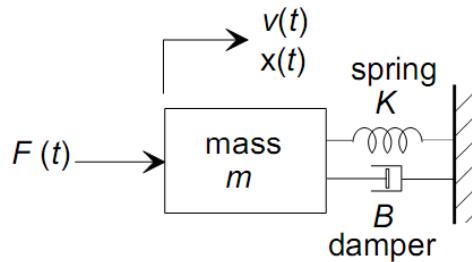


# Transfer Function and Block Diagram Approach to Modeling Dynamic Systems



# Solution of Differential Equation by Laplace Transformation



$$m\ddot{x} + B\dot{x} + Kx = F.$$

$$y'' + 2y' + 4y = 1 \quad y(0) = 0, \quad y'(0) = 2$$

L.T:  $s^2Y(s) - sy(0) - y'(0) + 2\{sY(s) - y(0)\} + 4Y(s) = \frac{1}{s}$

$$(s^2 + 2s + 4)Y(s) = \frac{1}{s} + 2 = \frac{2s+1}{s}$$

$$Y(s) = \frac{2s+1}{s(s^2 + 2s + 4)} = \frac{1}{4s} - \frac{1}{4} \frac{s+1-1}{(s+1)^2 + (\sqrt{3})^2}$$

$$\therefore y(t) = \frac{1}{4} - \frac{1}{4} e^{-t} \cos \sqrt{3}t + \frac{1}{4\sqrt{3}} e^{-t} \sin \sqrt{3}t$$



# The Concept of Transfer Function

the TF of the system is not a function of time. (except the input and output)

Consider the linear time-invariant system defined by the following differential

equation :

$$\begin{aligned} a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} \dot{y} + a_n y \\ = b_0 u^{(m)} + b_1 u^{(m-1)} + \cdots + b_{m-1} \dot{u} + b_m u \quad (n \geq m) \end{aligned}$$

Where  $y$  is the output of the system, and  $x$  is the input. And the Laplace transform of the equation is,

$$\begin{aligned} (a_0 S^n + a_1 S^{n-1} + \cdots + a_{n-1} S + a_n) Y(s) \\ = (b_0 S^m + b_1 S^{m-1} + \cdots + b_{m-1} S + b_m) U(s) \end{aligned}$$



# The Concept of Transfer Function

The ratio of the Laplace transform of the output (response function) to the Laplace Transform of the input (driving function) under the assumption that all initial conditions are Zero.

$$\begin{aligned} \text{Transfer Function} &= \frac{Y(s)}{U(s)} = G(s) = \frac{b_0 S^m + b_1 S^{m-1} + \dots + b_{m-1} S + b_m}{a_0 S^n + a_1 S^{n-1} + \dots + a_{n-1} S + a_n} \\ &= \frac{P(s)}{Q(s)} \quad (n \geq m) \end{aligned}$$



# Comments on Transfer Function

1. A mathematical model.
2. Property of system itself.
  - Independent of the input function and initial condition
  - Denominator of the transfer function is the characteristic polynomial,
  - TF tells us something about the intrinsic behavior of the model.
3. ODE equivalence
  - TF is equivalent to the ODE. We can reconstruct ODE from TF.
4. One TF for one input-output pair. : Single Input Single Output system.
  - If multiple inputs affect → Obtain TF for each input

