}(x)y'' + \tilde{p}(x)y' + \tilde{q}(x)y = \tilde{r}(x) \qquad (10)$$

Legendre's Equation. Legendre Polynomials $P_n(x)$

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- Legendre's Equation.

$$\left(1-x^2\right)y''-2xy'+n(n+1)y=0$$

n is a given real number

$$y=\sum_{m=0}^{\infty}a_m x^m,\quad y'=\sum_{m=1}^{\infty}m a_m x^{m-1},\quad y''=\sum_{m=2}^{\infty}m(m\text{-}1)a_m x^{m-2}$$

$$\Rightarrow\quad \left(1-x^2\right)\sum_{m=2}^{\infty}m(m\text{-}1)a_m x^{m-2}-2x\sum_{m=1}^{\infty}m a_m x^{m-1}+n(n+1)\sum_{m=0}^{\infty}a_m x^m=0$$

$$\Rightarrow\quad \sum_{m=2}^{\infty}m(m\text{-}1)a_m x^{m-2}-\sum_{m=2}^{\infty}m(m\text{-}1)a_m x^m-\sum_{m=1}^{\infty}2m a_m x^m+\sum_{m=0}^{\infty}n(n+1)a_m x^m=0$$

m-2 = *s*   *m* = *s*

$$\Rightarrow\quad \sum_{s=0}^{\infty}(s+2)(s+1)a_{s+2}x^s-\sum_{s=2}^{\infty}s(s\text{-}1)a_s x^s-\sum_{s=1}^{\infty}s a_s x^s+\sum_{s=0}^{\infty}n(n+1)a_s x^s=0$$

Legendre's Equation. Legendre Polynomials $P_n(x)$

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$$x^0: \; 2\cdot 1a_2 + n(n+1)a_0 = 0$$

$$x^1: \; 3\cdot 2a_3 + [-2 + n(n+1)]a_1 = 0$$

$$\vdots$$

$$(s+2)(s+1)a_{s+2} + [-s(s-1) - 2s + n(n+1)]a_s = 0$$

$$\therefore \; a_{s+2} = -\frac{(n-s)(n+s+1)}{(s+2)(s+1)}a_s \qquad (s = 0,\, 1,\, \cdots)$$

$$a_2 = -\frac{n(n+1)}{2!}a_0 \qquad a_3 = -\frac{(n-1)(n+2)}{3!}a_1$$

$$a_4 = -\frac{(n-2)(n+3)}{4\cdot 3}a_2 = \frac{(n-2)n(n+1)(n+3)}{4!}a_0 \qquad a_5 = -\frac{(n-3)(n+4)}{5\cdot 4}a_3 = \frac{(n-3)(n-1)(n+2)(n+4)}{5!}a_1$$

General solution: $y(x) = a_0 y_1(x) + a_1 y_2(x)$

$$y_1(x) = 1 - \frac{n(n+1)}{2!}x^2 + \frac{(n-2)n(n+1)(n+3)}{4!}x^4 - + \cdots$$

$$y_2(x) = x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!}x^5 - + \cdots$$

Legendre's Equation. Legendre Polynomials $P_n(x)$

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- Legendre Polynomials

$$a_n = \frac{(2n)!}{2^n (n!)^2} = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{n!}$$ 로 선택 ← $P_n(1)=1$ 이 되도록 선택

$$a_s = -\frac{(s+2)(s+1)}{(n-s)(n+s+1)} a_{s+2} \qquad (s \le n-2)$$

$$\Rightarrow \quad a_{n-2} = -\frac{n(n-1)}{2(2n-1)} a_n = -\frac{n(n-1)(2n)!}{2(2n-1)2^n (n!)^2}$$

$$= -\frac{n(n-1)2n(2n-1)(2n-2)!}{2(2n-1)2^n n(n-1)!n(n-1)(n-2)!} = -\frac{(2n-2)!}{2^n (n-1)!(n-2)!}$$

$$a_{n-4} = -\frac{(n-2)(n-3)}{4(2n-3)} a_{n-2} = \frac{(2n-4)!}{2^n 2!(n-2)!(n-4)!}$$

$$\therefore \quad a_{n-2m} = (-1)^m \frac{(2n-2m)!}{2^n m!(n-m)!(n-2m)!}$$

$$\Rightarrow \quad P_n(x) = \sum_{m=0}^{M} (-1)^m \frac{(2n-2m)!}{2^n m!(n-m)!(n-2m)!} x^{n-2m} \qquad \left(M = \frac{n}{2} \text{ 또는 } \frac{n-1}{2} \right)$$

Legendre's Equation. Legendre Polynomials $P_n(x)$

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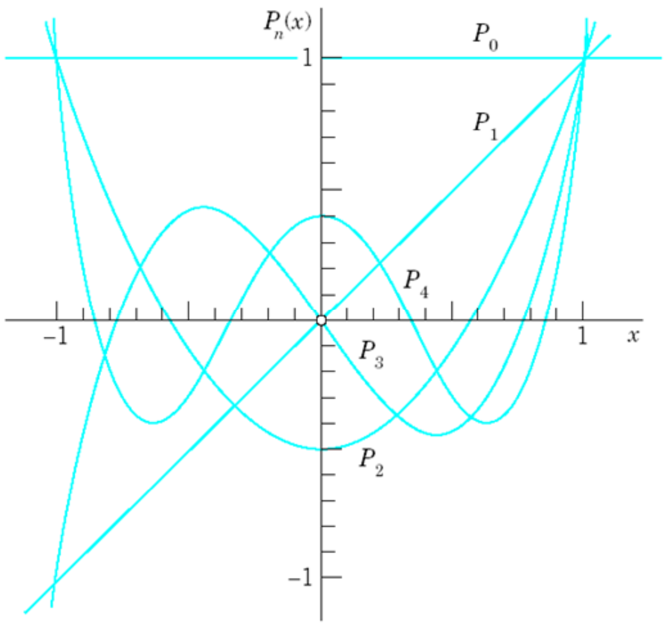
- Legendre Polynomials

$$P_0(x) = 1, \qquad P_1(x) = x,$$

$$P_2(x) = \frac{1}{2}\left(3x^2 - 1\right), \qquad P_3(x) = \frac{1}{2}\left(5x^3 - 3x\right),$$

$$P_4(x) = \frac{1}{8}\left(35x^4 - 30x^2 + 3\right) \qquad P_5(x) = \frac{1}{8}\left(63x^5 - 70x^3 + 15x\right)$$



Legendre's Equation. Legendre Polynomials $P_n(x)$

- Example. (problem set 5.3)

$$\left(1-x^2\right)y''-2xy'=0 \quad \leftarrow n=0$$

Frobenius Method

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- Theorem 1. Frobenius Method
  - Let *b*(*x*) and *c*(*x*) be any functions that are analytic at *x* = 0.

$$y'' + \frac{b(x)}{x} y' + \frac{c(x)}{x^2} y = 0$$

  - Then above ODE has at least one solution that can be represented in the form

$$y(x) = x^r \sum_{m=0}^{\infty} a_m x^m = x^r \left(a_0 + a_1 x + a_2 x^2 + \cdots\right) \qquad \left(a_0 \neq 0\right)$$

where the exponent *r* may be any number.

Frobenius Method

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$$x^2 y'' + xb_0 y' + c_0 y = 0$$

Euler-Cauchy eq.
← b, c가 상수일때

- Indicial Equation, Indicating the Form of Solutions

$$y'' + \frac{b(x)}{x} y' + \frac{c(x)}{x^2} y = 0 \xrightarrow{\text{Multiply } x^2} x^2 y'' + xb(x) y' + c(x) y = 0$$

$$\left(b(x) = b_0 + b_1 x + b_2 x^2 + \cdots, \quad c(x) = c_0 + c_1 x + c_2 x^2 + \cdots \right)$$

$$y(x) = x^r \sum_{m=0}^{\infty} a_m x^m = x^r \left(a_0 + a_1 x + a_2 x^2 + \cdots \right)$$

$$y'(x) = \sum_{m=0}^{\infty} (m+r) a_m x^{m+r-1} \qquad y''(x) = \sum_{m=0}^{\infty} (m+r)(m+r-1) a_m x^{m+r-2}$$

$$x^r \left[r(r-1) a_0 + \cdots \right] + \left(b_0 + b_1 x + \cdots \right) x^r \left[r a_0 + \cdots \right] + \left(c_0 + c_1 x + \cdots \right) x^r \left(a_0 + a_1 x + \cdots \right) = 0$$

최저차수 항의 계수 →

$$r(r-1) + b_0 r + c_0 = 0$$

Indicial equation

Frobenius Method

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- Theorem 2. **Frobenius Method. Basis of Solutions.**
  - Let r_1 and r_2 be the roots of the indicial equation. Then we have the following three cases.
  - Case 1. **Distinct Roots Not Differing by an Integer**. A basis is

$$y_1(x) = x^{r_1}\left(a_0 + a_1 x + a_2 x^2 + \cdots\right) \qquad y_2(x) = x^{r_2}\left(A_0 + A_1 x + A_2 x^2 + \cdots\right)$$

  - Case 2. **Double Root**. A basis is

$$y_1(x) = x^{r}\left(a_0 + a_1 x + a_2 x^2 + \cdots\right) \qquad y_2(x) = y_1(x)\ln x + x^{r}\left(A_1 x + A_2 x^2 + \cdots\right)$$

  - Case 3. **distinct Roots Differing by an Integer**. A basis is

$$y_1(x) = x^{r_1}\left(a_0 + a_1 x + a_2 x^2 + \cdots\right) \qquad y_2(x) = k y_1(x)\ln x + x^{r_2}\left(A_0 + A_1 x + A_2 x^2 + \cdots\right)$$

    where the roots are so denoted that $r_1 - r_2 > 0$

  - 결정방정식의 근이 결정되면, Frobenius 해법은 기술적으로 거듭제곱급수해법과 유사. 두 번째 해는 차수축소에 의해 보다 신속히 구할 수 있음.

