Engineering Mathematics I
- Chapter 6. Laplace Transforms
- 6장. 라플라스 변환

민기복

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schedule

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• 2 May, 4 May (Quiz):	Laplace Transform
• 9 May, 11 May :	Laplace Transform
• 16 May :	2nd Exam
• 18 May ~ :	Linear Algebra

Ch.5 Series Solutions of ODEs. Special Functions
상미분 방정식의 급수해법. 특수함수

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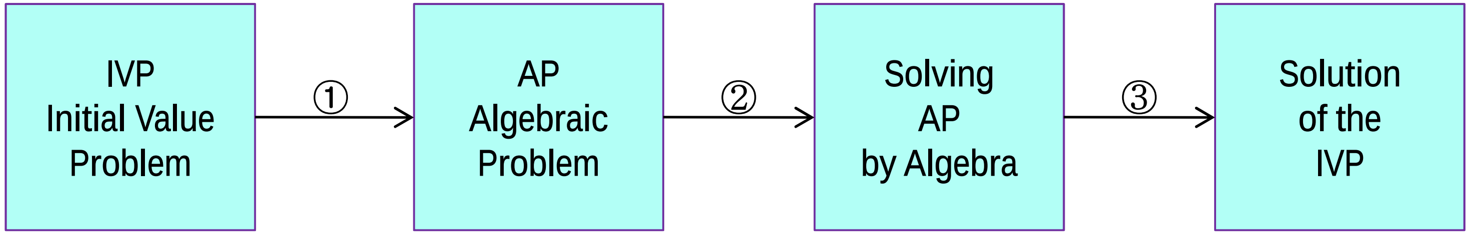
- 5.1 Power Series Method (거듭제곱 급수 해법)
- 5.2 Theory of the Power Series Method (거듭제곱 급수 해법의 이론)
- 5.3 Legendre's Equation. Legendre Polynomials $P_n(x)$ Legendre 방정식. Legendre 다항식 $P_n(x)$
- 5.4 Frobenius Method (Frobenius 해법)
- 5.5 Bessel's Equation. Bessel functions $J_\nu(x)$. Bessel의 방정식. Bessel 함수 $J_\nu(x)$
- 5.6 Bessel's Functions of the Second Kind $Y_\nu(x)$. 제2종 Bessel 함수 $Y_\nu(x)$
- 5.7 Sturm-Liouville Problems. Orthogonal Functions. Sturm-Liouville 문제. 직교함수
- 5.8 Orthogonal Eigenfunction Expansions. 직교 고유함수의 전개

Chapter 5. Laplace Transforms

- 6.1 Laplace Transforms. Inverse Transform. Linearity. *s*-Shifting
- 6.2 Transforms of Derivatives and Integrals. ODEs
- 6.3 Unit Step Function. t-Shifting
- 6.4 Short Impulses. Dirac's Delta function. Partial Fractions
- 6.5 Convolution. Integral Equations
- 6.6 Differentiation and Integration of Transforms
- 6.7 Systems of ODEs
- 6.8 Laplace Transforms. General Formulas
- 6.9 Table of Laplace Transforms

- The Laplace transform method
  - a powerful method for solving linear ODEs and corresponding initial value problems



Step 1 The given ODE is transformed into an algebraic equation(“subsidiary equation”).

Step 2 The subsidiary equation is solved by purely algebraic manipulations.

Step 3 The solution in Step 2 is transformed back, resulting in the solution of the given problem.

Laplace Transform. Inverse Transform. Linearity. s-Shifting

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- Laplace Transform:

$$F(s) = \mathscr{L}(f) = \int_0^\infty e^{-st} f(t)\,dt$$

  - 어떤 함수 f(t) (적분이 존재해야 함)에 e^{-st}를 곱하여 0에서 무한대까지 적분한 것
  - Integral Transform

$$F(s) = \int_0^\infty \underbrace{k(s,t)}_{\text{kernel}} f(t)\,dt$$

$$f(t) \text{ or } y(t) \xrightarrow{\text{Laplace Transform}} F(s) \text{ or } Y(s)$$

- Inverse Transform:

$$\mathscr{L}^{-1}(F) = f(t)$$

$$\mathscr{L}^{-1}(\mathscr{L}(f)) = f$$

$$\mathscr{L}\left(\mathscr{L}^{-1}(F)\right) = F$$

Laplace Transform. Inverse Transform. Linearity. s-Shifting

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- Example 1. Let $f(t)=1$ when $t \geq 0$. Find $F(s)$.

$$\mathscr{L}(f)=\mathscr{L}(1)=\int_0^\infty e^{-st}dt=-\frac{1}{s}e^{-st}\Big|_0^\infty=\frac{1}{s}\quad (s>0)$$

- Example 2. Let $f(t)=e^{at}$ when $t \geq 0$ where a is a constant. Find $\mathscr{L}(f)$

$$\mathscr{L}\left(e^{at}\right)=\int_0^\infty e^{-st}e^{at}dt=\frac{1}{a-s}e^{-(s-a)t}\Big|_0^\infty=\frac{1}{s-a}\qquad s-a>0$$

Laplace Transform. Inverse Transform. Linearity. s-Shifting

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Table 6.1 Some Functions $f(t)$ and Their Laplace Transforms $\mathcal{L}(f)$

	$f(t)$	$\mathcal{L}(f)$		$f(t)$	$\mathcal{L}(f)$
1	1	$1/s$	7	$\cos \omega t$	$\dfrac{s}{s^2 + \omega^2}$
2	t	$1/s^2$	8	$\sin \omega t$	$\dfrac{\omega}{s^2 + \omega^2}$
3	t^2	$2!/s^3$	9	$\cosh at$	$\dfrac{s}{s^2 - a^2}$
4	t^n $(n = 0, 1, \cdots)$	$\dfrac{n!}{s^{n+1}}$	10	$\sinh at$	$\dfrac{a}{s^2 - a^2}$
5	t^a (a positive)	$\dfrac{\Gamma(a+1)}{s^{a+1}}$	11	$e^{at} \cos \omega t$	$\dfrac{s-a}{(s-a)^2 + \omega^2}$
6	e^{at}	$\dfrac{1}{s-a}$	12	$e^{at} \sin \omega t$	$\dfrac{\omega}{(s-a)^2 + \omega^2}$

- Need to learn by heart.

Laplace Transform. Inverse Transform. Linearity. s-Shifting

● Theorem 1 Linearity of the Laplace Transform (라플라스 연산은 선형 연산이다)

The Laplace transform is a linear operation ; that is, for any functions *f* (*t*) and *g*(*t*) whose transforms exist and any constants *a* and *b* the transform of *af* (*t*)+ *bg*(*t*) exists, and

$$\mathscr{L}\left(af(t)+bg(t)\right)=a\mathscr{L}\left(f(t)\right)+b\mathscr{L}\left(g(t)\right)$$

- Ex.3 Find the transform of cosh*at* and sinh*at*

$$\cosh at=\frac{1}{2}\left(e^{at}+e^{-at}\right),\quad \text{inh}\, at=\frac{1}{2}\left(e^{at}-e^{-at}\right)\quad \Rightarrow\quad \mathscr{L}\left(e^{at}\right)=\frac{1}{s-a},\quad \mathscr{L}\left(e^{-at}\right)=\frac{1}{s+a}$$

$$\Rightarrow\quad \mathscr{L}\left(\cosh at\right)=\frac{1}{2}\left[\mathrm{L}\left(e^{at}\right)+\mathrm{L}\left(e^{-at}\right)\right]=\frac{1}{2}\left(\frac{1}{s-a}+\frac{1}{s+a}\right)=\frac{s}{s^2-a^2}$$

$$\mathscr{L}\left(\sinh at\right)=\frac{1}{2}\left[\mathrm{L}\left(e^{at}\right)-\mathrm{L}\left(e^{-at}\right)\right]=\frac{1}{2}\left(\frac{1}{s-a}-\frac{1}{s+a}\right)=\frac{a}{s^2-a^2}$$

Laplace Transform. Inverse Transform. Linearity. s-Shifting

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- Ex.4

$$\mathcal{L}(\cos \omega t) = \frac{s}{s^2 + \omega^2} \qquad \mathcal{L}(\sin \omega t) = \frac{\omega}{s^2 + \omega^2}$$

Laplace Transform. Inverse Transform. Linearity. s-Shifting

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● Theorem 2 First Shifting Theorem, s-shifting (제 1 이동정리)

If $f(t)$ has the transform $F(s)$ (where $s > k$ for some k), then $e^{at} f(t)$ has the transform $F(s - a)$ (where $s - a > k$). In formulas,

$$\mathscr{L}\left\{e^{at} f(t)\right\} = F(s-a)$$

or, if we take the inverse on both sides,

$$e^{at} f(t) = \mathscr{L}^{-1}\left\{F(s-a)\right\}$$

- Example 5. s-Shifting: Damped vibrations.

$$\mathcal{L}(\cos\omega t) = \frac{s}{s^2+\omega^2} \quad \Rightarrow \quad \mathcal{L}\left(e^{at}\cos\omega t\right) = \qquad \mathcal{L}(\sin\omega t) = \frac{\omega}{s^2+\omega^2} \quad \Rightarrow \quad \mathcal{L}\left(e^{at}\sin\omega t\right) =$$

Use these formulas to find the inverse of the transform

$$\mathcal{L}(f) = \frac{3s-137}{s^2+2s+401}$$

$$f = \mathcal{L}^{-1}\left(\frac{3(s+1)-140}{(s+1)^2+400}\right) = 3\mathcal{L}^{-1}\left(\frac{s+1}{(s+1)^2+400}\right) - 7\mathcal{L}^{-1}\left(\frac{20}{(s+1)^2+400}\right) = e^{-t}\left(3\cos 20t - 7\sin 20t\right)$$

Laplace Transform. Inverse Transform. Linearity. s-Shifting

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● Theorem 3 Existence Theorem for Laplace Transform

If *f* (*t*) is defined and piecewise continuous on every finite interval on the semi-axis $t \geq 0$ and satisfies $|f(t)| \leq Me^{kt}$ for all $t \geq 0$ and some constants *M* and *k*, then the Laplace transform $\mathscr{L}(f)$ exists for all *s* > *k*.