$$\ddot{x} + 6\dot{x} + 20x = 7f(t) + 3g(t)$$

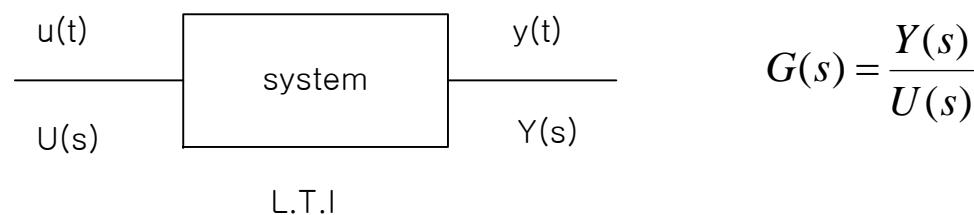
-If multiple outputs

$$\begin{aligned}\dot{x} &= -3x + 2y \\ \dot{y} &= -y - x + 3u(t)\end{aligned}$$



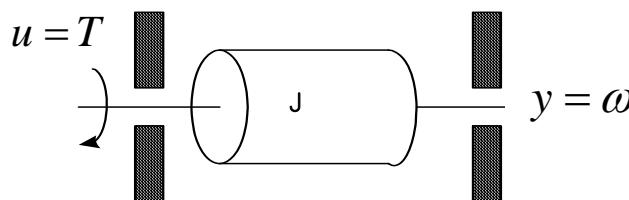
## Comments on Transfer Function

5. Analytic method and Experimental method



6. Different systems may have identical T.F

ex)



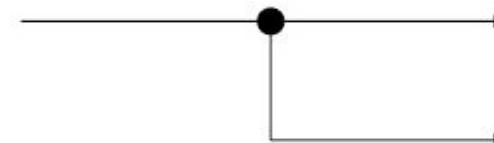
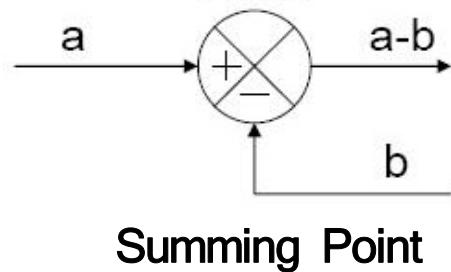
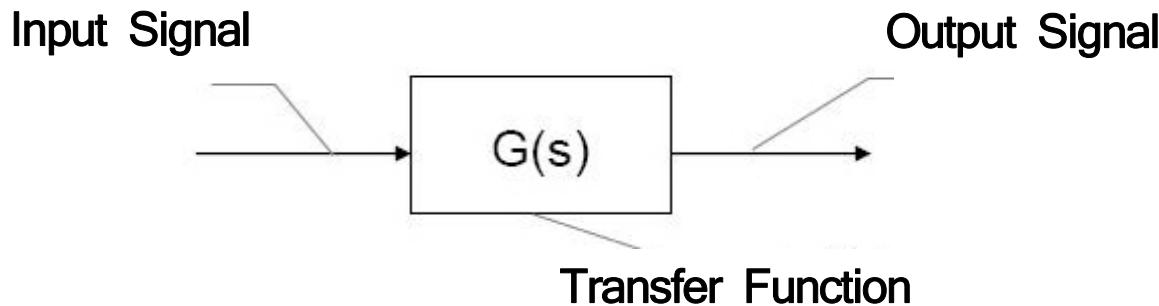
$$J\dot{\omega} + b\omega = T \quad \frac{Y}{U} = \frac{\omega}{T} = \frac{1}{Js + b}$$

Electrical System

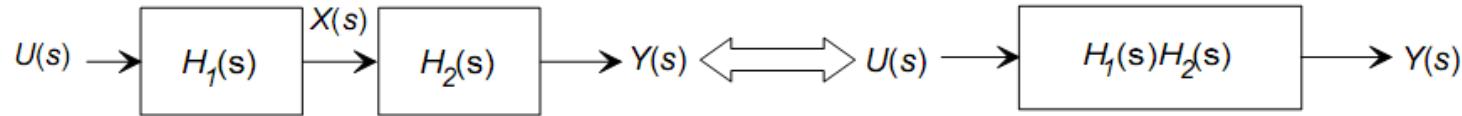
$$\frac{Y}{U} = \frac{R}{Ls + R} = \frac{1}{\frac{L}{R}s + 1} \quad (\text{if } b=1, J=L/R)$$



