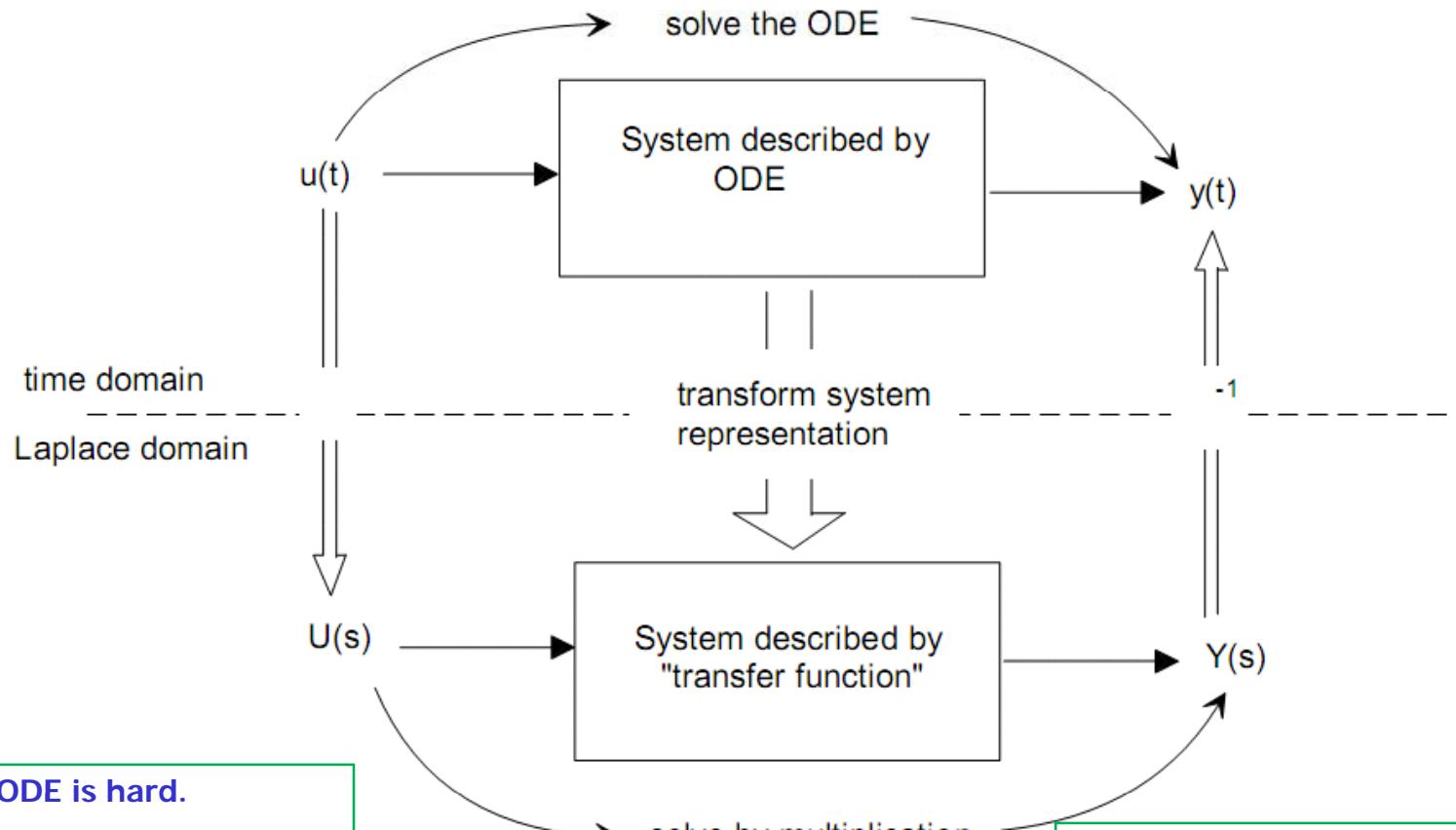


# Laplace Transformation I



# Why use Laplace Transform?

## Algebraic Manipulation of ODE



# Laplace Transformation

- Definition:  $\mathcal{L} [f(t)] = \int_{0^-}^{\infty} f(t) e^{-st} dt = F(s)$   
 $\mathcal{L}: f(t) \Rightarrow F(s), \quad s=\sigma + j\omega$  (complex variable)

f(t) : a time function such that  $f(t)=0$  for  $t<0$

- Inverse Laplace Transformation

$$\mathcal{L}^{-1}[F(s)] = f(t)$$



# Existence of Laplace Transformation

- $f(t)$  Laplace – transformable

if i)  $f(t)$  piecewise-continuous

ii)  $f(t)$  of exponential order as  $t$  approaches infinity

-  $e^{\alpha t} |f(t)|$  bounded,  $\alpha$  exist.