● **Existence Theorem for Laplace Transforms (라플라스 변환의 존재정리)**

함수 $f(t)$가 영역 $t \geq 0$상의 모든 유한구간에서 구분적 연속(piecewise continuous)인 함수. 어떤 상수 k 와 M 에 대해 $|f(t)| \leq Me^{kt}$ (너무 빠른 속도로 값이 커지지 않음) $\Rightarrow$ 모든$s > k$ 에 대해 $f(t)$의 라플라스 변환 $\mathcal{L}(f)$ 가 존재

● **Uniqueness (라플라스 변환의 유일성) – 주어진 변환의 역변환은 유일하다.**

연속인 두 함수가 같은 변환값을 가지면 두 함수는 동일.

구간연속인 두 함수가 같은 변환값을 가지면 일부 고립된 점에서 다른 값을 가질지언정 구간내에서는 다를 수 없다

Transform of Derivatives and Integrals. ODEs

- Laplace Transform of derivatives

$$\mathcal{L}(f') = s\mathcal{L}(f) - f(0)$$
$$\mathcal{L}(f'') = s^2\mathcal{L}(f) - sf(0) - f'(0)$$

$$\mathcal{L}\left(f^{(n)}\right) = s^n\mathcal{L}(f) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0)$$

– Ex.1 Let $f(t) = t\sin\omega t$ Find $\mathcal{L}(f)$

$$f(0) = 0, \quad f'(t) = \sin\omega t + \omega t\cos\omega t, \quad f'(0) = 0, \quad f''(t) = 2\omega\cos\omega t - \omega^2 t\sin\omega t$$

$$\Rightarrow \mathcal{L}(f'') = 2\omega\frac{s}{s^2+\omega^2} - \omega^2\mathcal{L}(f) = s^2\mathcal{L}(f) \Rightarrow \mathcal{L}(f) = \mathcal{L}(t\sin\omega t) = \frac{2\omega s}{\left(s^2+\omega^2\right)^2}$$

Transform of Derivatives and Integrals. ODEs

- Ex2. Derive the following formulas

$$\mathcal{L}(\cos \omega t) = \frac{s}{s^2 + \omega^2}$$

$$\mathcal{L}(\sin \omega t) = \frac{\omega}{s^2 + \omega^2}$$

Transform of Derivatives and Integrals. ODEs

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- Laplace Transform of Integral (적분의 라플라스 변환):

$$\mathcal{L}(f(t)) = F(s) \quad \Rightarrow \quad \mathcal{L}\left(\int_0^t f(\tau)d\tau\right) = \frac{1}{s}F(s), \quad \int_0^t f(\tau)d\tau = \mathcal{L}^{-1}\left(\frac{1}{s}F(s)\right)$$

$$g'(t) = f(t), \quad \mathcal{L}(f(t)) = \mathcal{L}(g'(t)) = s\mathcal{L}(g(t)) - g(0) = s\mathcal{L}(g(t))$$

If f(t) satisfy the growth restriction, so does the g(t)

- Ex.3. Find the inverse of $\dfrac{1}{s(s^2+\omega^2)}$ and $\dfrac{1}{s^2(s^2+\omega^2)}$

$$\mathcal{L}^{-1}\left(\frac{1}{s^2+\omega^2}\right) = \frac{1}{\omega}\sin\omega t \quad \Rightarrow \quad \mathcal{L}^{-1}\left(\frac{1}{s(s^2+\omega^2)}\right) = \int_0^t \frac{\sin\omega\tau}{\omega}d\tau = \frac{1}{\omega^2}(1-\cos\omega t)$$

$$\mathcal{L}^{-1}\left(\frac{1}{s^2(s^2+\omega^2)}\right) = \frac{1}{\omega^2}\int_0^t (1-\cos\omega\tau)d\tau = \frac{t}{\omega^2} - \frac{\sin\omega t}{\omega^3}$$

Transform of Derivatives and Integrals. ODEs

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- **Differential Equations. Initial Value Problems**

$$y''+ay'+by=r(t),\quad y(0)=K_0,\quad y'(0)=K_1$$

  – **Step 1.** Setting up the subsidiary equation (보조방정식의 도출):

$$Y=\mathcal{L}(y),\quad R=\mathcal{L}(r)$$

$$\left[s^2Y-sy(0)-y'(0)\right]+a\left[sY-y(0)\right]+bY=R(s)$$

$$\left(s^2+as+b\right)Y=(s+a)y(0)+y'(0)+R(s)$$

  – **Step 2.** Solution of the subsidiary equation by algebra:
  Transfer Function (전달함수):

$$Q(s)=\frac{1}{s^2+as+b}=\frac{1}{\left(s+\frac{1}{2}a\right)^2+b-\frac{1}{4}a^2}$$

  solution of the subsidiary equation:

$$Y(s)=[(s+a)y(0)+y'(0)]Q(s)+R(s)Q(s)$$

  – **Step 3.** Inversion of Y to obtain $y=\mathcal{L}^{-1}(Y)$

Transform of Derivatives and Integrals. ODEs

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- Ex.4 Solve the initial value problem (IVP)

$$y''-y=t, \quad y(0)=1, \quad y'(0)=1$$

  - **Step 1 subsidiary equation**

$$s^2Y-sy(0)-y'(0)-Y=\frac{1}{s^2} \quad \Rightarrow \quad (s^2-1)Y=s+1+\frac{1}{s^2}$$

  - **Step 2 transfer function** $Q=\dfrac{1}{s^2-1}$

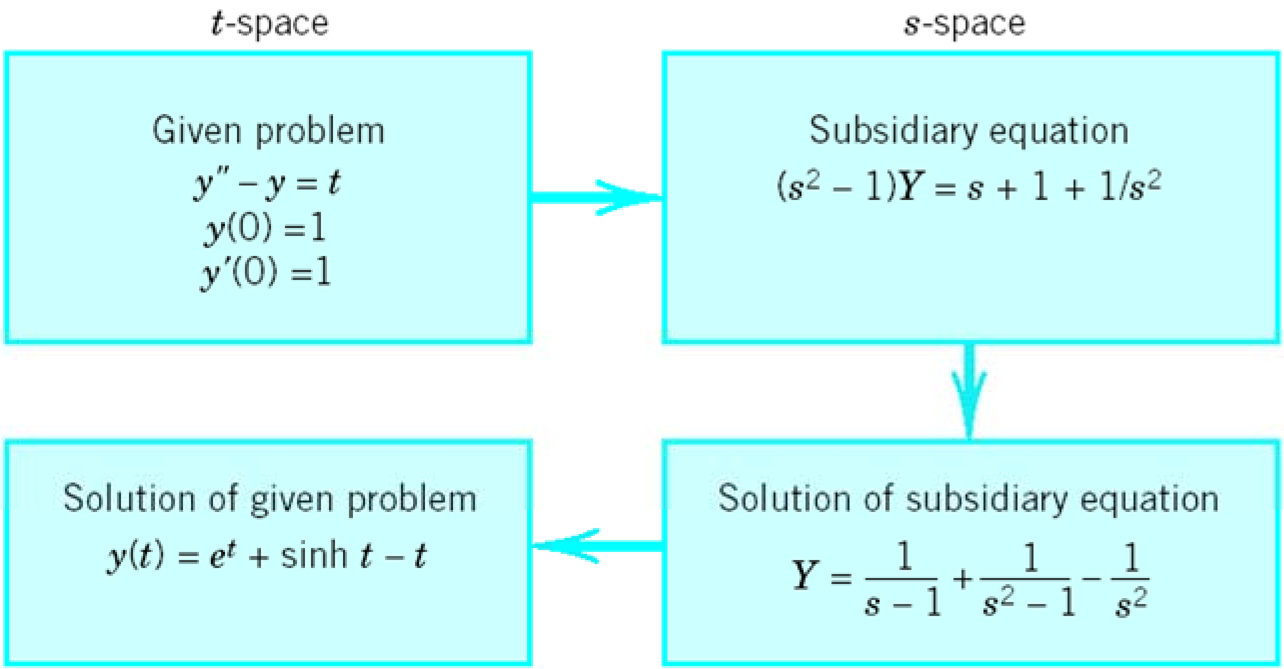
$$Y=(s+1)Q+\frac{1}{s^2}Q=\frac{s+1}{s^2-1}+\frac{1}{s^2(s^2-1)}=\frac{1}{s-1}+\left(\frac{1}{s^2-1}-\frac{1}{s^2}\right)$$

  - **Step 3 inversion**

$$y(t)=\mathcal{L}^{-1}(Y)=\mathcal{L}^{-1}\left(\frac{1}{s-1}\right)+\mathcal{L}^{-1}\left(\frac{1}{s^2-1}\right)-\mathcal{L}^{-1}\left(\frac{1}{s^2}\right)=e^t+\sinh t-t$$

Transform of Derivatives and Integrals. ODEs

- Laplace Transform Method



Transform of Derivatives and Integrals. ODEs

- Ex.5 Comparison with the usual method

$$y'' + y' + 9y = t, \quad y(0) = 0.16, \quad y'(0) = 0$$

- Ex.6 Shifted Data Problems

$$y'' + y = 2t, \quad y\left(\frac{\pi}{4}\right) = \frac{\pi}{2}, \quad y'\left(\frac{\pi}{4}\right) = 2 - \sqrt{2}$$

- **Advantages of the Laplace Method**
  - Solving a nonhomogeneous ODE does not require first solving the homogeneous ODE.
  - Initial values are automatically taken care of.
  - Complicated inputs $r(t)$ (right sides of linear ODEs) can be handled very efficiently, as we show in the next sections.

Unit Step Function. *t*-shifting

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Modeling: Forced Oscillations. Resonance
Free Motion vs. Forced Motion

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- Free Motion : Motions in the absence of external forces caused solely by internal forces.

$$my'' + cy' + ky = 0$$

- Forced Motion : Model by including an external force.

$$my'' + cy' + ky = r(t)$$

Input or driving force

y(t) : Output or response

- e.g., nonhomogeneous ODE with periodic external force

$$my'' + cy' + ky = \boxed{F_0 \cos \omega t}$$

짧은 충격!