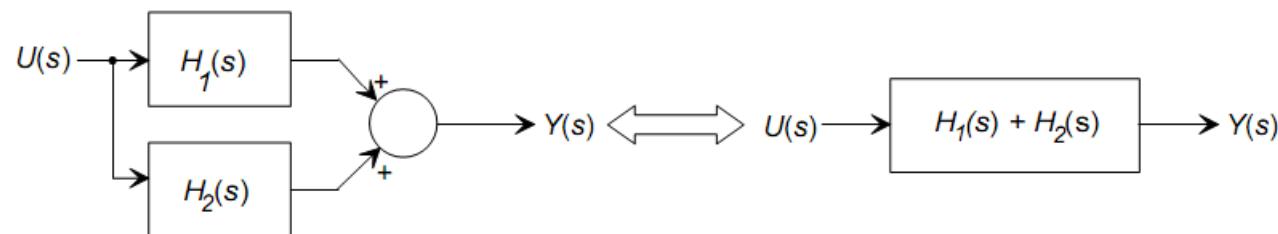
# Block Diagram



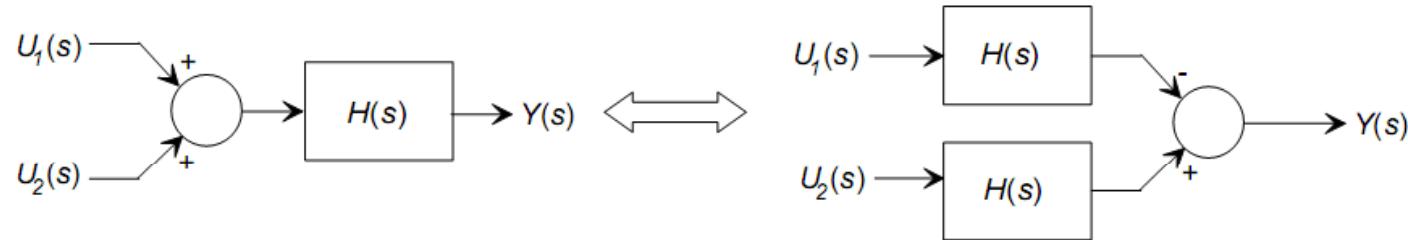
## Series(Cascade) Connection



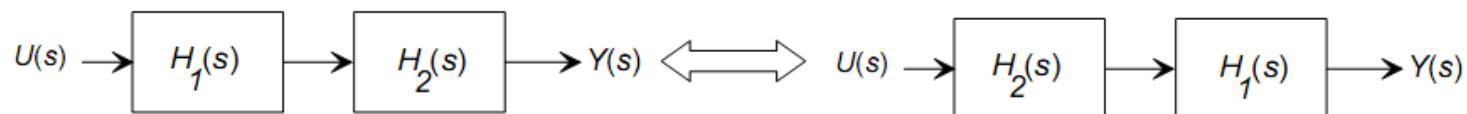
## Parallel Connection



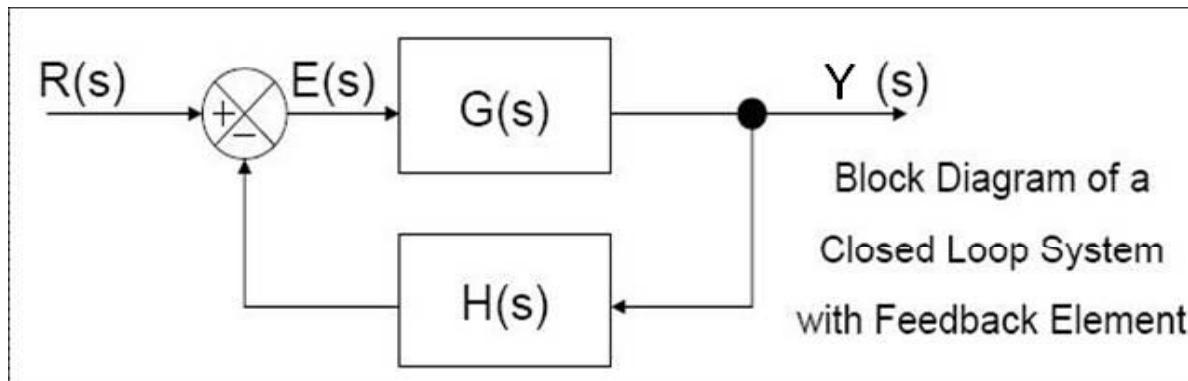
## Associative Rule



## Commutative Rule