Frobenius Method

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- Example 1. Euler-Cauchy Equtions.

$$x^2 y'' + b_0 x y' + c_0 y = 0$$

$$y(x) = x^r \sum_{m=0}^{\infty} a_m x^m = x^r \left(a_0 + a_1 x + a_2 x^2 + \cdots\right)$$

Special form

$$y = x^r$$

$$r(r-1) + b_0 r + c_0 = 0$$

Euler-Cauchy Equations

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- Euler-Cauchy Equations : $x^2 y'' + axy' + by = 0$

$$\downarrow \; y = x^m$$

  - Auxiliary Equation $m^2 + (a-1)m + b = 0$

- Three kinds of the general solution of the equation
  - Case 1 Two real roots $m_1,\ m_2 \quad \Rightarrow \quad y = c_1 x^{m_1} + c_2 x^{m_2}$
  - Case 2 A real double root $m = (1-a)/2 \quad \Rightarrow \quad y = (c_1 + c_2 \ln x) x^m$
  - Case 3 Complex conjugate roots

$$m = \mu \pm i\upsilon \quad \Rightarrow \quad y = x^{\mu}\left[A\cos(\upsilon \ln x) + B\sin(\upsilon \ln x)\right]$$

Frobenius Method

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- Example 2. $x(x-1)y''+(3x-1)y'+y=0$

$$y(x)=x^r\sum_{m=0}^{\infty}a_m x^m = x^r\left(a_0+a_1x+a_2x^2+\cdots\right)$$

$$\sum_{m=0}^{\infty}(m+r)(m+r-1)a_m x^{m+r}-\sum_{m=0}^{\infty}(m+r)(m+r-1)a_m x^{m+r-1}+3\sum_{m=0}^{\infty}(m+r)a_m x^{m+r}-\sum_{m=0}^{\infty}(m+r)a_m x^{m+r-1}+\sum_{m=0}^{\infty}a_m x^{m+r}=0$$

$$x^{r-1}:\left[-r(r-1)-r\right]a_0=0\Rightarrow r=0$$

$$x^s: s(s-1)a_s-(s+1)sa_{s+1}+3sa_s-(s+1)a_{s+1}+a_s=0 \quad \Rightarrow \quad a_{s+1}=a_s$$

$$a_0=1 \longrightarrow \therefore y_1(x)=\sum_{m=0}^{\infty}x^m=\frac{1}{1-x} \quad \left(|x|<1\right)$$

$$-\int p\,dx=-\int\frac{3x-1}{x(x-1)}dx=-\int\left(\frac{2}{x-1}+\frac{1}{x}\right)dx=-2\ln(x-1)-\ln x$$

$$u'=y_1^{-2}e^{-\int p\,dx}=\frac{(x-1)^2}{(x-1)^2x}=\frac{1}{x}, \qquad u=\ln x, \qquad \therefore y_2=uy_1=\frac{\ln x}{x-1}$$

Frobenius Method

Homogeneous Linear ODEs of 2nd order
Finding a basis if one solution is known

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- Method of reduction of order (by Lagrange)

$$\left(x^2 - x\right) y'' - xy' + y = 0$$

$$y_1 = x \longrightarrow y_2 = uy_1$$

Homogeneous
Finding a basis

- Method of reduction of order (by Lagrange)

$$y + p(x)y + q(x)y = 0$$

$$y = y_2 = uy_1 \qquad \left(y' = y_2' = u'y_1 + uy_1',\ y'' = y_2'' = u''y_1 + 2u'y_1' + uy_1''\right)$$

$$\Rightarrow \quad u''y_1 + u'(2y_1' + py_1) + u(y_1'' + py_1' + qy_1) = 0 \quad \Rightarrow \quad u'' + u'\frac{2y_1' + py_1}{y_1} = 0 \quad (\because\ y_1'' + py_1' + qy_1 = 0)$$

$$U = u', \quad U' = u'' \quad \Rightarrow \quad U + \left(2\frac{y_1'}{y_1} + p\right)U = 0$$

$$\Rightarrow \quad \frac{dU}{U} = -\left(2\frac{y_1'}{y_1} + p\right)dx \ \ \&\ \ \ln|U| = -2\ln|y_1| - \int p dx \quad \Rightarrow \quad \therefore\ U = \frac{1}{y_1^2}e^{-\int p dx}, \quad y_2 = uy_1 = y_1\int U dx$$

Frobenius Method

- Example 3. $\left(x^2 - x\right) y'' - xy' + y = 0$

Bessel's Equation. Bessel Functions *Jv*(x)
Introduction

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Bessel equation : $x^2 y'' + xy' + \left(x^2 - \nu^2\right) y = 0 \qquad \nu \geq 0$

- Applications: heat conduction, vibration…
- Cylindrical symmetry

Frobenius 해법 적용 : $y = \sum_{m=0}^{\infty} a_m x^{m+r}$과 그 도함수를 대입

$$\sum_{m=0}^{\infty} (m+r)(m+r-1) a_m x^{m+r} + \sum_{m=0}^{\infty} (m+r) a_m x^{m+r} + \sum_{m=0}^{\infty} a_m x^{m+r+2} - \nu^2 \sum_{m=0}^{\infty} a_m x^{m+r} = 0$$

$s = 0$일 때, $\quad r(r-1)a_0 + ra_0 - \nu^2 a_0 = 0$

$s = 1$일 때, $\quad (r+1)ra_1 + (r+1)a_1 - \nu^2 a_1 = 0$

$s = 2,\ 3,\ \cdots$일 때, $\quad (s+r)(s+r-1)a_s + (s+r)a_s + a_{s-2} - \nu^2 a_s = 0$

$\therefore$ indicial equation (결정방정식) : $(r+\nu)(r-\nu) = 0$

$r_1 = \nu (\geq 0), \quad r_2 = -\nu$

Bessel's Equation. Bessel Functions *Jv*(x)

Introduction

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$r = r_1 = \nu$ 에 대한 계수 점화(Coefficient Recursion)

$$(2\nu+1)a_1 = 0 \quad \Rightarrow \quad a_1 = 0$$

$$(s+2\nu)sa_s + a_{s-2} = 0 \qquad \Rightarrow \qquad a_3 = a_5 = \cdots = 0$$

$s = 2m$을 대입하면 $(2m+2\nu)2ma_{2m} + a_{2m-2} = 0$

$$\Rightarrow \quad a_{2m} = -\frac{1}{2^2 m(m+\nu)} a_{2m-2}, \quad m = 1,\ 2,\ \cdots$$

$$\Rightarrow \quad a_2 = -\frac{1}{2^2(\nu+1)} a_0$$

$$a_4 = -\frac{1}{2^2 2(\nu+2)} a_2 = \frac{1}{2^4 2!(\nu+1)(\nu+2)} a_0$$

$$\Rightarrow \quad \therefore\ a_{2m} = \frac{(-1)^m}{2^{2m} m!(\nu+1)(\nu+2)\cdots(\nu+m)} a_0, \qquad m = 1,\ 2,\ \cdots$$

Bessel's Equation. Bessel Functions *Jv*(x)

For Integer *v* = n

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정수 $\nu = n$에 대한 Bessel 함수 $J_n(x)$

$$a_{2m} = \frac{(-1)^m}{2^{2m} m!(n+1)(n+2)\cdots(n+m)} a_0, \qquad m = 1,\ 2,\ \cdots$$

$a_0 = \frac{1}{2^n n!}$으로 선택하면 $a_{2m} = \frac{(-1)^m}{2^{2m+n} m!(n+m)!}, \qquad m = 1,\ 2,\ \cdots$

$$J_n(x) = x^n \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+n} m!(n+m)!}$$

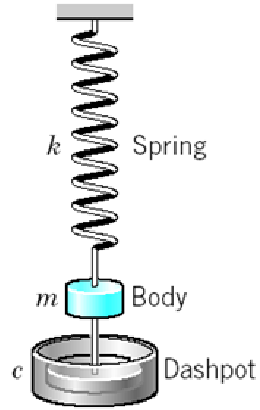
Bessel Function of the 1st kind of order n
(n차 제1종 Bessel 함수)

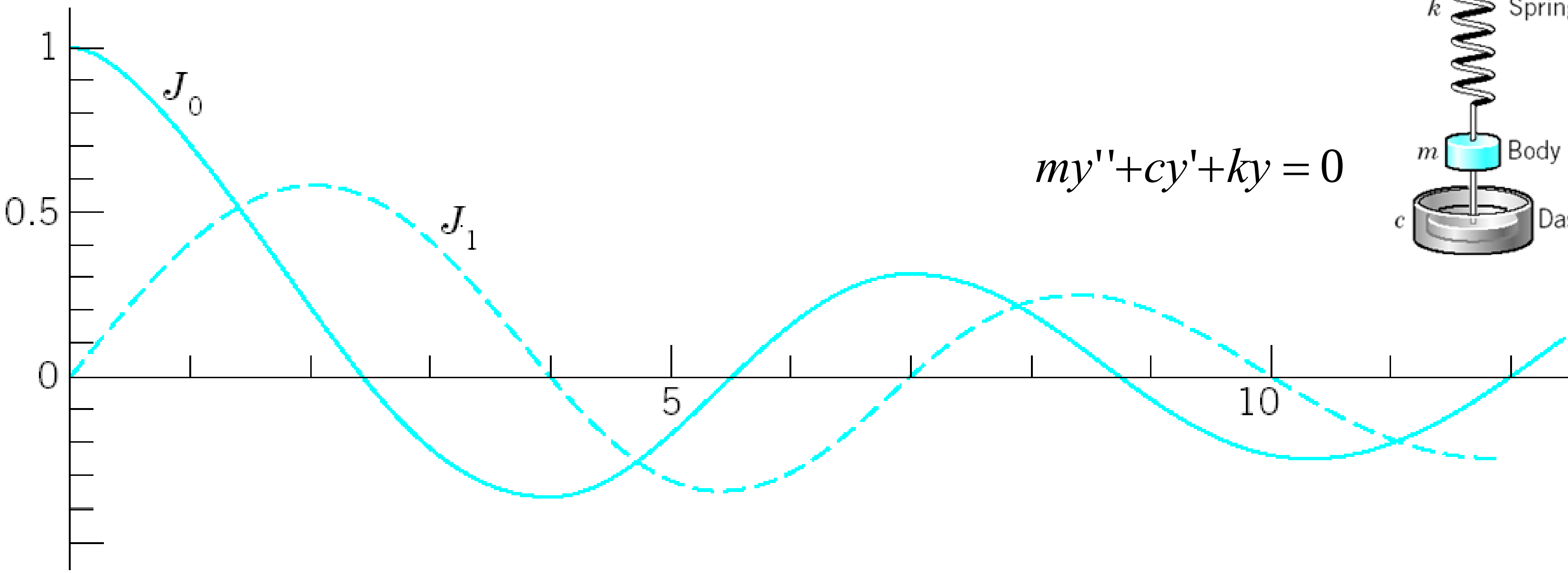
Bessel's Equation. Bessel Functions *Jv*(x)
For Integer *v* = n