- Complicated driving force: → Unit Step Function or Dirac Delta Function
  - Single wave, discontinuous input, impulsive force (hammerblows)

Unit Step Function. *t*-shifting

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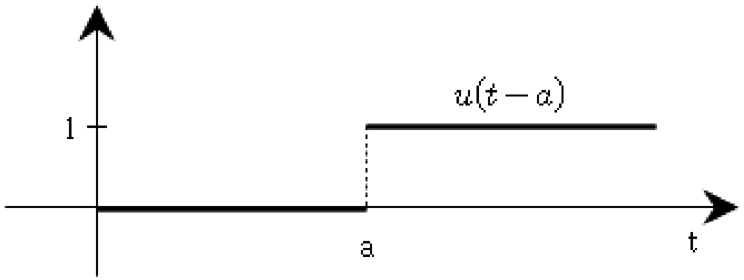
- Unit Step Function (단위계단함수) or Heaviside function:

$$u(t-a)=\begin{cases}0 & (t<a)\\ 1 & (t>a)\end{cases}$$

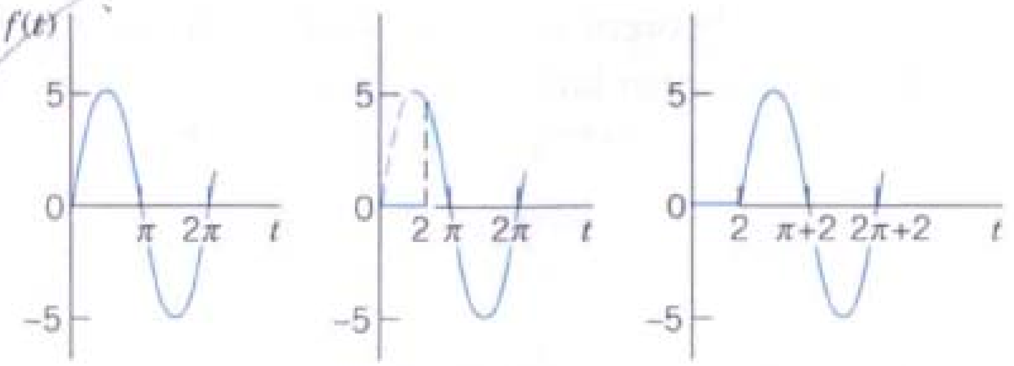


- Laplace Transform of Unit Step Function:

$$\mathcal{L}\{u(t-a)\}=\frac{e^{-as}}{s}$$

Unit Step Function. *t*-shifting

- On and off of functions → $f(t)u(t-a)$



(A) $f(t) = 5\sin t$ (B) $f(t)u(t-2)$ (C) $f(t-2)u(t-2)$

Fig. 119. Effects of the unit step function: (A) Given function. (B) Switching off and on. (C) Shift.

(A) $k[u(t-1) - 2u(t-4) + u(t-6)]$ (B) $4\sin(\frac{1}{2}\pi t)[u(t) - u(t-2) + u(t-4) - + \cdots]$

Fig. 120. Use of many unit step functions.

Unit Step Function. *t*-shifting

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- Time shifting (t-Shifting) – 제 2 이동정리

$$\mathcal{L}\left(f(t)\right) = F(s)$$

$$\mathcal{L}\left(f(t-a)u(t-a)\right) = e^{-as}F(s)$$

$$f(t-a)u(t-a) = \mathcal{L}^{-1}\left\{e^{-as}F(s)\right\}$$

$$e^{-as}F(s) = e^{-as}\int_0^\infty e^{-s\tau}f(\tau)d\tau = \int_0^\infty e^{-s(\tau+a)}f(\tau)d\tau$$

$$e^{-as}F(s) = \int_a^\infty e^{-st}f(t-a)dt$$

$$e^{-as}F(s) = \int_0^\infty e^{-st}f(t-a)u(t-a)dt$$

Unit Step Function. *t*-shifting

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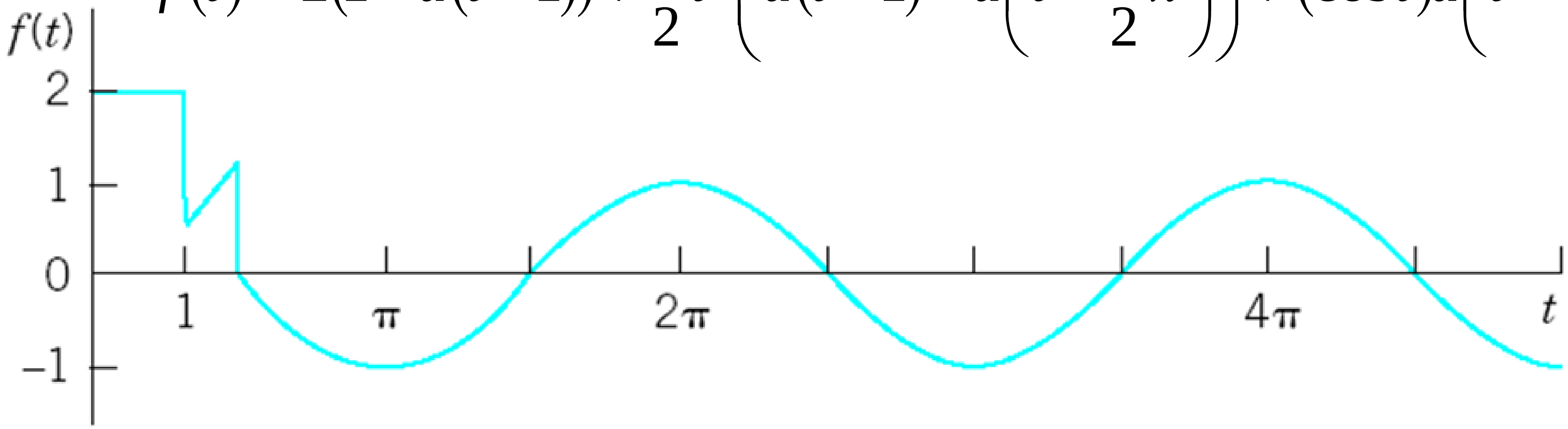
- Ex. 1 Write the following function using unit step function and find its transform.

$$f(t) = \begin{cases} 2 & (0 < t < 1) \\ t^2/2 & (1 < t < \pi/2) \\ \cos t & (t > \pi/2) \end{cases}$$

$$f(t) = 2(1 - u(t-1)) + \frac{1}{2}t^2\left(u(t-1) - u\left(t - \frac{1}{2}\pi\right)\right) + (\cos t)u\left(t - \frac{1}{2}\pi\right)$$



Unit Step Function. *t*-shifting

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$$\mathcal{L}\left\{\frac{1}{2}t^2u(t-1)\right\}=\mathcal{L}\left\{\left(\frac{1}{2}(t-1)^2+(t-1)+\frac{1}{2}\right)u(t-1)\right\}=\left(\frac{1}{s^3}+\frac{1}{s^2}+\frac{1}{2s}\right)e^{-s}$$

$$\mathcal{L}\left\{\frac{1}{2}t^2u\left(t-\frac{1}{2}\pi\right)\right\}=\mathcal{L}\left\{\left(\frac{1}{2}\left(t-\frac{1}{2}\pi\right)^2+\frac{\pi}{2}\left(t-\frac{1}{2}\pi\right)+\frac{\pi^2}{8}\right)u\left(t-\frac{1}{2}\pi\right)\right\}=\left(\frac{1}{s^3}+\frac{\pi}{2s^2}+\frac{\pi^2}{8s}\right)e^{-\pi s/2}$$

$$\mathcal{L}\left\{(\cos t)u\left(t-\frac{1}{2}\pi\right)\right\}=\mathcal{L}\left\{-\left(\sin\left(t-\frac{1}{2}\pi\right)\right)u\left(t-\frac{1}{2}\pi\right)\right\}=-\frac{1}{s^2+1}e^{-\pi s/2}$$

$$\Rightarrow \quad L(f)=\frac{2}{s}-\frac{2}{s}e^{-s}+\left(\frac{1}{s^3}+\frac{1}{s^2}+\frac{1}{2s}\right)e^{-s}-\left(\frac{1}{s^3}+\frac{\pi}{2s^2}+\frac{\pi^2}{8s}\right)e^{-\pi s/2}-\frac{1}{s^2+1}e^{-\pi s/2}$$

Unit Step Function. *t*-shifting

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- Ex.2 Find the inverse transform of $F(s)=\dfrac{e^{-s}}{s^2+\pi^2}+\dfrac{e^{-2s}}{s^2+\pi^2}+\dfrac{e^{-3s}}{(s+2)^2}$

$$\mathcal{L}^{-1}\left(\frac{1}{s^2+\pi^2}\right)=\frac{\sin \pi t}{\pi}$$

$$\mathcal{L}^{-1}\left(\frac{1}{s^2}\right)=t \qquad \Rightarrow \qquad \mathcal{L}^{-1}\left(\frac{1}{(s+2)^2}\right)=te^{-2t}$$

(제 1이동정리) s-shifting

$$\Rightarrow \qquad f(t)=\frac{1}{\pi}\sin(\pi(t-1))u(t-1)+\frac{1}{\pi}\sin(\pi(t-2))u(t-2)+\underline{(t-3)e^{-2(t-3)}u(t-3)}$$

(제 2이동정리) t-shifting

$$=\begin{cases} 0 & (0<t<1) \\ -(\sin \pi t)/\pi & (1<t<2) \\ 0 & (2<t<3) \\ (t-3)e^{-2(t-3)} & (t>3) \end{cases}$$

Short Impulses. Dirac's Delta Function

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- Dirac's Delta Function or unit impulse function (단위충격함수)