# Closed Loop Transfer Function



$$Y(s) = G(s)E(s) = G(s)[R(s) - H(s)Y(s)]$$

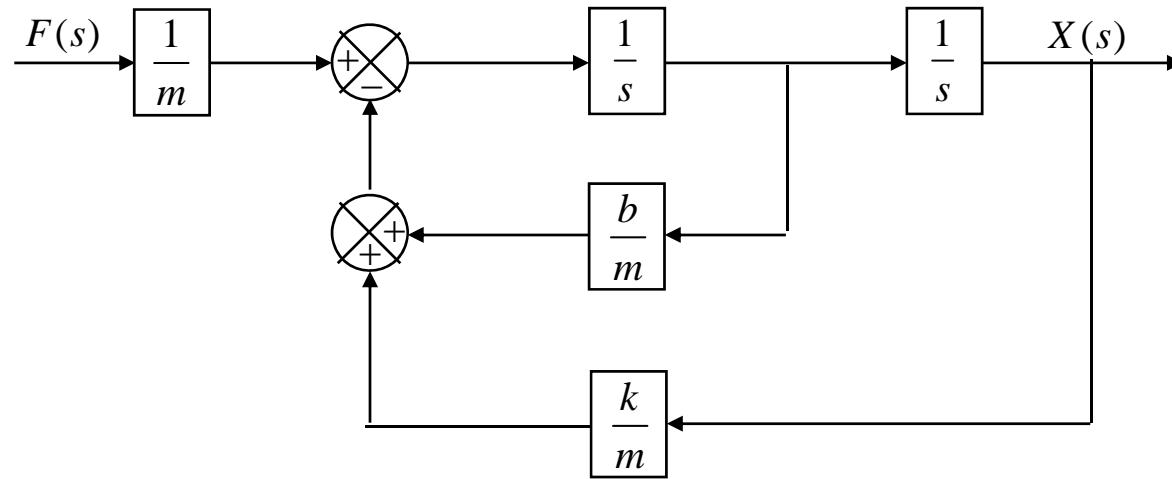
$$[1 + G(s)H(s)]Y(s) = G(s)R(s)$$

$$\therefore \text{Transfer Function} = \frac{Y(s)}{R(s)} = \frac{G_1(s)}{1 + G_1(s)H(s)}$$



# Closed Loop Transfer Function

ex)