- or  $e^{-\sigma t} |f(t)|$  approaches zero as  $t$  approaches infinity.

- If  $\lim_{t \rightarrow \infty} e^{-\sigma t} |f(t)| = \begin{cases} 0 & \text{for } \sigma > \sigma_c \\ \infty & \text{for } \sigma < \sigma_c \end{cases}$

the  $\sigma_c$  : the abscissa of convergence



# Existence of Laplace Transformation

example : 1)  $t, \sin \omega t, t \sin \omega t\dots$

$$\lim_{t \rightarrow \infty} e^{-\sigma t} |t \sin \omega t| = \begin{cases} 0 & \text{if } \sigma > 0 \\ \infty & \text{if } \sigma < 0 \end{cases} \quad \text{the abscissa of convergence } \sigma_c = 0$$

2)  $e^{-ct}, te^{-ct}, e^{-ct} \sin \omega t, c = \text{const.}$

$$\lim_{t \rightarrow \infty} e^{-\sigma t} |te^{-ct}| = \begin{cases} 0 & \text{if } \sigma > -c \\ \infty & \text{if } \sigma < -c \end{cases} \quad \text{the abscissa of convergence } \sigma_c = -c$$

- $e^{t^2}, te^{t^2}$  does not possess L. T.
- $f(t) = \begin{cases} e^{t^2} & \text{for } 0 \leq t \leq T < \infty \\ 0 & \text{for } t < 0, T < t \end{cases}$   $L[f(t)]$  exists.

- The signals that can be physically generated always have corresponding  
Laplace transforms



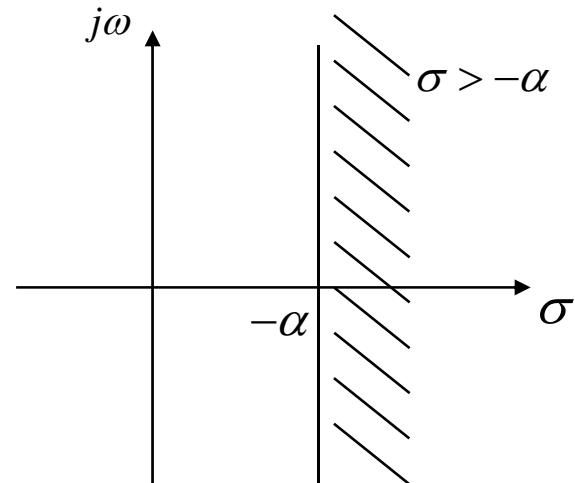
# Laplace Transformation of simple function

- Exponential function

$$f(t) = \begin{cases} 0 & \text{for } t < 0 \\ Ae^{-\alpha t} & \text{for } t \geq 0 \end{cases}$$

$A, \alpha$  : constants

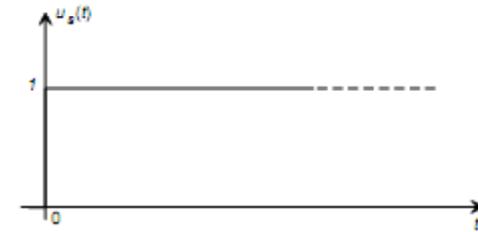
$$\begin{aligned}\mathcal{L}[Ae^{-\alpha t}] &= \int_{0^-}^{\infty} Ae^{-\alpha t} \cdot e^{-st} dt \\ &= \int_{0^-}^{\infty} Ae^{-(\alpha+s)t} dt \\ &= A \left[ -\frac{1}{s+\alpha} e^{-(s+\alpha)\cdot\infty} + \frac{1}{s+\alpha} e^0 \right] \\ &= A \left[ -\frac{1}{s+\alpha} e^{-(\sigma+\alpha)\cdot\infty - j\omega\cdot\infty} + \frac{1}{s+\alpha} \right] \quad (s = \sigma + j\omega) \\ &= A \left[ 0 + \frac{1}{s+\alpha} \right] = \frac{A}{s+\alpha}\end{aligned}$$



# Laplace Transformation of simple function

- step function

$$f(t) = \begin{cases} A & t > 0 \\ 0 & t < 0 \end{cases}$$



$$\begin{aligned}\mathcal{L}[f(t)] &= \int_{0^-}^{\infty} Ae^{-st} dt = A\left[-\frac{1}{s}e^{-s\cdot\infty} + \frac{1}{s}e^{-s\cdot 0}\right] \\ &= A\left[0 + \frac{1}{s}\right] \quad \text{if } \operatorname{Re}[s] > 0\end{aligned}$$