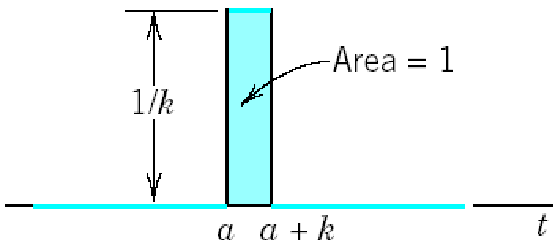
$$\delta(t-a)=\begin{cases}\infty & (t=a)\\ 0 & otherwise\end{cases}$$

$$\int_0^\infty \delta(t-a)dt=1$$

$$\int_0^\infty g(t)\delta(t-a)dt=g(a)$$

$$f_k(t-a)=\begin{cases}1/k & (a\le t\le a+k)\\ 0 & \end{cases} \Rightarrow \delta(t-a)=\lim_{k\to 0} f_k(t-a)$$

$$\int_0^\infty f_k(t-a)dt=\int_a^{a+k}\frac{1}{k}dt=1 \Rightarrow \int_0^\infty \delta(t-a)dt=1$$



Short Impulses. Dirac's Delta Function

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- Laplace Transform of Dirac's Delta Function

$$\mathcal{L}\left\{\delta\left(t-a\right)\right\}=e^{-as}$$

$$f_k(t-a)=\frac{1}{k}\left[u(t-a)-u(t-(a+k))\right]$$

$$\Rightarrow \mathcal{L}\left(f_k(t-a)\right)=\frac{1}{ks}\left[e^{-as}-e^{-(a+k)s}\right]=e^{-as}\frac{1-e^{-ks}}{ks}$$ Put k→ 0

Short Impulses. Dirac's Delta Function

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- Ex.1 Mass-Spring System Under a Square Wave

$$y''+3y'+2y=r(t)=u(t-1)-u(t-2),\quad y(0)=0,\quad y'(0)=0$$

$$y=\begin{cases} 0 & (0<t<1) \\ \frac{1}{2}-e^{-(t-1)}+\frac{1}{2}e^{-2(t-1)} & (1<t<2) \\ -e^{-(t-1)}+e^{-(t-2)}+\frac{1}{2}e^{-2(t-1)}-\frac{1}{2}e^{-2(t-2)} & (t>2) \end{cases}$$

Short Impulses. Dirac's Delta Function

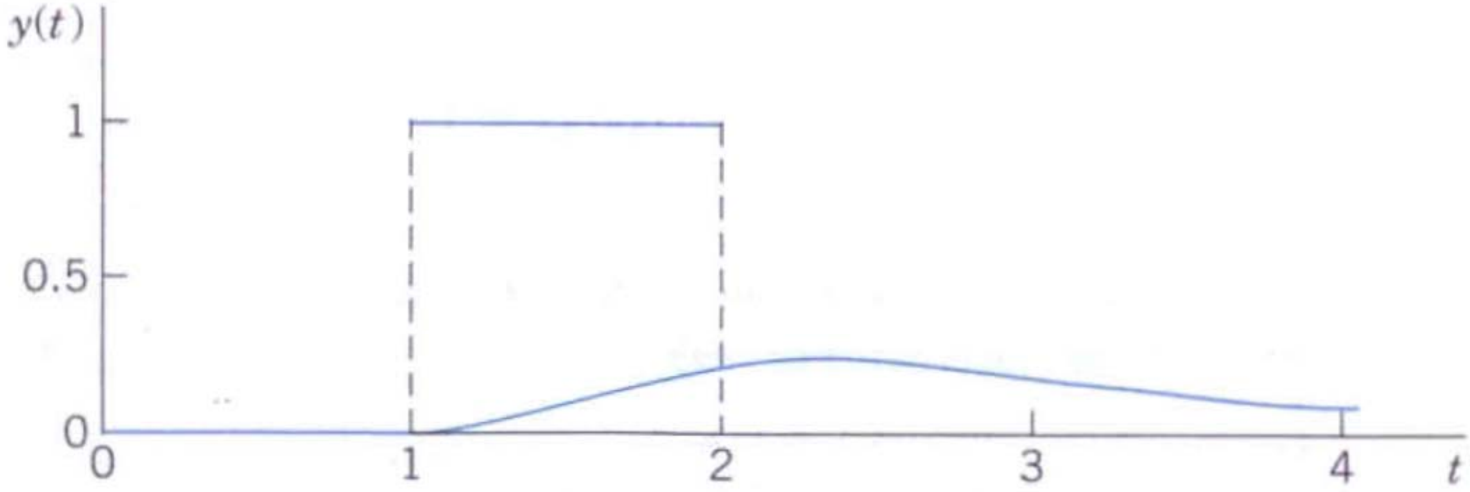

Fig. 131. Square wave and response in Example 1

Short Impulses. Dirac's Delta Function

- Ex.2 Hammerblow Response of a Mass-Spring System

$$y''+3y'+2y=\delta(t-1), \quad y(0)=0, \quad y'(0)=0$$

$$s^2Y+3sY+2Y=e^{-s}$$

$$Y(s)=\frac{e^{-s}}{(s+1)(s+2)}=\left(\frac{1}{s+1}-\frac{1}{s+2}\right)e^{-s}$$

$$y(t)=\mathcal{L}^{-1}(Y)=\begin{cases}0 & (0<\mathrm{t}<1)\\ e^{-(t-1)}-e^{-2(t-1)} & (\mathrm{t}>1)\end{cases}$$