$$\frac{1}{m} F(s) - \frac{k}{m} X(s) - \frac{b}{m} s X(s) = s^2 X(s)$$

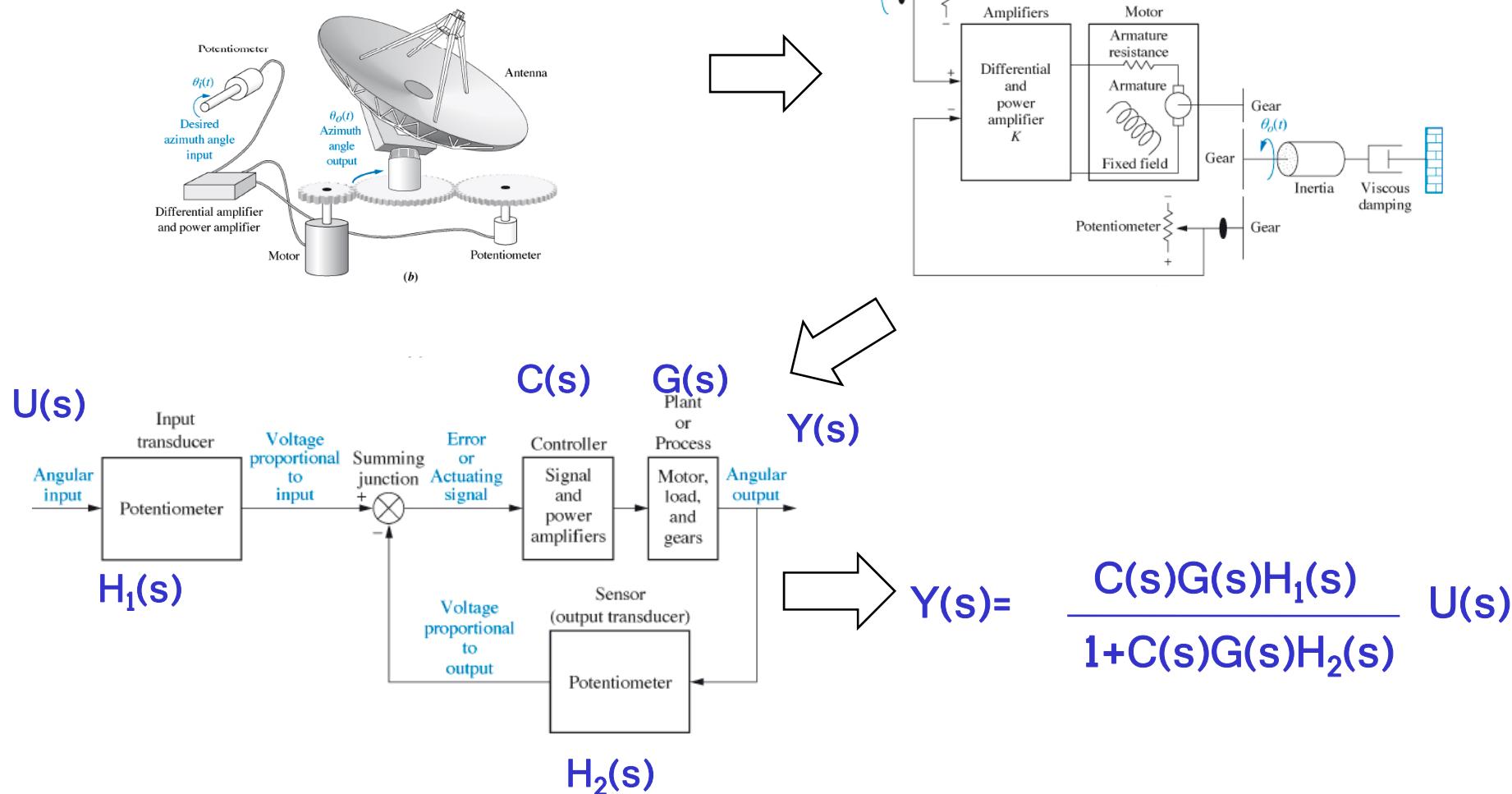
$$F(s) - [kX(s) + bsX(s)] = ms^2 X(s)$$

$$(ms^2 + bs)X(s) = F(s) - kX(s), \quad (ms^2 + bs + k)X(s) = F(s)$$

$$\therefore \frac{X(s)}{F(s)} = \frac{1}{ms^2 + bs + k} \quad \text{Transfer Function}$$

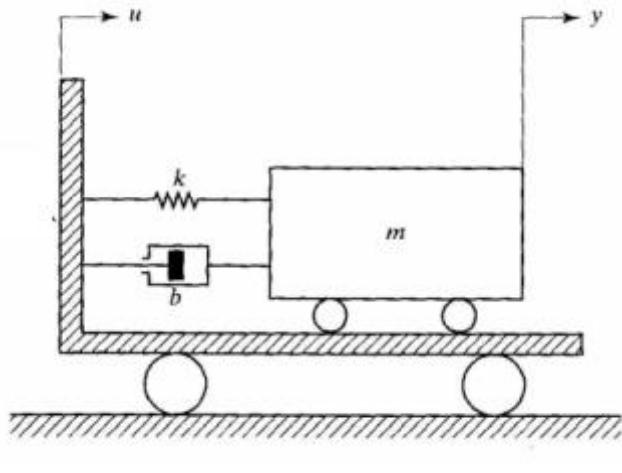


# System → Schematic → Block Diagram → Transfer Functions



## Example

ex)



$$m \frac{d^2 y}{dt^2} = -b \left( \frac{dy}{dt} - \frac{du}{dt} \right) - k(y - u)$$

$$\text{or } m \frac{d^2 y}{dt^2} + b \frac{dy}{dt} + ky = b \frac{du}{dt} + ku$$