- unit step input function

$$1(t - t_0) = \begin{cases} 1 & t > t_0 \\ 0 & t < t_0 \end{cases}$$

$$1(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$$

$$\mathcal{L}[1(t)] = \frac{1}{s}$$



# Laplace Transformation of simple function

- Sinusoidal Functions

$$f(t) = \begin{cases} A \sin \omega t & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$\cos \omega t = \frac{1}{2}(e^{j\omega t} + e^{-j\omega t})$$

$$\sin \omega t = \frac{1}{2j}(e^{j\omega t} - e^{-j\omega t})$$

$$\begin{aligned}\mathcal{L}[A \sin \omega t] &= \int_{0^-}^{\infty} \frac{A}{2j}(e^{j\omega t} - e^{-j\omega t}) e^{-st} dt \\ &= \frac{1}{2j} \left( \frac{1}{s - j\omega} - \frac{1}{s + j\omega} \right) \\ &= \frac{A\omega}{s^2 + \omega^2} \\ \mathcal{L}[A \cos \omega t] &= \frac{As}{s^2 + \omega^2}\end{aligned}$$

- Ramp function

$$f(t) = \begin{cases} At & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$\begin{aligned}\mathcal{L}[At] &= A \int_{0^-}^{\infty} t e^{-st} dt \\ &= A \left( t \frac{e^{-st}}{-s} \Big|_{0^-}^{\infty} - \int_{0^-}^{\infty} \frac{e^{-st}}{-s} dt \right) \\ &= A \cdot \frac{1}{s} \int_{0^-}^{\infty} e^{-st} dt = A \cdot \frac{1}{s^2}\end{aligned}$$



# Laplace Transformation of simple function

- Pulse function

$$f(t) = \begin{cases} \frac{A}{t_0} & \text{for } 0 \leq t \leq t_0 \\ 0 & \text{for } t < 0, t_0 < t \end{cases}$$

$$f(t) = \frac{A}{t_0} 1(t) - \frac{A}{t_0} 1(t - t_0)$$

$$\begin{aligned}\mathcal{L}[f(t)] &= \mathcal{L}\left[\frac{A}{t_0} 1(t)\right] - \mathcal{L}\left[\frac{A}{t_0} 1(t - t_0)\right] \\ &= \frac{A}{t_0} \frac{1}{s} (1 - e^{-t_0 s})\end{aligned}$$

- Impulse function

$$f(t) = \begin{cases} \lim_{t_0 \rightarrow 0} \frac{A}{t_0} & \text{for } 0 \leq t < t_0 \\ 0 & \text{for } t < 0, t_0 < t \end{cases}$$

$$\begin{aligned}\mathcal{L}[f(t)] &= \lim_{t_0 \rightarrow 0} \left[ \frac{A}{t_0} \frac{1}{s} (1 - e^{-t_0 s}) \right] \\ &= \lim_{t_0 \rightarrow 0} \frac{\frac{d}{dt_0} \left[ A(1 - e^{-t_0 s}) \right]}{\frac{d}{dt_0} (t_0 s)} = \frac{A \cdot s}{s} = A\end{aligned}$$



# Laplace Transformation of simple function

- Unit impulse function ; impulse function of magnitude 1

$$f(t) = \begin{cases} \lim_{t_0 \rightarrow 0} \frac{1}{t_0} & \text{for } 0 \leq t < t_0 \\ 0 & \text{for } t < 0, \quad t_0 < t \end{cases}$$

$$\delta(t) = \begin{cases} \infty & t = 0 \\ 0 & t \neq 0 \end{cases} \quad \mathcal{L}[\delta(t)] = \int_0^{\infty} \delta(t) e^{-st} dt = \int_{0-}^{0+} \delta(t) dt = 1$$

The unit-impulse function occurring at  $t = t_0$

$$\delta(t - t_0) = \begin{cases} \infty & t = t_0 \\ 0 & t \neq t_0 \end{cases} \quad \mathcal{L}[\delta(t - t_0)] = \int_0^{\infty} \delta(t - t_0) e^{-st} dt = \int_{t_0-}^{t_0+} \delta(t - t_0) e^{-st_0} dt = e^{-t_0 s} \cdot 1$$

$$\delta(t - t_0) = \frac{d}{dt} 1(t - t_0)$$