$$f(t-a)u(t-a)=\mathcal{L}^{-1}\left\{e^{-as}F(s)\right\}$$

Short Impulses. Dirac's Delta Function

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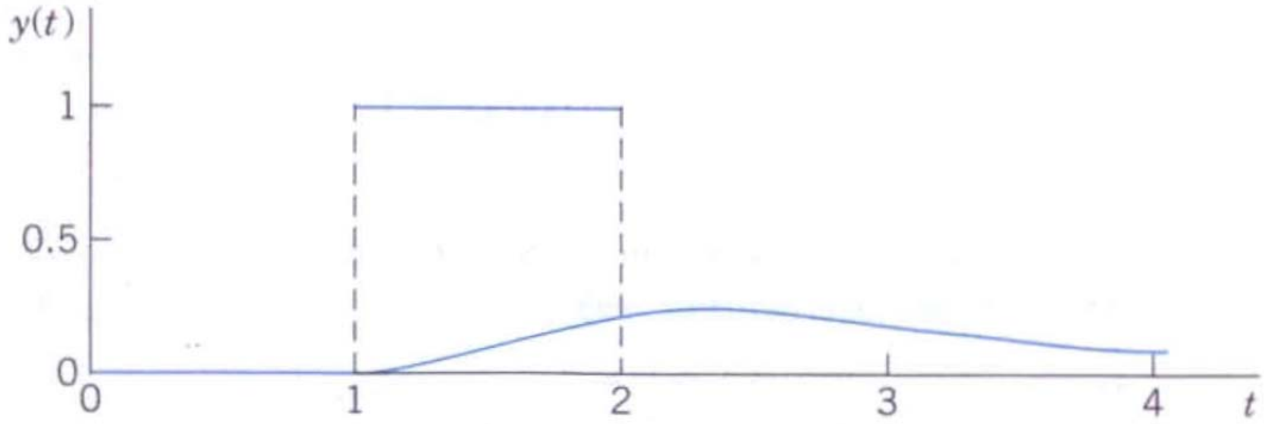
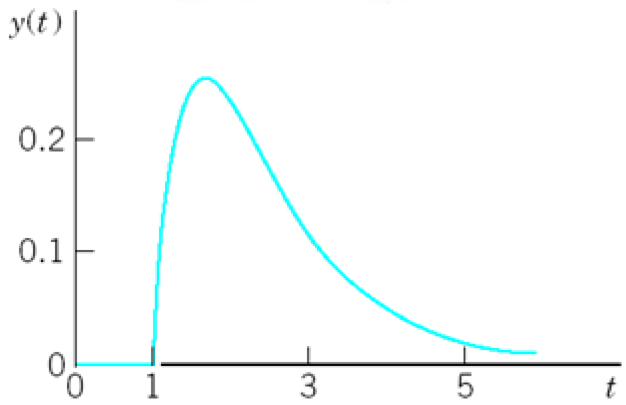
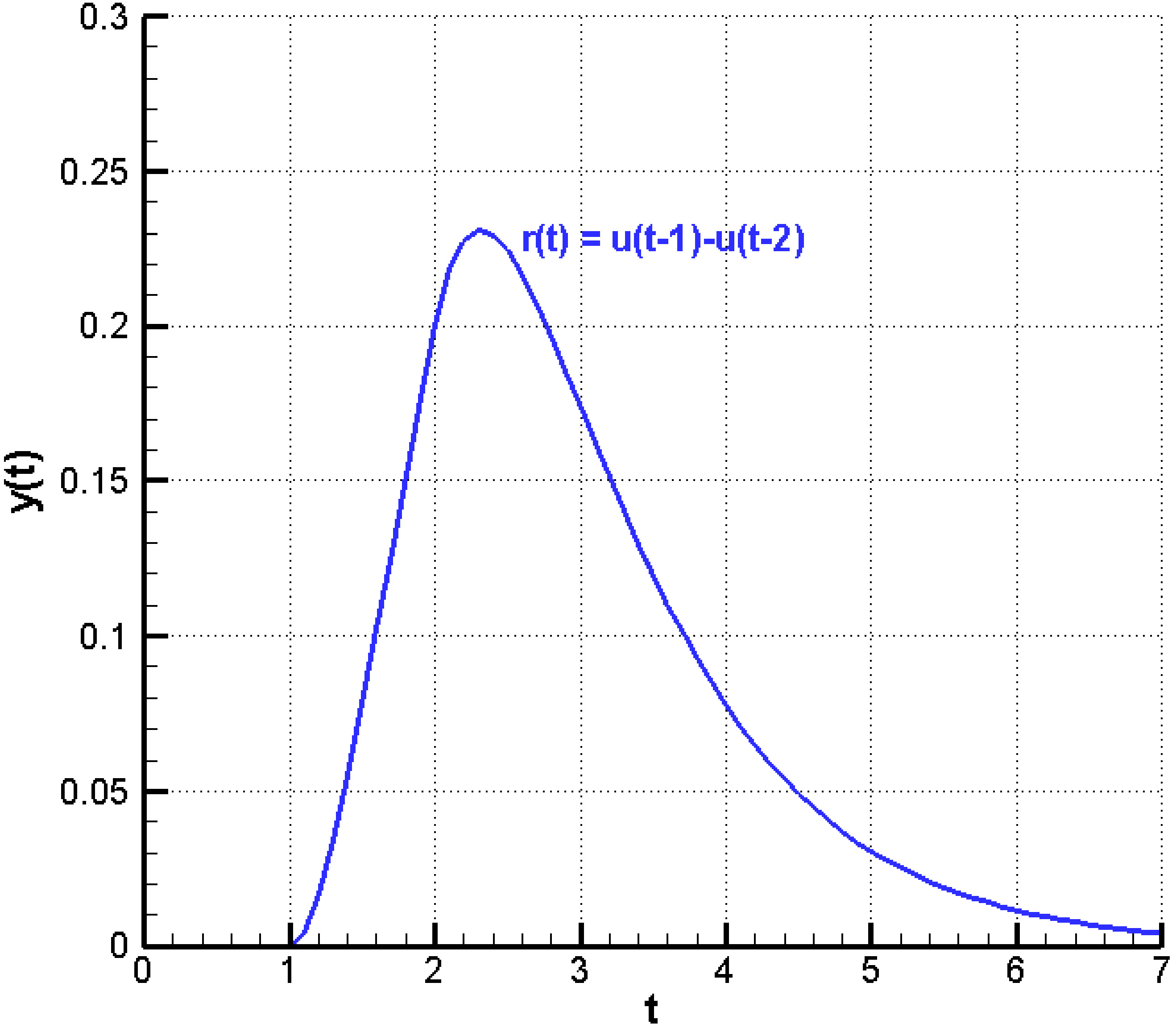
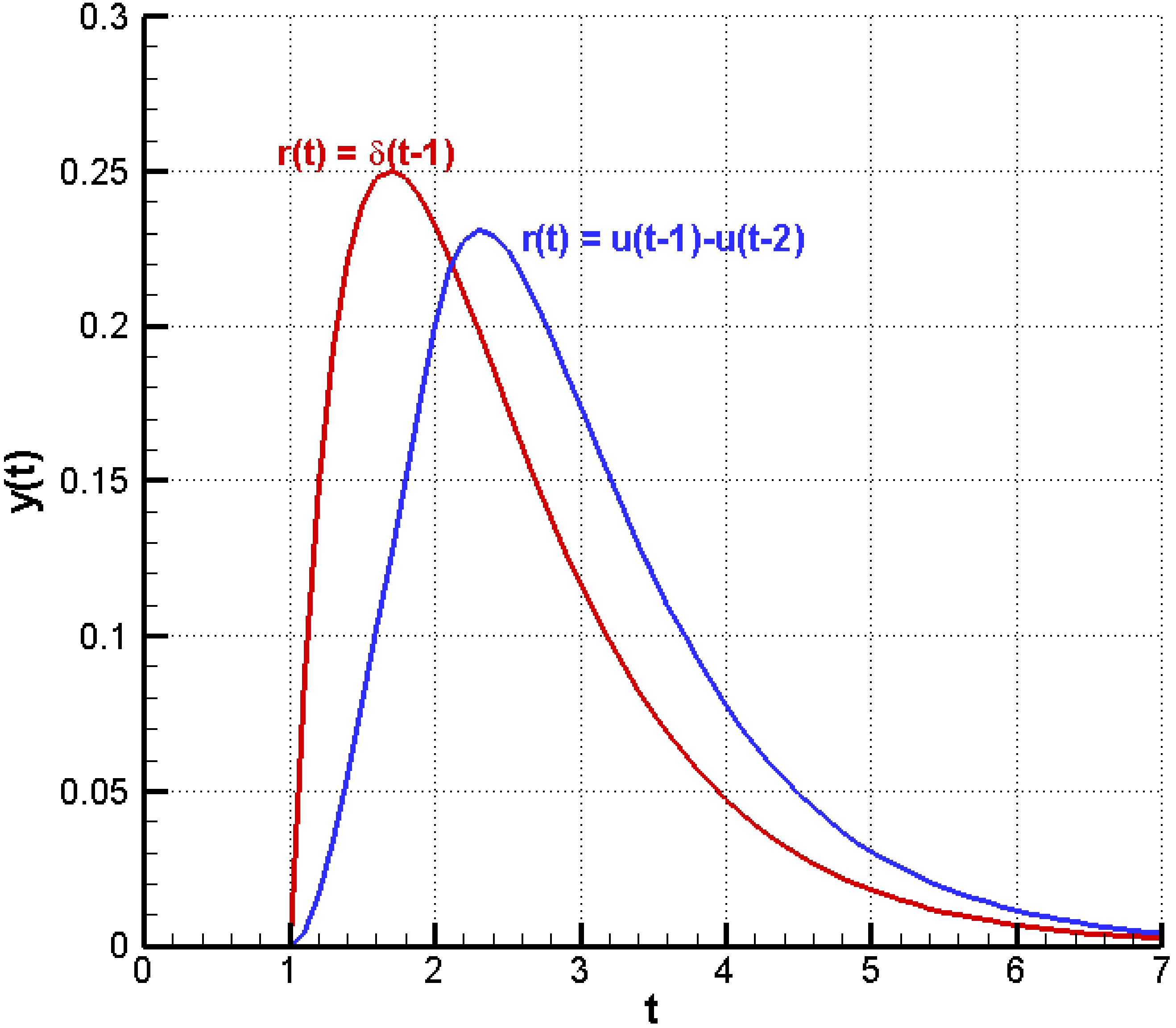


Fig. 131. Square wave and response in Example 1




r(t) = u(t-1)-u(t-2)
y(t)
t
0.3
0.25
0.2
0.15
0.1
0.05
0
0
1
2
3
4
5
6
7



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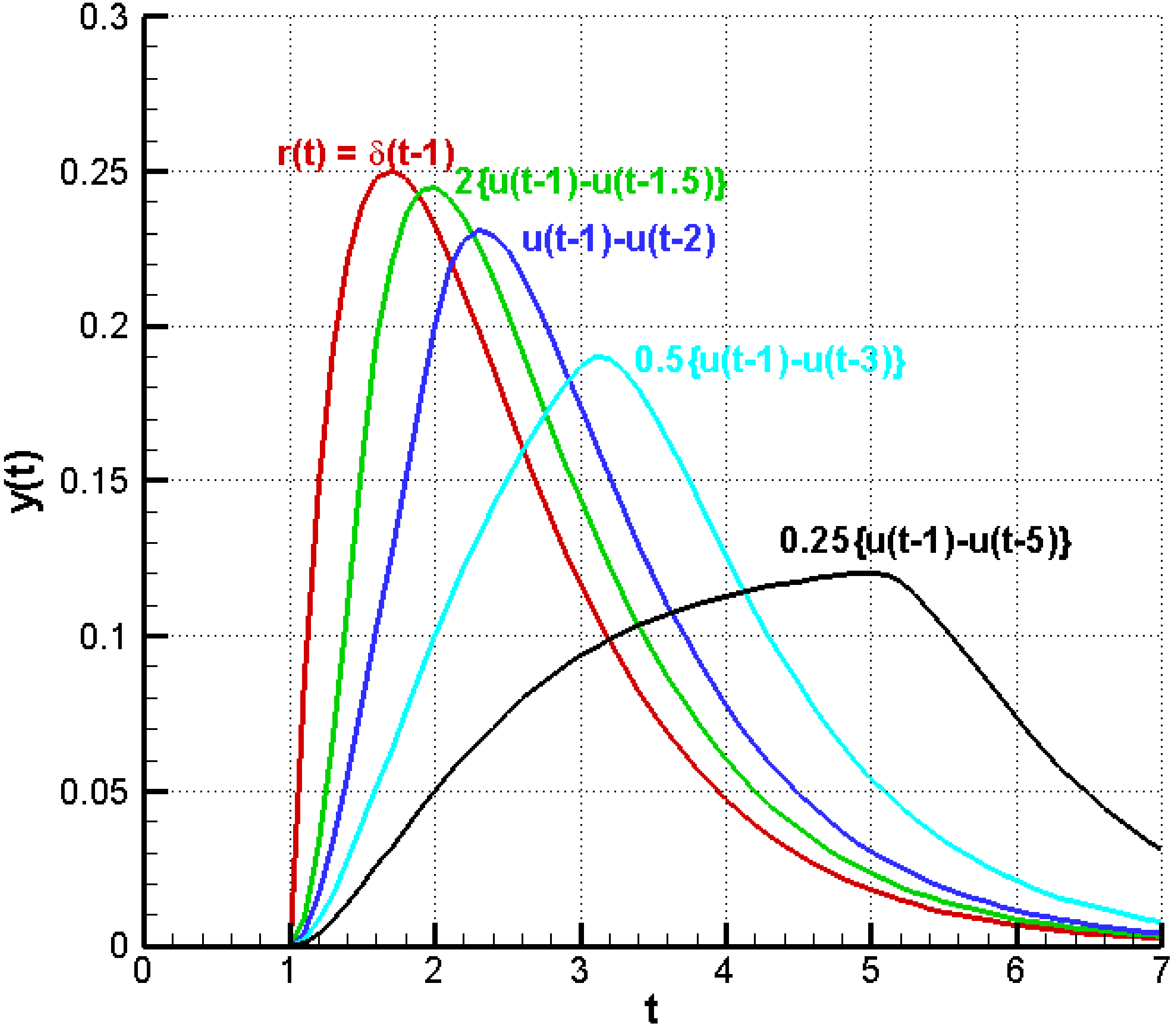
r(t) = δ(t-1)
2{u(t-1)-u(t-1.5)}
u(t-1)-u(t-2)
0.5{u(t-1)-u(t-3)}
0.25{u(t-1)-u(t-5)}
y(t)
t
0
0.05
0.1
0.15
0.2
0.25
0.3
0
1
2
3
4
5
6
7

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Short Impulses. Dirac's Delta Function