Assume the initial condition is 0,

$$(ms^2 + bs + k)Y(s) = (bs + k)U(s)$$

$$\text{Transfer Function} = \frac{Y(s)}{U(s)} = \frac{bs + k}{ms^2 + bs + k}$$

If,  $m=10\text{kg}$ ,  $b=20\text{N}\cdot\text{s}/\text{m}$ ,  $k=100\text{N}/\text{m}$

$$\frac{Y(s)}{U(s)} = \frac{20s + 100}{10s^2 + 20s + 100} = \frac{2s + 10}{s^2 + 2s + 10}, \quad U(s) = \frac{1}{s}$$

$$\therefore Y(s) = \frac{2s + 10}{s^2 + 2s + 10} \frac{1}{s} = \frac{2s + 10}{s^3 + 2s^2 + 10s}$$



# The law of conservation of momentum

By Newton's 2nd law,

$$F = ma = m \frac{dv}{dt} = \frac{d}{dt}(mv) \rightarrow F \cdot dt = d(mv)$$

$$\int_{t_1}^{t_2} F dt = \int_{v_1}^{v_2} d(mv) = mv_2 - mv_1$$

And if There is no input force, then we get

$$d(mv) = 0, \quad mv = \text{const.} \quad \text{The law of conservation of momentum}$$

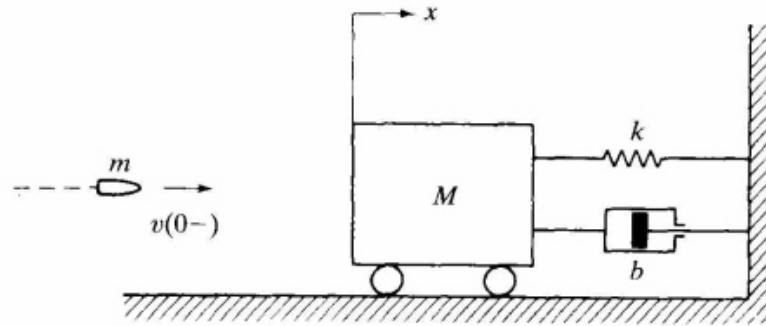
And also,

$$I\omega = \text{const.}$$

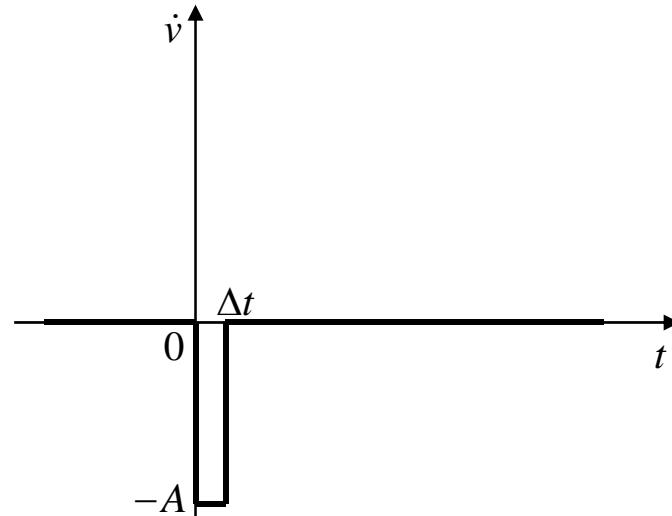
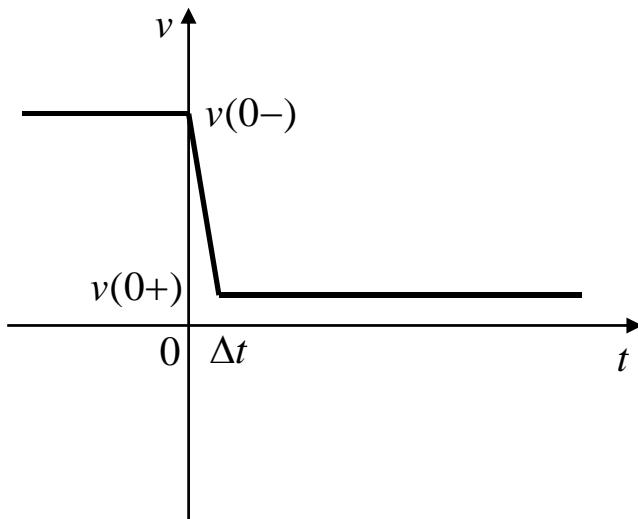
The law of conservation of angular momentum