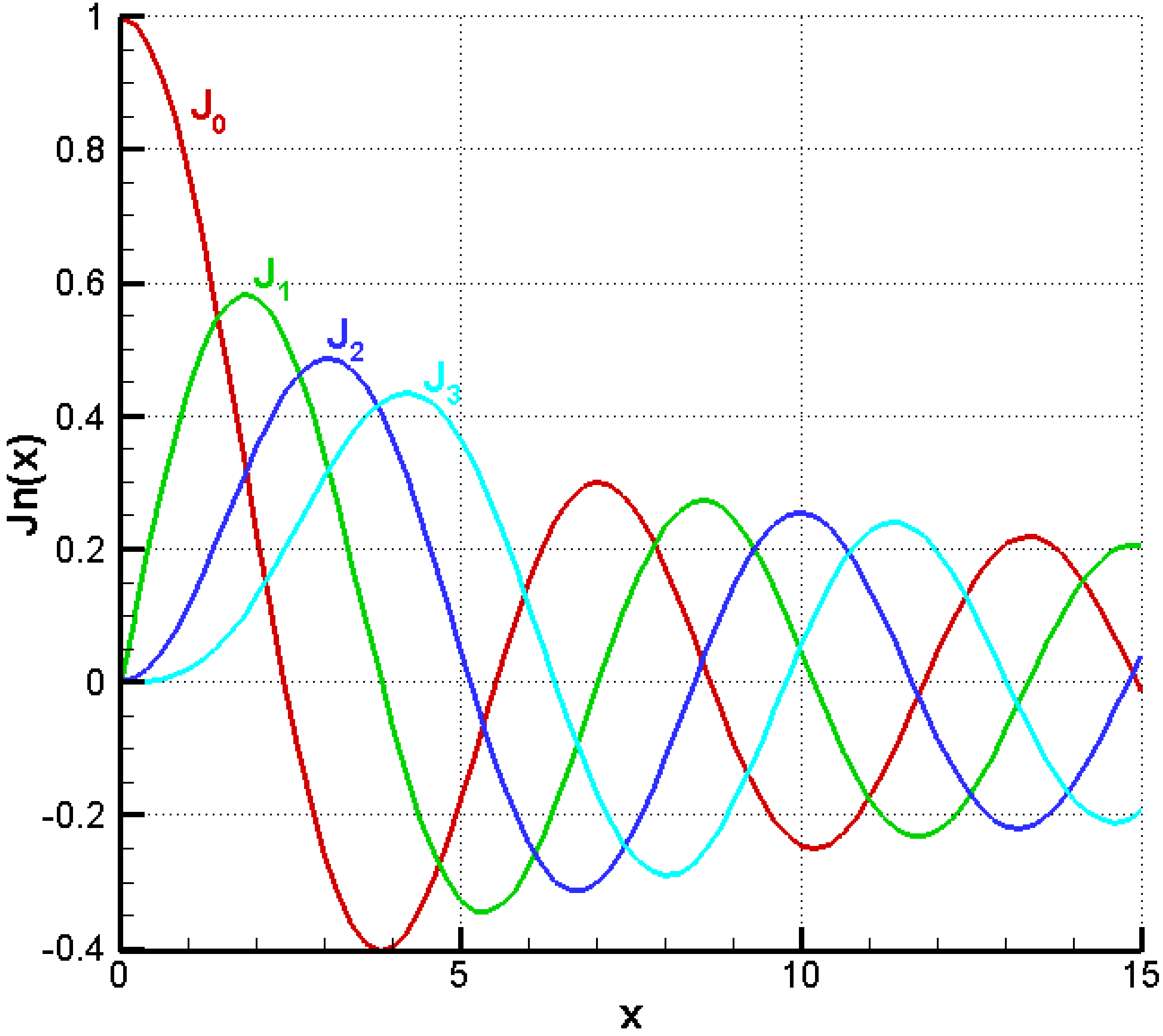
$$J_n(x) = x^n \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+n} m!(n+m)!} \sim \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right) \quad \text{for large } x$$

$$x^2 y'' + xy' + (x^2 - \nu^2)y = 0$$

$$my'' + cy' + ky = 0$$






J0
J1
J2
J3
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
Jn(x)
0
5
10
15
x