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- Ex.4 Damped Forced Vibrations

$$y''+2y'+2y=r(t),\ r(t)=10\sin 2t\ \text{if}\ 0<t<\pi\ \text{and}\ 0\ \text{if}\ t>\pi,\quad y(0)=1,\quad y'(0)=-5$$

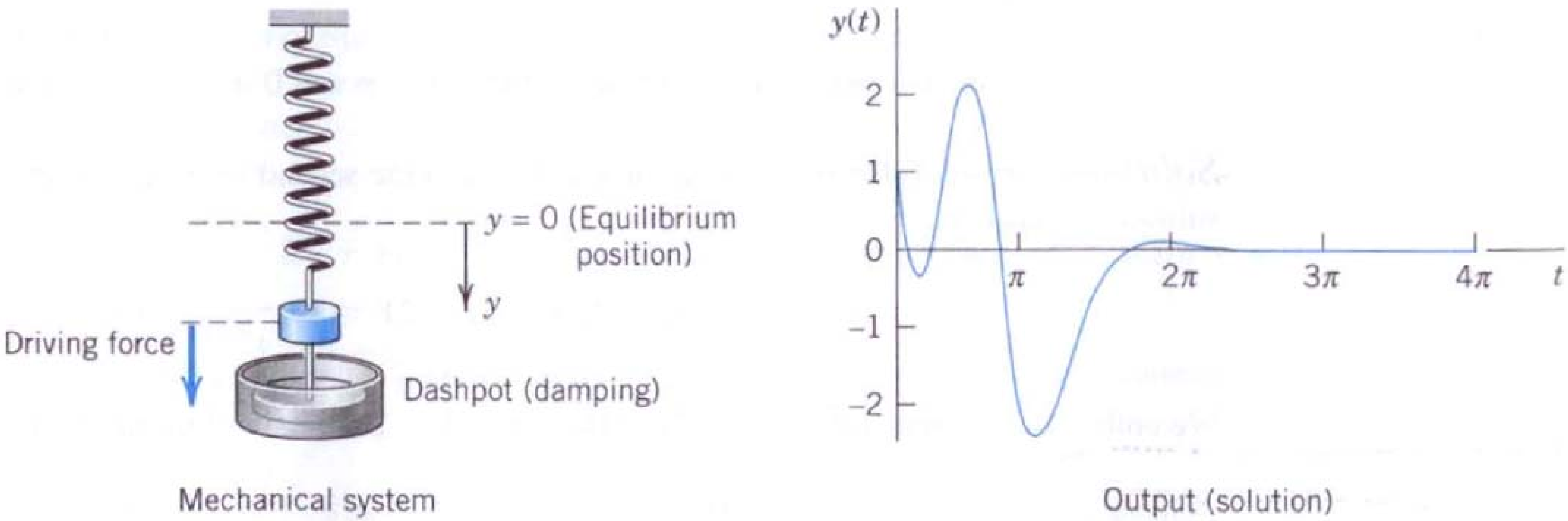

Fig. 134. Example 4

Convolution. Integral Equation

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- Convolution (합성곱)

$$\mathcal{L}(fg) \neq \mathcal{L}(f)\mathcal{L}(g) \quad \Rightarrow \quad \mathcal{L}^{-1}(\mathcal{L}(f)\mathcal{L}(g)) = ?$$

ex) $f = e^t, \quad g = 1$

Table 6.1 Some Functions $f(t)$ and Their Laplace Transforms $\mathcal{L}(f)$

	$f(t)$	$\mathcal{L}(f)$		$f(t)$	$\mathcal{L}(f)$
1	1	$1/s$	7	$\cos \omega t$	$\dfrac{s}{s^2 + \omega^2}$
2	t	$1/s^2$	8	$\sin \omega t$	$\dfrac{\omega}{s^2 + \omega^2}$
3	t^2	$2!/s^3$	9	$\cosh at$	$\dfrac{s}{s^2 - a^2}$
4	t^n $(n = 0, 1, \cdots)$	$\dfrac{n!}{s^{n+1}}$	10	$\sinh at$	$\dfrac{a}{s^2 - a^2}$
5	t^a (a positive)	$\dfrac{\Gamma(a+1)}{s^{a+1}}$	11	$e^{at} \cos \omega t$	$\dfrac{s-a}{(s-a)^2 + \omega^2}$
6	e^{at}	$\dfrac{1}{s-a}$	12	$e^{at} \sin \omega t$	$\dfrac{\omega}{(s-a)^2 + \omega^2}$

Convolution. Integral Equation

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- Convolution

$$h(t) = (f * g)(t) = \int_0^t f(\tau) g(t-\tau)\, d\tau$$

- Properties of convolution
  - Commutative law $\quad f * g = g * f$
  - Distributive law $\quad f * (g_1 + g_2) = f * g_1 + f * g_2$
  - Associative law $\quad (f * g) * v = f * (g * v)$

$$f * 0 = 0 * f = 0$$

$$f * 1 \neq f \qquad \cancel{(f * f) \geq 0}$$

Unusual Properties of convolution (특이성질)

- Convolution Theorem

$$\mathcal{L}(f * g) = \mathcal{L}(f)\mathcal{L}(g) \qquad H = FG$$

Convolution. Integral Equation

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- Ex.1 Let $H(s) = 1/[(s-a)s]$ Find $h(t)$

$$\mathcal{L}^{-1}\left(\frac{1}{s-a}\right) = e^{at}, \quad \mathcal{L}^{-1}\left(\frac{1}{s}\right) = 1$$

$$\Rightarrow \quad h(t) = e^{at} * 1 = \int_0^t e^{a\tau} \cdot 1 d\tau = \frac{1}{a}\left(e^{at} - 1\right)$$

Convolution. Integral Equation

- Ex.2 Let $H(s) = 1 \Big/ \left(s^2 + w^2\right)^2$ Find $h(t)$

$$\mathcal{L}^{-1}\left(\frac{1}{s^2+w^2}\right) = (\sin wt)/w$$

$$\sin x \sin y = \frac{1}{2}\left[-\cos(x+y) + \cos(x-y)\right]$$

$$\Rightarrow \quad h(t) = \frac{\sin wt}{w} * \frac{\sin wt}{w} = \frac{1}{w^2}\int_0^t \sin w\tau \sin w(t-\tau)\,d\tau$$

$$= \frac{1}{2w^2}\int_0^t \left[-\cos wt + \cos(2w\tau - wt)\right] d\tau$$

$$= \frac{1}{2w^2}\left[-\tau\cos wt + \frac{\sin(2w\tau - wt)}{2w}\right]_{\tau=0}^{t}$$

$$= \frac{1}{2w^2}\left[-t\cos wt + \frac{\sin wt}{w}\right]$$

Convolution. Integral Equation

- Ex.4 Resonance.
  - In an undamped mass-spring system, resonance occurs if the frequency of the driving force equals the natural frequency of the system (Sec. 2.8).

$$y'' + \omega_0^2 y = K \sin \omega_0 t, \quad y(0) = 0, \quad y'(0) = 0$$

$$\mathcal{L}(f') = s\mathcal{L}(f) - f(0), \quad \mathcal{L}(f'') = s^2\mathcal{L}(f) - sf(0) - f'(0)$$

$$Y = \frac{K\omega_0}{(s^2 + \omega_0^2)^2}$$

$$\mathcal{L}^{-1}\left(1 \big/ (s^2 + w^2)^2 \right) = \frac{1}{2w^2}\left[-t\cos wt + \frac{\sin wt}{w} \right]$$

$$\Rightarrow \quad y(t) = \frac{K}{2\omega_0^2}\left(-\omega_0 t \cos \omega_0 t + \sin \omega_0 t \right)$$

Convolution. Integral Equation

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- Integral Equation
  - Unknown function y(t) appear in an integral
- Ex. 6 A Volterra Integral Equation of the Second Kind

$$y(t)-\int_0^t y(\tau)\sin(t-\tau)d\tau = t$$

$$y - y * \sin t = t$$

$$Y(s)-Y(s)\frac{1}{s^2+1}=\frac{1}{s^2}$$

$$Y(s)=\frac{s^2+1}{s^4}=\frac{1}{s^2}+\frac{1}{s^4} \quad \Rightarrow \quad y(t)=t+\frac{t^3}{6}$$

Differentiation and Integration of Transforms.

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- Differentiation of *F(s)*

$$F(s) = \mathcal{L}(f) = \int_0^\infty e^{-st} f(t)\,dt$$

$$\Rightarrow \qquad F'(s) = \frac{dF}{ds} = -\int_0^\infty e^{-st} t f(t)\,dt = -\mathcal{L}(tf)$$

$$\mathcal{L}(tf(t)) = -F'(s), \quad \mathcal{L}^{-1}\{F'(s)\} = -tf(t)$$

Differentiation and Integration of Transforms.

- Ex. 1 Derive the following three formulas

$\mathscr{L}(f)$	$f(t)$
$\dfrac{1}{(s^2+\beta^2)^2}$	$\dfrac{1}{2\beta^3}(\sin\beta t - \beta t\cos\beta t)$
$\dfrac{s}{(s^2+\beta^2)^2}$	$\dfrac{t}{2\beta}\sin\beta t$
$\dfrac{s^2}{(s^2+\beta^2)^2}$	$\dfrac{1}{2\beta}(\sin\beta t + \beta t\cos\beta t)$

$$\mathcal{L}(tf(t)) = -F'(s)$$

$$\mathcal{L}(\sin\beta t) = \frac{\beta}{s^2+\beta^2} \xrightarrow{\text{미분에 의하여}} \mathcal{L}(t\sin\beta t) = \frac{2\beta s}{(s^2+\beta^2)^2}$$

$$\longrightarrow \mathcal{L}\left(\frac{t}{2\beta}\sin\beta t\right) = \frac{s}{(s^2+\beta^2)^2}$$

$$\mathcal{L}(\cos\beta t) = \frac{s}{s^2+\beta^2} \xrightarrow{\text{미분에 의하여}} \mathcal{L}(t\cos\beta t) = -\frac{(s^2+\beta^2)-2s^2}{(s^2+\beta^2)^2} = \frac{s^2-\beta^2}{(s^2+\beta^2)^2}$$

$$\mathcal{L}\left(t\cos\beta t \pm \frac{1}{\beta}\sin\beta t\right) = \frac{s^2-\beta^2}{(s^2+\beta^2)^2} \pm \frac{1}{s^2+\beta^2} = \frac{(s^2-\beta^2)\pm(s^2+\beta^2)}{(s^2+\beta^2)^2}$$

Differentiation and Integration of Transforms.