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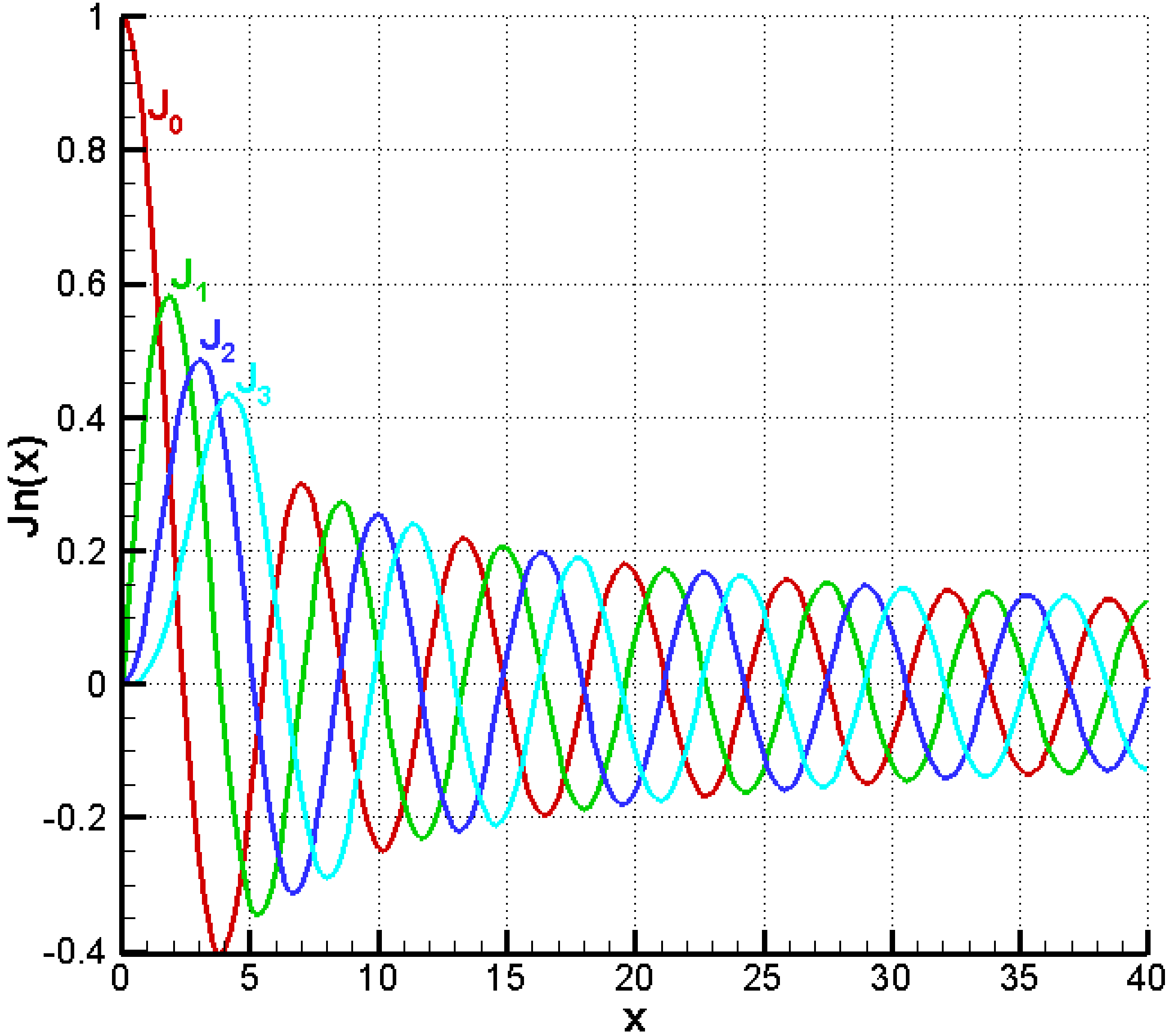
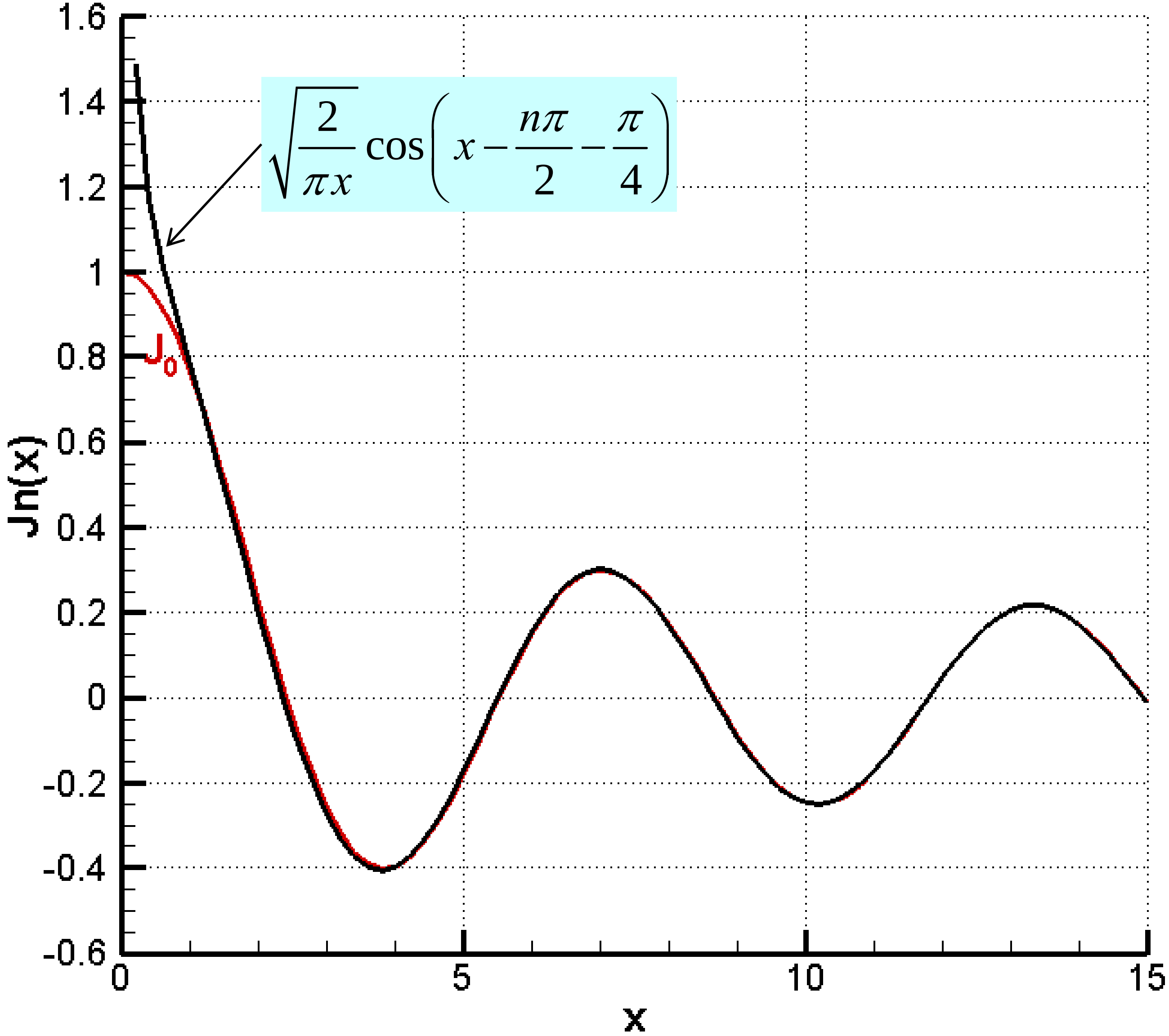
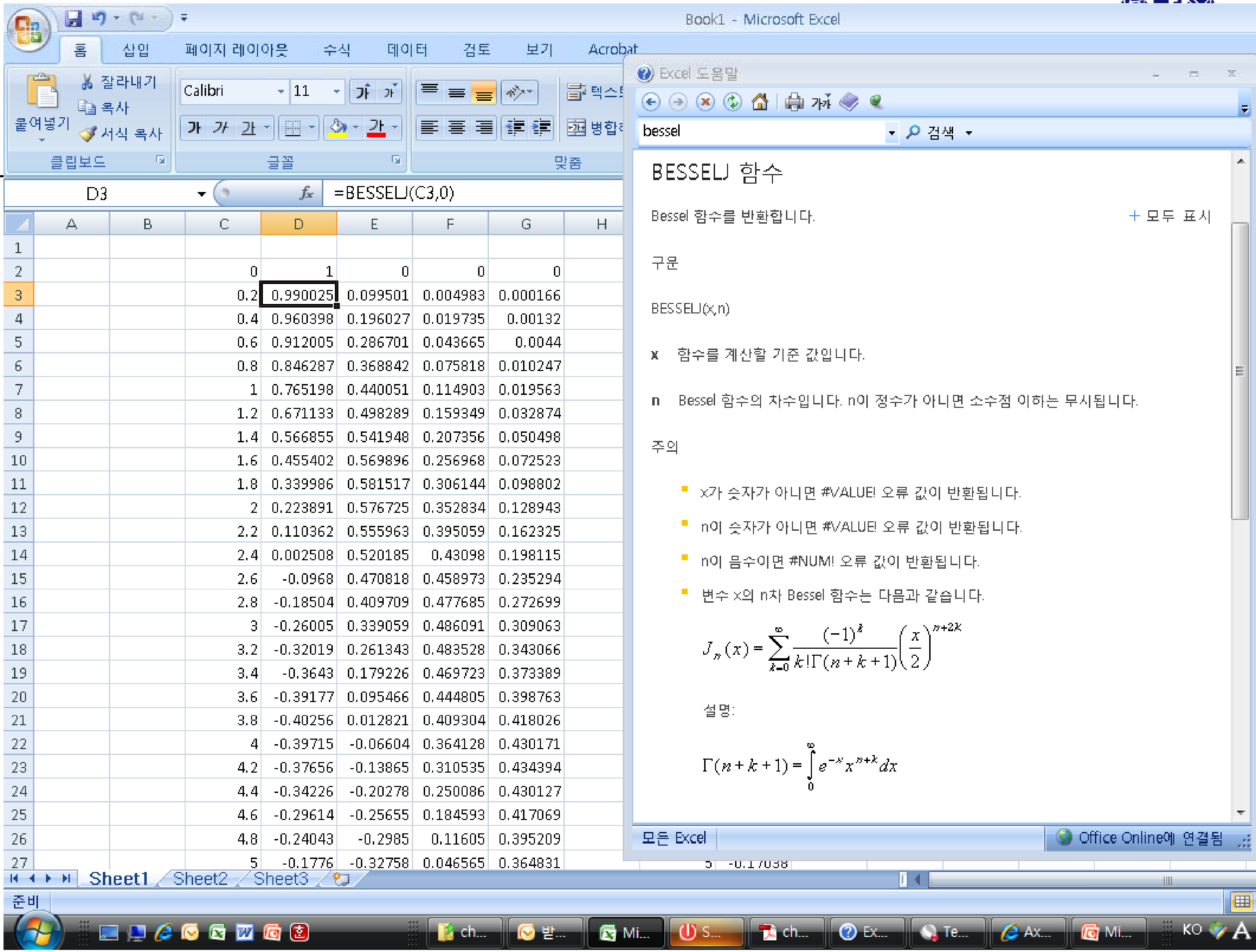



$\sqrt{\frac{2}{\pi x}}\cos\left(x-\frac{n\pi}{2}-\frac{\pi}{4}\right)$
J_0
Jn(x)
x
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
0
5
10
15



D3 =BESSELJ(C3,0)

	A	B	C	D	E	F	G	H
1								
2			0	1	0	0	0	
3			0.2	0.990025	0.099501	0.004983	0.000166	
4			0.4	0.960398	0.196027	0.019735	0.00132	
5			0.6	0.912005	0.286701	0.043665	0.0044	
6			0.8	0.846287	0.368842	0.075818	0.010247	
7			1	0.765198	0.440051	0.114903	0.019563	
8			1.2	0.671133	0.498289	0.159349	0.032874	
9			1.4	0.566855	0.541948	0.207356	0.050498	
10			1.6	0.455402	0.569896	0.256968	0.072523	
11			1.8	0.339986	0.581517	0.306144	0.098802	
12			2	0.223891	0.576725	0.352834	0.128943	
13			2.2	0.110362	0.555963	0.395059	0.162325	
14			2.4	0.002508	0.520185	0.43098	0.198115	
15			2.6	-0.0968	0.470818	0.458973	0.235294	
16			2.8	-0.18504	0.409709	0.477685	0.272699	
17			3	-0.26005	0.339059	0.486091	0.309063	
18			3.2	-0.32019	0.261343	0.483528	0.343066	
19			3.4	-0.3643	0.179226	0.469723	0.373389	
20			3.6	-0.39177	0.095466	0.444805	0.398763	
21			3.8	-0.40256	0.012821	0.409304	0.418026	
22			4	-0.39715	-0.06604	0.364128	0.430171	
23			4.2	-0.37656	-0.13865	0.310535	0.434394	
24			4.4	-0.34226	-0.20278	0.250086	0.430127	
25			4.6	-0.29614	-0.25655	0.184593	0.417069	
26			4.8	-0.24043	-0.2985	0.11605	0.395209	
27			5	-0.1776	-0.32758	0.046565	0.364831	

BESSELJ 함수

Bessel 함수를 반환합니다.

구문

BESSELJ(x,n)

x 함수를 계산할 기준 값입니다.

n Bessel 함수의 차수입니다. n이 정수가 아니면 소수점 이하는 무시됩니다.

주의

- x가 숫자가 아니면 #VALUE! 오류 값이 반환됩니다.
- n이 숫자가 아니면 #VALUE! 오류 값이 반환됩니다.
- n이 음수이면 #NUM! 오류 값이 반환됩니다.
- 변수 x의 n차 Bessel 함수는 다음과 같습니다.

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(n+k+1)} \left(\frac{x}{2}\right)^{n+2k}$$

설명:

$$\Gamma(n+k+1) = \int_0^{\infty} e^{-x} x^{n+k} dx$$

Bessel's Equation. Bessel Functions *Jv*(x)
For any *v* >= 0. Gamma Function

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- Gamma Function (감마함수)

$$\Gamma(\nu+1)=\int_0^\infty e^{-t}t^{\nu}dt = -e^{-t}t^{\nu}\Big|_0^\infty + \nu\int_0^\infty e^{-t}t^{\nu-1}dt$$

$$\Gamma(\nu)=\int_0^\infty e^{-t}t^{\nu-1}dt \qquad (\nu>0)$$

  - Functional Relationship (감마함수의 성질)

$$\Gamma(\nu+1)=\nu\Gamma(\nu), \quad \Gamma(n+1)=n! \qquad (n=0,1,\cdots)$$

$$a_0=\frac{1}{2^n n!}\rightarrow\frac{1}{2^\nu\Gamma(\nu+1)} \quad\rightarrow\quad a_{2m}=\frac{(-1)^m}{2^{2m+\nu}m!\Gamma(\nu+m+1)}, \qquad m=1,\ 2,\ \cdots$$

$$\therefore\ J_\nu(x)=x^\nu\sum_{m=0}^{\infty}\frac{(-1)^m x^{2m}}{2^{2m+\nu}m!\Gamma(\nu+m+1)}$$: ν 차 제 1종 **Bessel** 함수

Bessel's Equation. Bessel Functions *Jv*(x)
Gamma Function

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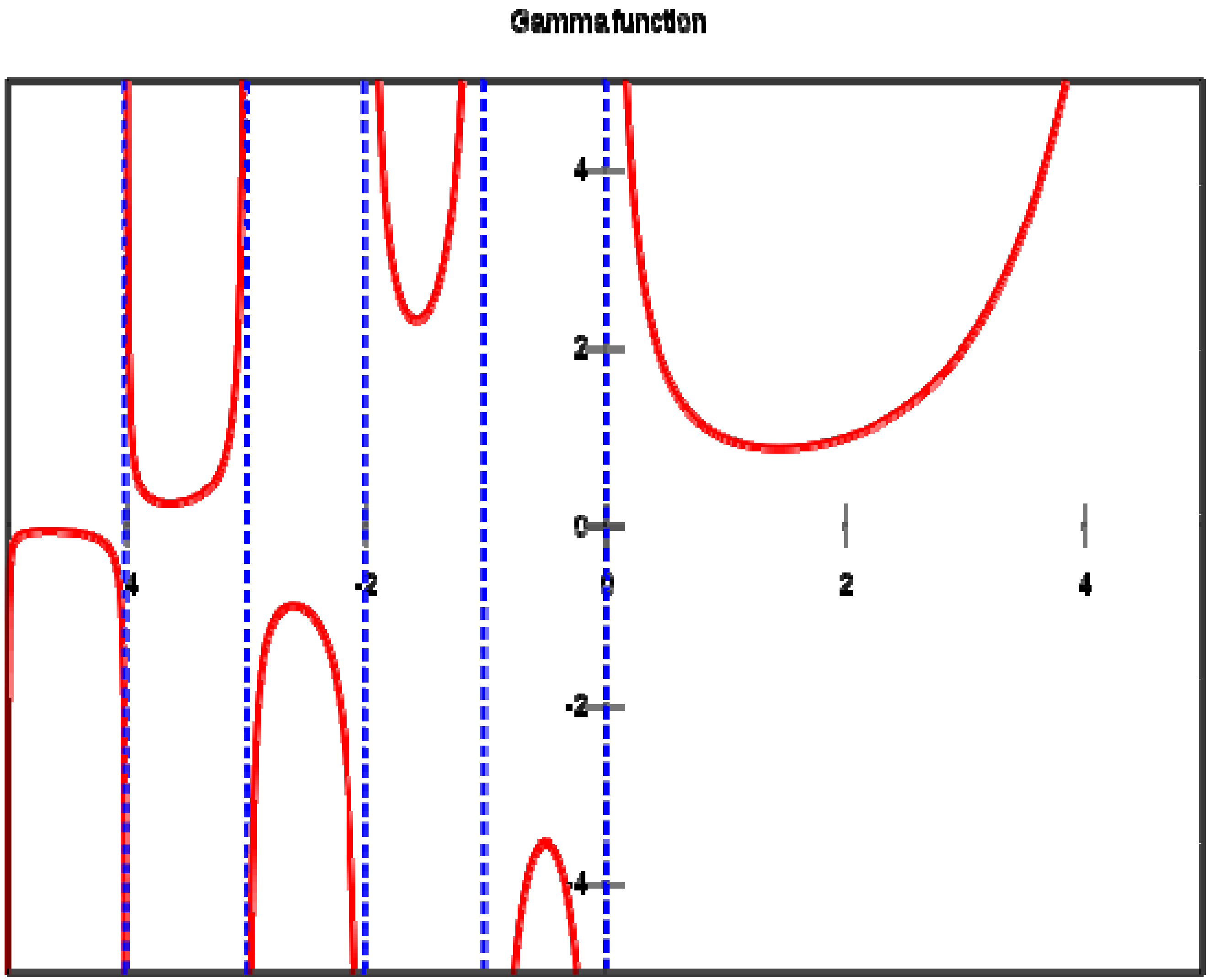

www.en.wikipedia.org

Bessel's Equation. Bessel Functions $J\nu(x)$
General Solution for noninteger v

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- Second linearly independent solution

$$J_{-\nu}(x) = x^{-\nu} \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m-\nu} m! \Gamma(m-\nu+1)}$$

- Theorem 1. General Solution of Bessel Equation (v is NOT an integer)

$$y(x) = c_1 J_\nu(x) + c_2 J_{-\nu}(x) \qquad x \neq 0$$

- Theorem 2. Linear Dependence of J_n and J_{-n} (v is an integer)

$$J_{-n}(x) = (-1)^n J_n(x)$$

$$J_{-n}(x) = \sum_{m=n}^{\infty} \frac{(-1)^m x^{2m-n}}{2^{2m-n} m!(m-n)!} = \sum_{s=0}^{\infty} \frac{(-1)^{n+s} x^{2s+n}}{2^{2s+n} (n+s)! s!} \quad (m = n+s)$$

Frobenius Method

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- Theorem 2. **Frobenius Method. Basis of Solutions.**
  - Let r_1 and r_2 be the roots of the indicial equation. Then we have the following three cases.
  - **Case 1. Distinct Roots Not Differing by an Integer**. A basis is

$$y_1(x) = x^{r_1}\left(a_0 + a_1 x + a_2 x^2 + \cdots\right) \qquad y_2(x) = x^{r_2}\left(A_0 + A_1 x + A_2 x^2 + \cdots\right)$$

  - **Case 2. Double Root**. A basis is

$$y_1(x) = x^{r}\left(a_0 + a_1 x + a_2 x^2 + \cdots\right) \qquad y_2(x) = y_1(x)\ln x + x^{r}\left(A_1 x + A_2 x^2 + \cdots\right)$$

  - **Case 3. distinct Roots Differing by an Integer**. A basis is

$$y_1(x) = x^{r_1}\left(a_0 + a_1 x + a_2 x^2 + \cdots\right) \qquad y_2(x) = k y_1(x)\ln x + x^{r_2}\left(A_0 + A_1 x + A_2 x^2 + \cdots\right)$$

    where the roots are so denoted that $r_1 - r_2 > 0$

  - 결정방정식의 근이 결정되면, Frobenius 해법은 기술적으로 거듭제곱급수해법과 유사. 두 번째 해는 차수축소에 의해 보다 신속히 구할 수 있음.

Bessel's Equation. Bessel Functions *Jv*(x)
Properties from Series

- Derivatives and recursions

$$\left[x^{\nu} J_{\nu}(x)\right]' = x^{\nu} J_{\nu-1}(x)$$

$$\left[x^{-\nu} J_{\nu}(x)\right]' = -x^{-\nu} J_{\nu+1}(x)$$

$$J_{\nu-1}(x) + J_{\nu+1}(x) = \frac{2\nu}{x} J_{\nu}(x)$$

$$J_{\nu-1}(x) - J_{\nu+1}(x) = 2J_{\nu}'(x)$$

$$J_{\nu}(x) = x^{\nu} \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+\nu} m! \Gamma(\nu+m+1)}$$

Bessel's Equation. Bessel Functions *Jv*(x)
Elementary *Jv* for half-integer order v

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Elementray Bessel Functions J_ν for Half integer order (반정수 차수) ν

Bessel function J_ν of order $\nu = \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \cdots \rightarrow$ elementary function(초등함수)

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x, \qquad J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos x$$

$$J_{3/2}(x) = \sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x} - \cos x\right), \qquad J_{-3/2}(x) = -\sqrt{\frac{2}{\pi x}}\left(\frac{\cos x}{x} + \sin x\right)$$

$$\cos x = \sum_{m=0}^{\infty}\frac{(-1)^m x^{2m}}{(2m)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - + \cdots$$

$$\sin x = \sum_{m=0}^{\infty}\frac{(-1)^m x^{2m+1}}{(2m+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - + \cdots$$

Bessel function of the second kind Y*v*(x)
Introduction (when *v* is 0)