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- Integration of Transform

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(\tilde{s})\,d\tilde{s} \quad \Rightarrow \quad \mathcal{L}^{-1}\left\{\int_s^\infty F(\tilde{s})\,d\tilde{s}\right\} = \frac{f(t)}{t}$$

Differentiation and Integration of Transforms.

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- Ex.2 Find the inverse transform of $\ln\left(1+\frac{\omega^2}{s^2}\right)=\ln\frac{s^2+\omega^2}{s^2}$

$$\ln\left(1+\frac{\omega^2}{s^2}\right)=\ln\frac{s^2+\omega^2}{s^2}\xrightarrow{\text{미분}}\frac{d}{ds}\left(\ln\left(s^2+\omega^2\right)-\ln s^2\right)=\frac{2s}{s^2+\omega^2}-\frac{2s}{s^2}$$

Case 1) 변환의 미분이용 $\mathcal{L}(tf(t))=-F'(s)$

$$F(s)=\mathcal{L}(f)=\ln\left(1+\frac{\omega^2}{s^2}\right)\quad\Rightarrow\quad\mathcal{L}^{-1}(F'(s))=\mathcal{L}^{-1}\left(\frac{2s}{s^2+\omega^2}-\frac{2s}{s^2}\right)=2\cos\omega t-2=-tf(t)$$

$$\therefore\ f(t)=\frac{2}{t}(1-\cos\omega t)$$

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\}=\int_s^\infty F(\tilde{s})d\tilde{s}$$

Case 2) 적분이용 $\mathcal{L}^{-1}\left(\ln\left(1+\frac{\omega^2}{s^2}\right)\right)=-\mathcal{L}^{-1}\left(\int_s^\infty G(\tilde{s})\,d\tilde{s}\right)=-\frac{g(t)}{t}=\frac{2}{t}(1-\cos\omega t)$

$$G(s)=-\left(\frac{2s}{s^2+\omega^2}-\frac{2}{s}\right)\quad\Rightarrow\quad g(t)=\mathcal{L}^{-1}(G)=-2(\cos\omega t-1)$$

Differentiation and Integration of Transforms.

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- Special Linear ODEs with variable coefficients

$$\mathcal{L}(tf(t)) = -F'(s)$$

$$\mathcal{L}(ty') = -\frac{d}{ds}[sY - y(0)] = -Y - s\frac{dY}{ds}$$

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^{\infty} F(\tilde{s})d\tilde{s}$$

$$\mathcal{L}(ty'') = -\frac{d}{ds}\left[s^2Y - sy(0) - y'(0)\right] = -2sY - s^2\frac{dY}{ds} + y(0)$$

- Ex.3 Laguerre's Equation. Laguerre Polynomials

$$ty'' + (1-t)y' + ny = 0 \quad (n = 0, 1, 2, \cdots)$$

$$\left[-2sY - s^2\frac{dY}{ds} + y(0)\right] + sY - y(0) - \left(-Y - s\frac{dY}{ds}\right) + nY = 0 \quad \Rightarrow \quad \left(s\text{-}s^2\right)\frac{dY}{ds} + (n+1-s)Y = 0$$

$$\Rightarrow \quad \frac{dY}{Y} = -\frac{n+1-s}{s\text{-}s^2}ds = \left(\frac{n}{s-1} - \frac{n+1}{s}\right)ds \quad \Rightarrow \quad Y = \frac{(s-1)^n}{s^{n+1}}$$

$$l_n(t) = \mathcal{L}^{-1}(Y) = \begin{cases} 1, & n = 0 \\ \dfrac{e^t}{n!}\dfrac{d^n}{dt^n}\left(t^n e^{-t}\right), & n = 1, 2, \cdots \end{cases}$$

Systems of ODEs

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- Ex.1 Mixing Problem Involving Two Tanks
  - Balance Law (mass conservation)

Time rate of change = Inflow/min – Outflow/min

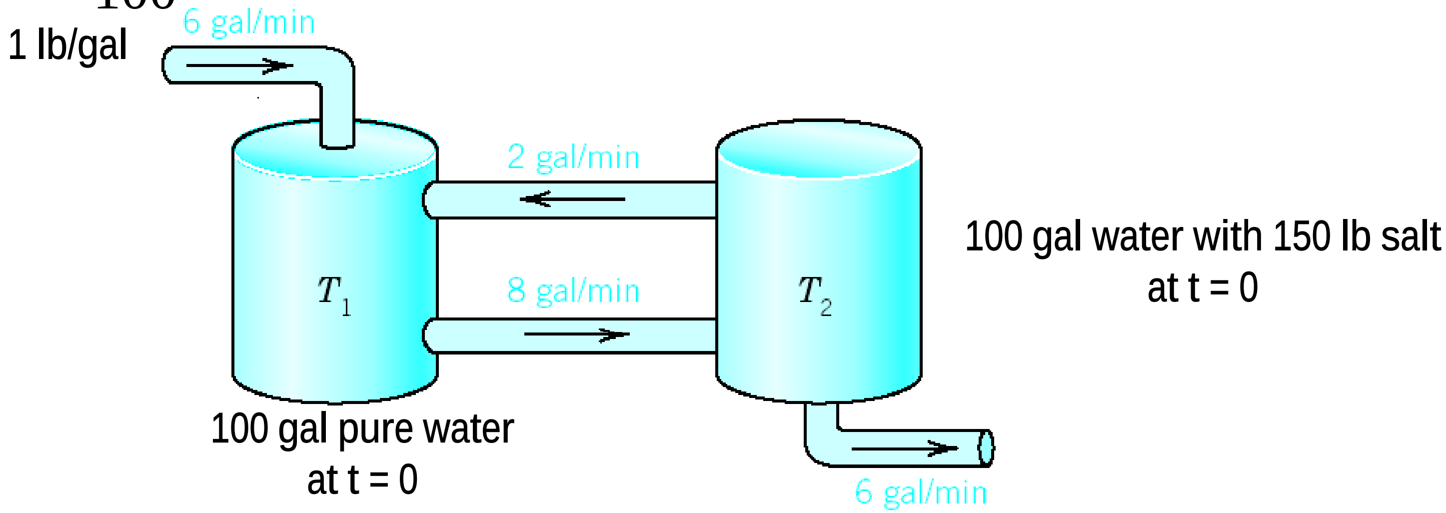
$$y_1' = -\frac{8}{100}y_1 + \frac{2}{100}y_2 + 6$$

$$y_2' = \frac{8}{100}y_1 - \frac{(2+6)}{100}y_2$$

$\xrightarrow{\text{라플라스 변환}}$

$$(-0.08 - s)Y_1 + 0.02Y_2 = -\frac{6}{s}$$

$$0.08Y_1 + (-0.08 - s)Y_2 = -150$$



Systems of ODEs

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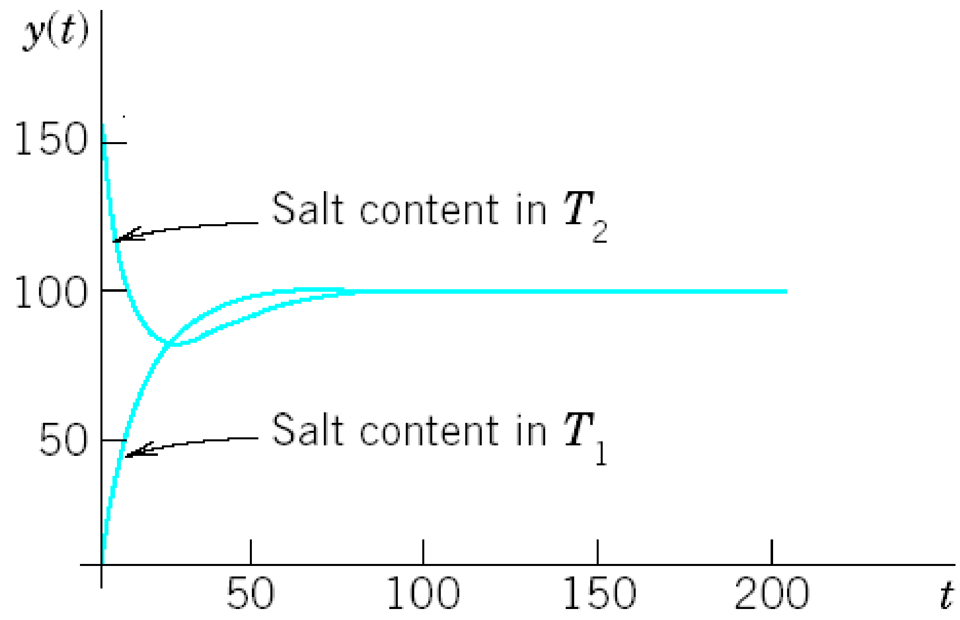
$$Y_1 = \frac{9s+0.48}{s(s+0.12)(s+0.04)} = \frac{100}{s} - \frac{62.5}{s+0.12} - \frac{37.5}{s+0.04}$$

$$Y_2 = \frac{150s^2+12s+0.48}{s(s+0.12)(s+0.04)} = \frac{100}{s} + \frac{125}{s+0.12} - \frac{75}{s+0.04}$$

$$y_1 = 100 - 62.5e^{-0.12t} - 37.5e^{-0.04t}$$

$$y_2 = 100 + 125e^{-0.12t} - 75e^{-0.04t}$$



Systems of ODEs

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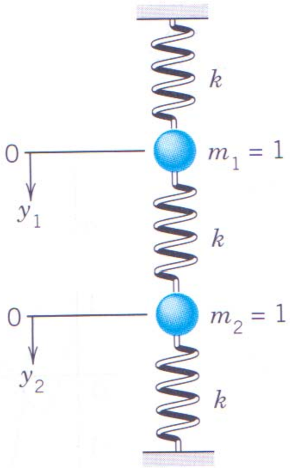
- Ex.3 Model of Two Masses on Springs

$$y_1(0)=y_2(0)=1, \quad y_1'(0)=-y_2'(0)=\sqrt{3k}$$



지배방정식:

$$y_1''=-ky_1+k(y_2-y_1)$$

$$y_2''=-k(y_2-y_1)-ky_2$$

라플라스 변환 →

$$s^2Y_1-s-\sqrt{3k}=-kY_1+k(Y_2-Y_1)$$

$$s^2Y_2-s+\sqrt{3k}=-k(Y_2-Y_1)-kY_2$$

↓ **Cramer의 법칙 또는 소거법 적용**

$$Y_1=\frac{\left(s+\sqrt{3k}\right)\left(s^2+2k\right)+k\left(s-\sqrt{3k}\right)}{\left(s^2+2k\right)^2-k^2}=\frac{s}{s^2+k}+\frac{\sqrt{3k}}{s^2+3k}$$

$$Y_2=\frac{\left(s-\sqrt{3k}\right)\left(s^2+2k\right)+k\left(s-\sqrt{3k}\right)}{\left(s^2+2k\right)^2-k^2}=\frac{s}{s^2+k}-\frac{\sqrt{3k}}{s^2+3k}$$

← **역변환**

$$y_1(t)=\mathcal{L}^{-1}(Y_1)=\cos\sqrt{k}t+\sin\sqrt{3k}t$$

$$y_2(t)=\mathcal{L}^{-1}(Y_2)=\cos\sqrt{k}t-\sin\sqrt{3k}t$$

Laplace Transform. General Formulas

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Formula	Name, Comments	Sec.
$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$ $f(t) = \mathcal{L}^{-1}\{F(s)\}$	Definition of Transform Inverse Transform	6.1
$\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$	Linearity	6.1
$\mathcal{L}\{e^{at} f(t)\} = F(s - a)$ $\mathcal{L}^{-1}\{F(s - a)\} = e^{at} f(t)$	s-Shifting (First Shifting Theorem)	6.1

Laplace Transform. General Formulas

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$\mathcal{L}(f') = s\mathcal{L}(f) - f(0)$ $\mathcal{L}(f'') = s^2\mathcal{L}(f) - sf(0) - f'(0)$ $\mathcal{L}(f^{(n)}) = s^n\mathcal{L}(f) - s^{(n-1)}f(0) - \cdots$ $\cdots - f^{(n-1)}(0)$	Differentiation of Function	6.2
$\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{1}{s}\mathcal{L}(f)$	Integration of Function	
$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau$ $= \int_0^t f(t-\tau)g(\tau)\,d\tau$ $\mathcal{L}(f * g) = \mathcal{L}(f)\mathcal{L}(g)$	Convolution	6.5

Laplace Transform. General Formulas

$\mathcal{L}\{f(t-a)\,u(t-a)\} = e^{-as}F(s)$ $\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\,u(t-a)$	t-Shifting (Second Shifting Theorem)	6.3
$\mathcal{L}\{tf(t)\} = -F'(s)$ $\mathcal{L}\left\{\dfrac{f(t)}{t}\right\} = \displaystyle\int_s^\infty F(\tilde{s})\,d\tilde{s}$	Differentiation of Transform Integration of Transform	6.6
$\mathcal{L}(f) = \dfrac{1}{1-e^{-ps}}\displaystyle\int_0^p e^{-st}f(t)\,dt$	f Periodic with Period p	6.4 Project 16

Table of Laplace Transforms

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	$F(s) = \mathcal{L}\{f(t)\}$	$f(t)$	Sec.
1	$1/s$	1	6.1
2	$1/s^2$	t	
3	$1/s^n \quad (n = 1, 2, \cdots)$	$t^{n-1}/(n-1)!$	
4	$1/\sqrt{s}$	$1/\sqrt{\pi t}$	
5	$1/s^{3/2}$	$2\sqrt{t/\pi}$	
6	$1/s^a \quad (a > 0)$	$t^{a-1}/\Gamma(a)$	
7	$\dfrac{1}{s-a}$	e^{at}	6.1
8	$\dfrac{1}{(s-a)^2}$	te^{at}	
9	$\dfrac{1}{(s-a)^n} \quad (n = 1, 2, \cdots)$	$\dfrac{1}{(n-1)!} t^{n-1}e^{at}$	
10	$\dfrac{1}{(s-a)^k} \quad (k > 0)$	$\dfrac{1}{\Gamma(k)} t^{k-1}e^{at}$	

Table of Laplace Transforms

	$F(s) = \mathcal{L}\{f(t)\}$	$f(t)$	Sec.
11	$\dfrac{1}{(s-a)(s-b)} \quad (a \neq b)$	$\dfrac{1}{(a-b)}(e^{at} - e^{bt})$	
12	$\dfrac{s}{(s-a)(s-b)} \quad (a \neq b)$	$\dfrac{1}{(a-b)}(ae^{at} - be^{bt})$	
13	$\dfrac{1}{s^2 + \omega^2}$	$\dfrac{1}{\omega} \sin \omega t$	6.1
14	$\dfrac{s}{s^2 + \omega^2}$	$\cos \omega t$	
15	$\dfrac{1}{s^2 - a^2}$	$\dfrac{1}{a} \sinh at$	
16	$\dfrac{s}{s^2 - a^2}$	$\cosh at$	
17	$\dfrac{1}{(s-a)^2 + \omega^2}$	$\dfrac{1}{\omega} e^{at} \sin \omega t$	
18	$\dfrac{s-a}{(s-a)^2 + \omega^2}$	$e^{at} \cos \omega t$	

Table of Laplace Transforms

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	$F(s) = \mathscr{L}\{f(t)\}$	$f(t)$	Sec.
19	$\dfrac{1}{s(s^2 + \omega^2)}$	$\dfrac{1}{\omega^2}(1 - \cos \omega t)$	6.2
20	$\dfrac{1}{s^2(s^2 + \omega^2)}$	$\dfrac{1}{\omega^3}(\omega t - \sin \omega t)$	
21	$\dfrac{1}{(s^2 + \omega^2)^2}$	$\dfrac{1}{2\omega^3}(\sin \omega t - \omega t \cos \omega t)$	6.6
22	$\dfrac{s}{(s^2 + \omega^2)^2}$	$\dfrac{t}{2\omega} \sin \omega t$	6.6
23	$\dfrac{s^2}{(s^2 + \omega^2)^2}$	$\dfrac{1}{2\omega}(\sin \omega t + \omega t \cos \omega t)$	
24	$\dfrac{s}{(s^2 + a^2)(s^2 + b^2)} \quad (a^2 \neq b^2)$	$\dfrac{1}{b^2 - a^2}(\cos at - \cos bt)$	

Table of Laplace Transforms

	$F(s) = \mathcal{L}\{f(t)\}$	$f(t)$	Sec.
25	$\dfrac{1}{s^4 + 4k^4}$	$\dfrac{1}{4k^3}(\sin kt \cos kt - \cos kt \sinh kt)$	
26	$\dfrac{s}{s^4 + 4k^4}$	$\dfrac{1}{2k^2} \sin kt \sinh kt$	
27	$\dfrac{1}{s^4 - k^4}$	$\dfrac{1}{2k^3}(\sinh kt - \sin kt)$	
28	$\dfrac{s}{s^4 - k^4}$	$\dfrac{1}{2k^2}(\cosh kt - \cos kt)$	
29	$\sqrt{s-a} - \sqrt{s-b}$	$\dfrac{1}{2\sqrt{\pi t^3}}(e^{bt} - e^{at})$	
30	$\dfrac{1}{\sqrt{s+a}\sqrt{s+b}}$	$e^{-(a+b)t/2} I_0\left(\dfrac{a-b}{2} t\right)$	5.6
31	$\dfrac{1}{\sqrt{s^2+a^2}}$	$J_0(at)$	5.5

Table of Laplace Transforms

	$F(s) = \mathcal{L}\{f(t)\}$	$f(t)$	Sec.
32	$\dfrac{s}{(s-a)^{3/2}}$	$\dfrac{1}{\sqrt{\pi t}} e^{at}(1 + 2at)$	
33	$\dfrac{1}{(s^2 - a^2)^k} \qquad (k > 0)$	$\dfrac{\sqrt{\pi}}{\Gamma(k)} \left(\dfrac{t}{2a}\right)^{k-1/2} I_{k-1/2}(at)$	5.6
34	e^{-as}/s	$u(t - a)$	6.3
35	e^{-as}	$\delta(t - a)$	6.4
36	$\dfrac{1}{s} e^{-k/s}$	$J_0(2\sqrt{kt})$	5.5
37	$\dfrac{1}{\sqrt{s}} e^{-k/s}$	$\dfrac{1}{\sqrt{\pi t}} \cos 2\sqrt{kt}$	
38	$\dfrac{1}{s^{3/2}} e^{k/s}$	$\dfrac{1}{\sqrt{\pi k}} \sinh 2\sqrt{kt}$	
39	$e^{-k\sqrt{s}} \qquad (k > 0)$	$\dfrac{k}{2\sqrt{\pi t^3}} e^{-k^2/4t}$	

Table of Laplace Transforms

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	$F(s) = \mathcal{L}\{f(t)\}$	$f(t)$	Sec.
40	$\frac{1}{s} \ln s$	$-\ln t - \gamma \quad (\gamma \approx 0.5772)$	5.6
41	$\ln \frac{s-a}{s-b}$	$\frac{1}{t}(e^{bt} - e^{at})$	
42	$\ln \frac{s^2 + \omega^2}{s^2}$	$\frac{2}{t}(1 - \cos \omega t)$	6.6
43	$\ln \frac{s^2 - a^2}{s^2}$	$\frac{2}{t}(1 - \cosh at)$	
44	$\arctan \frac{\omega}{s}$	$\frac{1}{t} \sin \omega t$	
45	$\frac{1}{s} \operatorname{arccot} s$	$\operatorname{Si}(t)$	App. A3.1