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- n = 0, Bessel function of the second kind $Y_0(x)$

$$xy'' + y' + xy = 0 \quad \leftarrow \text{n = 0 (double root)}$$

first solution : $J_0(x)$

second solution: $y_2(x) = J_0(x)\ln x + \sum_{m=1}^{\infty} A_m x^m$

$$y_2'(x) = J_0'\ln x + \frac{J_0}{x} + \sum_{m=1}^{\infty} mA_m x^{m-1}, \quad y_2''(x) = J_0''\ln x + \frac{2J_0'}{x} - \frac{J_0}{x^2} + \sum_{m=1}^{\infty} m(m-1)A_m x^{m-2}$$

$$\Rightarrow \quad 2J_0' + \sum_{m=1}^{\infty} m(m-1)A_m x^{m-1} + \sum_{m=1}^{\infty} mA_m x^{m-1} + \sum_{m=1}^{\infty} A_m x^{m+1} = 0$$

$$J_0(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m}(m!)^2} \quad \Rightarrow \quad J_0'(x) = \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m-1}}{2^{2m-1} m!(m-1)!}$$

$$\Rightarrow \quad \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m-1}}{2^{2m-2} m!(m-1)!} + \sum_{m=1}^{\infty} m^2 A_m x^{m-1} + \sum_{m=1}^{\infty} A_m x^{m+1} = 0$$

Bessel function of the second kind Y*v*(x)
Introduction (when *v* is 0)

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$$\sum_{m=1}^{\infty}\frac{(-1)^m x^{2m-1}}{2^{2m-2}m!(m-1)!}+\sum_{m=1}^{\infty}m^2A_m x^{m-1}+\sum_{m=1}^{\infty}A_m x^{m+1}=0$$

$$\Rightarrow \quad A_1=0$$

x^{2s} 의 계수들의 합: $(2s+1)^2 A_{2s+1}+A_{2s-1}=0 \quad (s=1,\ 2,\ \cdots)$

$$\Rightarrow \quad A_3=A_5=\cdots=0,$$

x^{2s+1} 의 계수들의 합: $\dfrac{(-1)^{s+1}}{2^{2s}(s+1)!s!}+(2s+2)^2 A_{2s+2}+A_{2s}=0$

$$\Rightarrow \quad s=0:\ -1+4A_2=0,\quad A_2=1/4$$

$$\Rightarrow \quad s=1:\ 1/8+16A_4+A_2=0,\quad A_4=-3/128$$

$$\Rightarrow \quad A_{2m}=\frac{(-1)^{m-1}}{2^{2m}(m!)^2}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{m}\right)=\frac{(-1)^{m-1}h_m}{2^{2m}(m!)^2}\quad (m=1,\ 2,\ \cdots)$$

$$\therefore\ y_2(x)=J_0(x)\ln x+\sum_{m=1}^{\infty}\frac{(-1)^{m-1}h_m}{2^{2m}(m!)^2}x^{2m}=J_0(x)\ln x+\frac{1}{4}x^2-\frac{3}{128}x^4+\frac{11}{13824}x^6-+\cdots$$

Bessel function of the second kind Y*v*(x)

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- Bessel function of the second kind of order zero (or Neumann's function of order zero)

$$Y_0(x) = \frac{2}{\pi}\left[J_0(x)\left(\ln\frac{x}{2}+\gamma\right)+\sum_{m=1}^{\infty}\frac{(-1)^{m-1}h_m}{2^{2m}(m!)^2}x^{2m}\right]$$

$$y_2 \to a(y_2 + bJ_0),\quad a = 2/\pi,\quad b = \gamma - \ln 2$$

$$\gamma = 1 + \frac{1}{2} + \cdots + \frac{1}{s} - \ln s$$

γ: Euler constant

- Bessel function of the second kind of order v (or Neumann's function of order v) for all v.

$$Y_\nu(x) = \frac{1}{\sin\nu\pi}\left[J_\nu(x)\cos\nu\pi - J_{-\nu}(x)\right]$$

$$Y_n(x) = \lim_{\nu\to n} Y_\nu(x) = \frac{2}{\pi}J_n(x)\left(\ln\frac{x}{2}+\gamma\right)+\frac{x^n}{\pi}\sum_{m=0}^{\infty}\frac{(-1)^{m-1}(h_m+h_{m+n})}{2^{2m+n}m!(m+n)!}x^{2m} - \frac{x^{-n}}{\pi}\sum_{m=0}^{n-1}\frac{(n-m-1)!}{2^{2m-n}m!}x^{2m}$$

- General Solution of Bessel's Equation (for all v, and x>0)

$$y(x) = C_1 J_\nu(x) + C_2 Y_\nu(x)$$

Bessel function of the second kind Y*v*(x)

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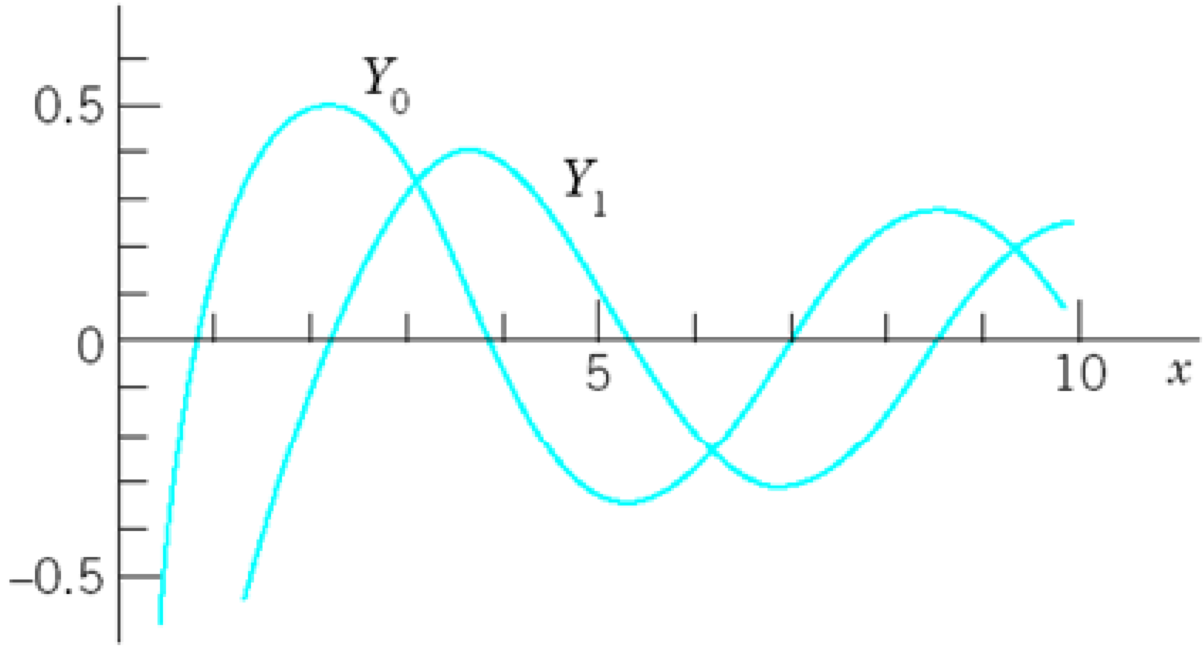

Bessel function of the second kind $Y_\nu(x)$

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Bessel equation : $x^2y''+xy'+\left(x^2-\nu^2\right)y=0 \qquad \nu\geq 0$

v = not integer

$$y(x)=c_1J_\nu(x)+c_2J_{-\nu}(x) \qquad x\neq 0$$

or

$$y(x)=C_1J_\nu(x)+C_2Y_\nu(x)$$

v = integer

$$y(x)=C_1J_n(x)+C_2Y_n(x)$$

- Sturm-Liouville Problems (← boundary value problem)
  - Sturm-Liouville Equation:

$$\left[p(x)y'\right]' + \left[q(x) + \lambda r(x)\right]y = 0$$

λ: parameter (매개변수)

  - Legendre's, Bessel's, and other ODEs of importance of engineering can be written as S-L equation.
  - Sturm-Liouville Boundary Conditions

$$k_1 y(a) + k_2 y'(a) = 0$$
$$l_1 y(b) + l_2 y'(b) = 0$$

k_1과 k_2 둘다 0은 아님.
l_1과 l_2 둘다 0은 아님.

  - p, q, r, and p' are continuous, on a ≤ x ≤ b, and

$$r(x) > 0$$

Sturm-Liouville Problems. Orthogonal Functions
Introduction

$$\left[p(x)y'\right]' + \left[q(x) + \lambda r(x)\right]y = 0$$

- Ex. 1 Legendre's and Bessel's Equations are Sturm-Liouville Equations
  - Legendre's equation

$$\left(1-x^2\right)y'' - 2xy' + n(n+1)y = 0 \quad \longrightarrow \quad \left[\left(1-x^2\right)y'\right]' + \lambda y = 0, \quad \lambda = n(n+1)$$

$$p = 1 - x^2, \quad q = 0, \quad r = 1$$

  - Bessel's equation

$$\tilde{x}^2 \ddot{y} + \tilde{x}\dot{y} + \left(\tilde{x}^2 - n^2\right)y = 0 \quad \left(\dot{y} = \frac{dy}{dt}\right) \quad \xrightarrow{\text{set } \tilde{x} = kx} \quad x^2 y'' + xy' + \left(k^2x^2 - n^2\right)y = 0$$

$$\xrightarrow{\text{division by } x} \quad \left[xy'\right]' + \left(-\frac{n^2}{x} + \lambda x\right)y = 0$$

$$p = x, \quad q = -n^2/x, \quad r = x$$

- Eigenfunctions, Eigenvalues
  - Eigenfunctions y(x): solution of S-L Equation without being zero (trivial solution: 무용한 해)
  - λ: eigenvalues (고유값)of S-L problem

Sturm-Liouville Problems. Orthogonal Functions

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Introduction

- Ex.2 Trigonometric Functions as Eigenfunctions. Vibrating String.
  - Find the eigenvalues and eigenfunctions of the Sturm-Liouville problem

$$y'' + \lambda y = 0, \qquad y(0) = 0, \qquad y(\pi) = 0$$

$$p = 1, \ \ q = 0, \ \ r = 1, \text{ and } a = 0, \ \ b = \pi, \ \ k_1 = l_1 = 1, \ \ k_2 = l_2 = 0$$

  - Case 1. Negative eigenvalue ($\lambda = -\nu^2$). General solution of ODE $y(x) = c_1 e^{\nu x} + c_2 e^{-\nu x}$
    Apply Boundary condition $(c_1 = c_2 = 0)$: $y \equiv 0$
  - Case 2. $\lambda = 0$ $\qquad y \equiv 0$
  - Case 3. Positive eigenvalue ($\lambda = \nu^2$). General solution of ODE $y(x) = A\cos\nu x + B\sin\nu x$
    Apply boundary conditions. $y(0) = A = 0$ $\quad y(\pi) = B\sin\nu\pi = 0 \quad \text{thus} \quad \nu = 0, \ \pm 1, \ \pm 2, \ \cdots$
    For v = 0, $y \equiv 0$
    For $\lambda = \nu^2 = 1, 4, 9, 16, \cdots, \ y(x) = \sin\nu x \quad (\nu = 1, 2, 3, 4, \cdots)$
    Hence the eigenvalues of the problem are $\lambda = \nu^2 (\nu = 1, 2, 3, 4, \cdots)$ and corresponding eigenfunctions are $y(x) = \sin\nu x \quad (\nu = 1, 2, 3, 4, \cdots)$

Sturm-Liouville Problems. Orthogonal Functions

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Introduction

- Existence of eigenvalues (고유값의 존재성)
  - Eigenvalues of a Sturm-Liouville , even infinitely many, exist under rather general conditions on *p*, *q*, *r*.
  - E.g., 1,4, 9,… in ex.2
- Reality of Eigenvalues (고유값의 실수성)
  - If *p*, *q*, *r*, and *p*\` are real-valued and continuous on the interval $a \le x \le b$ and *r* is positive throughout that interval, then all the eigenvalues of the Sturm-Liouville problem are real.
  - 고유값이 주파수 등과 같은 물리적인 값에 연관

Sturm-Liouville Problems. Orthogonal Functions

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Introduction

- Orthogonality (직교성)
  - Functions $y_1(x), y_2(x), \cdots$ defined on some interval $a \le x \le b$ are called **orthogonal** in this interval with respect to the **weight function** *r*(*x*) > 0 if for all *m* and all *n* different from *m*,

$$\int_a^b r(x)\, y_m(x) y_n(x)\, dx = 0 \quad (m \ne n)$$

  - The **norm** $\|y_m\|$ of y_m is defined by

$$\|y_m\| = \sqrt{\int_a^b r(x)\, y_m^2(x) dx}$$

  - Note that this is the square root of the integral with *n* = *m*.
  - The functions $y_1, y_2, \cdots$ are called **orthonormal** (정규직교) on $a \le x \le b$ if they are orthogonal on this interval and all have norm 1.
  - If *r*(*x*) = 1, we more briefly call the functions orthogonal instead of orthogonal with respect to *r*(*x*) = 1; similarly for orthonormality. Then

$$\int_a^b y_m(x) y_n(x)\, dx = 0 \quad (m \ne n), \qquad \|y_m\| = \sqrt{\int_a^b y_m^2(x) dx}$$

Sturm-Liouville Problems. Orthogonal Functions

Introduction

- Example 3. Orthogonal Functions. Orthonormal (정규직교) functions

$$y_m = \sin mx, \quad m = 1, 2, \cdots$$

$$\int_{-\pi}^{\pi} y_m(x) y_n(x) dx = \int_{-\pi}^{\pi} \sin mx \sin nx dx = 0$$

$$\|y_m\|^2 = \int_{-\pi}^{\pi} \sin^2 mx dx = \pi$$

$$\frac{\sin mx}{\sqrt{\pi}}$$

Sturm-Liouville Problems. Orthogonal Functions

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Introduction

- Theorem 1. Orthogonality of Eigenfunctions.
  - Suppose that Suppose that the functions *p*, *q*, *r*, and *p*` in the Sturm-Liouville equation are real-valued and continuous and *r*(*x*) > 0 on the interval $a \le x \le b$. Let $y_m(x)$ and $y_n(x)$ be eigenfunctions of the Sturm-Liouville problem that correspond to different eigenvalues λ_m and λ_n, respectively. Then y_m, y_n are orthogonal on that interval with respect to the weight functions *r*, that is,

$$\int_a^b r(x)\, y_m(x) y_n(x)\, dx = 0$$

  - If *p*(*a*) = 0, then Sturm-Liouville first boundary condition can be dropped from the problem. If *p*(*b*) = 0, then Sturm-Liouville second boundary condition can be dropped.
    If *p*(*a*) = *p*(*b*), then Sturm-Liouville boundary condition can be replaced by the "**periodic boundary conditions**"

$$y(a) = y(b), \quad y'(a) = y'(b)$$

Sturm-Liouville Problems. Orthogonal Functions

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Introduction

- Example 4. Application of Theorem 1. Vibrating elastic string

$$y_m = \sin mx, \quad m = 1, 2, \cdots \qquad \int_{-\pi}^{\pi} y_m(x) y_n(x)\,dx = \int_{-\pi}^{\pi} \sin mx \sin nx\,dx = 0$$

- Example 5. Orthogonality of the Legendre Polynomials

$$\left(1-x^2\right) y'' - 2xy' + n(n+1) y = 0 \longrightarrow \left[\left(1-x^2\right) y'\right]' + \lambda y = 0, \quad \lambda = n(n+1)$$

$$\int_{-1}^{1} P_m(x) P_n(x)\,dx = 0$$

- Example 6. Orthogonality of the Bessel Functions Jn(x)

Sturm-Liouville Problems. Orthogonal Functions

Introduction

- **Theorem 2 Orthogonality of Bessel Functions**

For each fixed nonnegative integer *n* the sequence of Bessel functions of the first kind $J_n(k_{n,1}x),\ J_n(k_{n,2}x),\ \cdots$ forms an orthogonal set on the interval $0 \le x \le R$ with respect to the weight function *r*(*x*) = *x*, that is,

$$\int_0^R x J_n(k_{n,m}x) J_n(k_{n,j}x)\, dx = 0 \qquad (j \neq m,\ n \text{ fixed})$$

Orthogonal Eigenfunction Expansions

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- Standard Notation for Orthogonality and Orthonormality
  - For orthonormal functions $y_0,\ y_1,\ y_2,\ \cdots$ with respect to weight function $r(x)$ (> 0) on

$$(y_m\ ,\ y_n) = \int_a^b r(x) y_m(x)\, y_n(x)\, dx = \delta_{mn} = \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases}$$

Kronecker's delta

$$\|y_m\| = \sqrt{(y_m, y_m)} = \sqrt{\int_a^b r(x)\, {y_m}^2(x)\, dx}$$

Orthogonal Eigenfunction Expansions

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- Orthogonal Series (직교전개)
  - Orthogonal expansion or generalized Fourier series (일반화된 푸리에급수)

$$f(x)=\sum_{m=0}^{\infty} a_m y_m(x) = a_0 y_0(x) + a_1 y_1(x) + \cdots$$

If ym = eigenfunction of SL equation, eigenfunction expansion

$$(f, y_n) = \int_a^b r f y_n dx = \int_a^b r\left(\sum_{m=0}^{\infty} a_m y_m\right) y_n dx = \sum_{m=0}^{\infty} a_m (y_m, y_n)$$

$$a_n (y_n, y_n) = a_n \|y_n\|^2$$

$$a_m = \frac{(f, y_m)}{\|y_m\|^2} = \frac{1}{\|y_m\|^2}\int_a^b r(x) f(x) y_m(x) dx \quad (m = 0, 1, 2, \cdots)$$

Orthogonal Eigenfunction Expansions

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- Example 1. Fourier Series

$$y''+\lambda y=0,\qquad y(\pi)=y(-\pi),\qquad y'(\pi)=y'(-\pi)$$

$$y(x)=A\cos kx+B\sin kx,\quad k=\sqrt{\lambda}$$

$$k=m$$

$$f(x)=a_0+\sum_{m=1}^{\infty}(a_m\cos mx+b_m\sin mx)$$

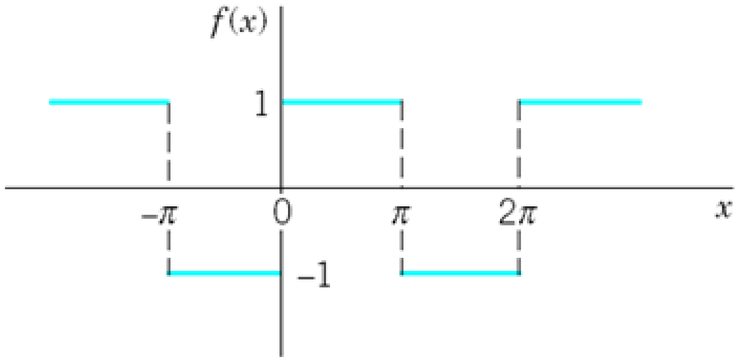
$$a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)dx,\quad a_m=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos mxdx,\quad b_m=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin mxdx$$

$$\leftarrow\quad a_m=\frac{(f,y_m)}{\|y_m\|^2}=\frac{1}{\|y_m\|^2}\int_a^b r(x)f(x)y_m(x)dx\quad (m=0,\ 1,\ 2,\ \cdots)$$

Fourier series, Fourier coefficients (Ch. 11, 12)

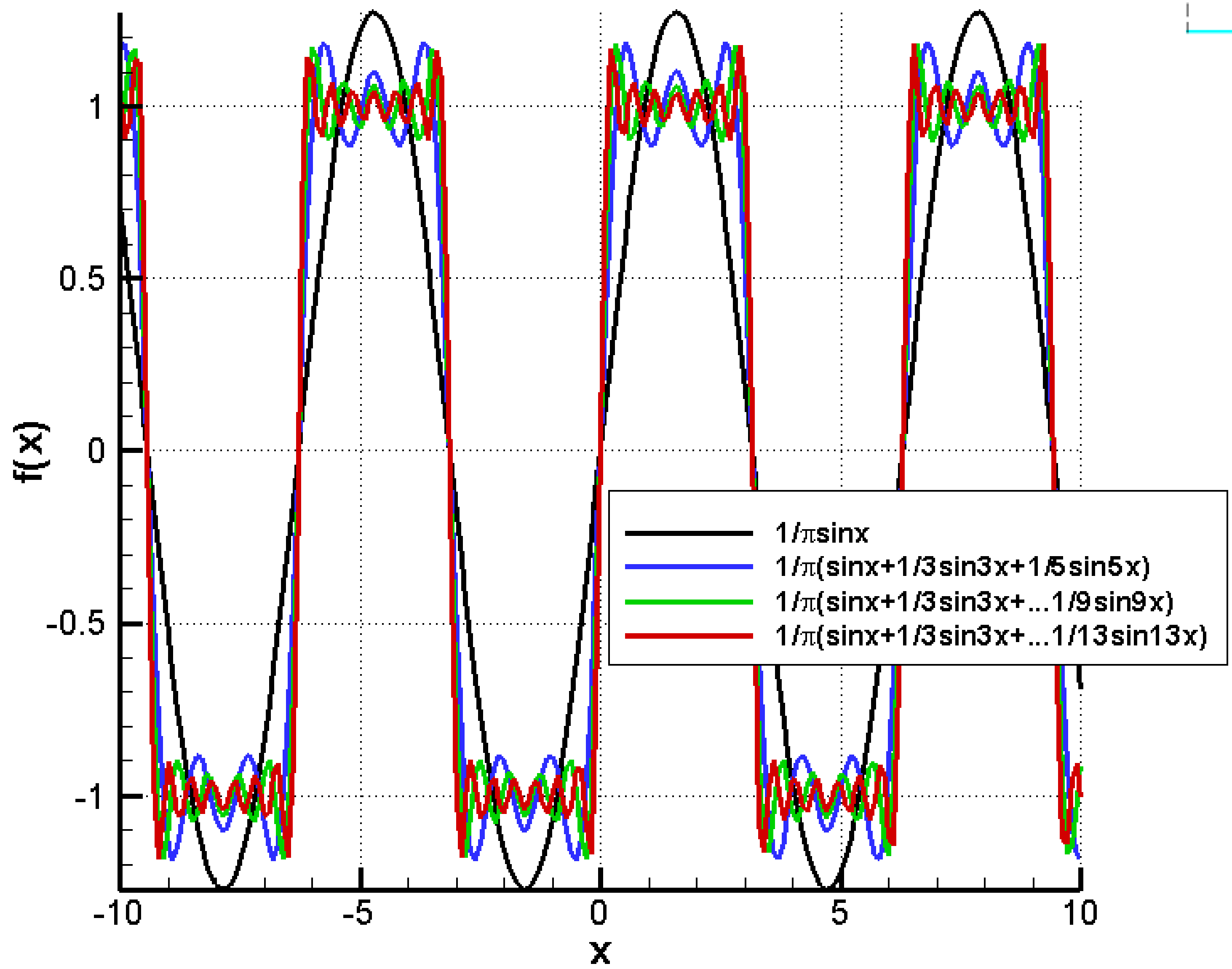
Orthogonal Eigenfunction Expansions

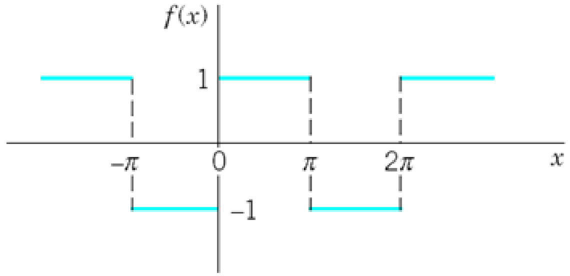
- Fourier series of a periodic rectangular wave

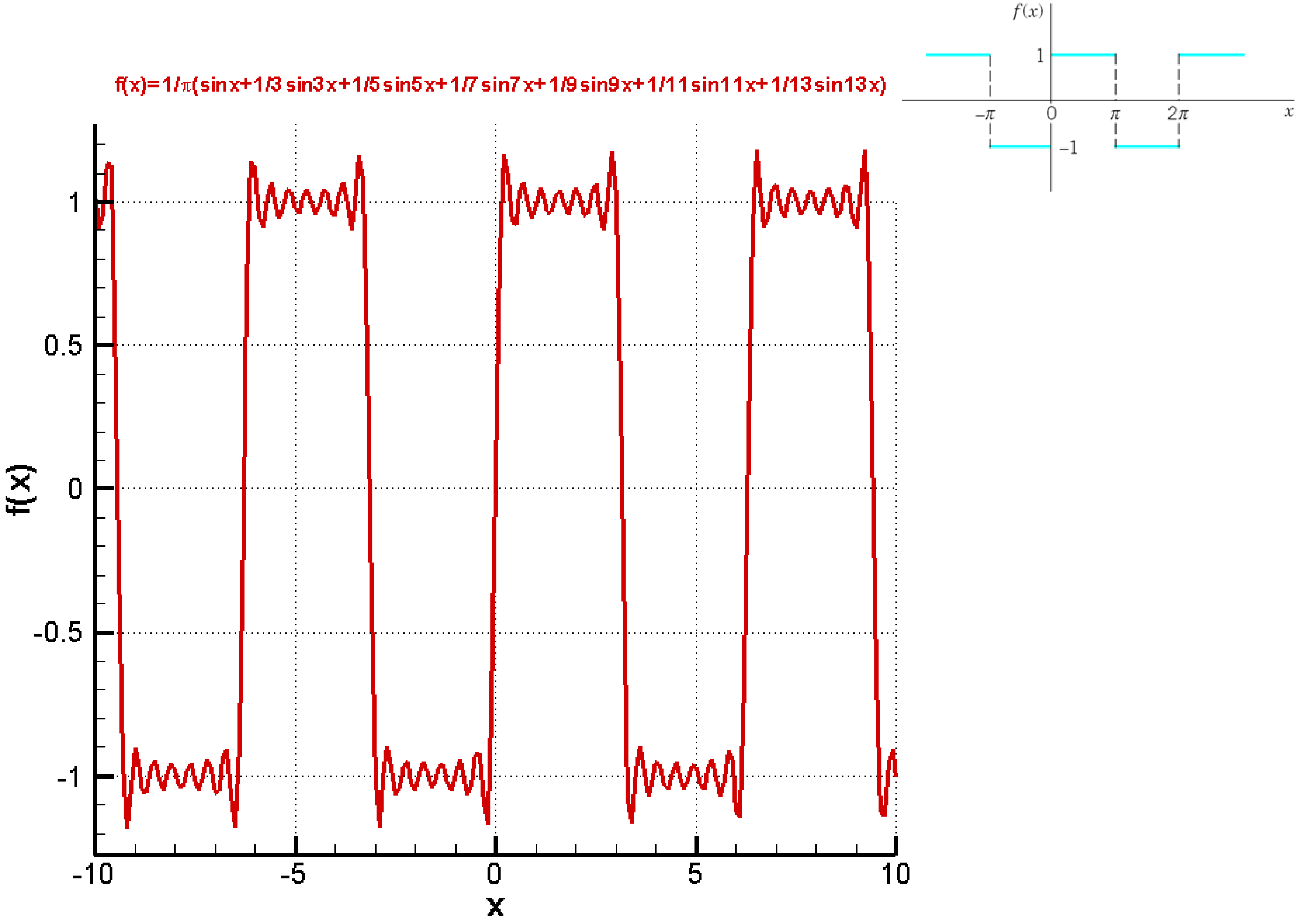


$$f(x) = \begin{cases} -1 & \text{if} \quad -\pi < x < 0 \\ 1 & \text{if} \quad -\pi < x < 0 \end{cases} \quad \text{and} \quad f(x+2\pi) = f(x)$$

$$f(x) = \frac{4}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots \right)$$


f(x)
x
1/πsinx
1/π(sinx+1/3sin3x+1/5sin5x)
1/π(sinx+1/3sin3x+...1/9sin9x)
1/π(sinx+1/3sin3x+...1/13sin13x)


f(x)
1
−π
0
π
2π
x
−1


f(x)=1/π(sinx+1/3 sin3x+1/5sin5x+1/7 sin7x+1/9sin9x+1/11sin11x+1/13sin13x)
f(x)
x
1
0.5
0
-0.5
-1
-10
-5
0
5
10
f(x)
x
1
−1
−π
0
π
2π

Orthogonal Eigenfunction Expansions

- Fourier-Legendre and Fourier-Bessel

$$f(x) = \sum_{m=0}^{\infty} a_m P_m(x)$$

$$a_m = \frac{2m+1}{2} \int_{-1}^{1} f(x) P_m(x) dx$$

$$f(x) = \sum_{m=1}^{\infty} a_m J_n(k_{n,m} x)$$

$$a_m = \frac{2}{R^2 J_{n+1}^2(\alpha_{n,m})} \int_{0}^{R} x f(x) J_n(k_{n,m} x) dx$$

Orthogonal Eigenfunction Expansions

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● **Theorem 1 Completeness (완비성)**

Let $y_0, y_1, \cdots$ be a complete orthonormal set on $a \le x \le b$ in a set of function *S*. Then if a function *f* belongs to *S* and is orthogonal to every y_m, it must have norm zero. In particular, if *f* is continuous, then *f* must be identically zero.

- "충분히 많은" 함수들로 구성된 정규직교집합만을 이용하여 다양한 종류의 함수를 일반화된 푸리에 급수로 나타낼 수 있다. → 정규직교집합이 완비하다(complete)