# Impulse Input



A bullet stuck in mass  $M$



Sudden change in the velocity and acceleration of bullet



# Impulse Input

System equation :  $(M + m)\ddot{x} + b\dot{x} + kx = F(t)$

Impulse force :  $F(t) = -m\dot{v} = A\Delta t \delta(t), \quad A\Delta t$  : Magnitude of impulse force

$$\int_{0-}^{0+} A\Delta t \delta(t) dt = -m \int_{0-}^{0+} \dot{v} dt, \quad A\Delta t = mv(0-) - mv(0+)$$

Noting that, Magnitude of impulse force is equal to change of momentum!!

$v(0+) = \dot{x}(0+) =$  Initial velocity of M+m

Thus the system equation becomes

$$(M + m)\ddot{x} + b\dot{x} + kx = F(t) = [mv(0-) - m\dot{x}(0+)]\delta(t)$$



## Impulse Input

By taking the Laplace Transform,

$$(M + m)[s^2 X(s) - sx(0-) - \dot{x}(0-)] + b[sX(s) - x(0-)] + kX(s) = mv(0-) - m\dot{x}(0+)$$

Noting that  $\dot{x}(0-) = x(0-) = 0$  we obtain  $X(s) = \frac{mv(0-) - m\dot{x}(0+)}{(M + m)s^2 + bs + k}$

$$\dot{x}(0+) = \lim_{t \rightarrow 0+} \dot{x}(t) = \lim_{s \rightarrow \infty} s[X(s)] = \lim_{s \rightarrow \infty} \frac{s^2 [mv(0-) - m\dot{x}(0+)]}{(M + m)s^2 + bs + k} = \frac{mv(0-) - m\dot{x}(0+)}{M + m}$$

$$\rightarrow \dot{x}(0+) = \frac{m}{M + 2m} v(0-)$$

$$\therefore X(s) = \frac{(M + m)\dot{x}(0+)}{(M + m)s^2 + bs + k} = \frac{1}{(M + m)s^2 + bs + k} \frac{(M + m)mv(0-)}{M + 2m}$$



# Partial Fraction Expansion with MATLAB

$$\frac{B(s)}{A(s)} = \frac{\text{num}}{\text{den}} = \frac{b(1)S^h + b(2)S^{h-1} + \dots + b(h+1)}{a(1)S^n + a(2)S^{n-1} + \dots + a(n+1)}$$

$$\text{num} = [b(1) \quad b(2) \quad \dots \quad b(h)], \quad \text{den} = [a(1) \quad a(2) \quad \dots \quad a(h)]$$

[r, p, k] = residue (num, den)

$$\frac{B(s)}{A(s)} = k(s) + \frac{r(1)}{s - p(1)} + \frac{r(2)}{s - p(2)} + \dots + \frac{r(n)}{s - p(n)}$$

ex)  $\frac{B(s)}{A(s)} = \frac{s^4 + 8s^3 + 16s^2 + 9s + 6}{s^3 + 6s^2 + 11s + 6}$

$$= s + 2 + \frac{-6}{s + 3} + \frac{-4}{s + 2} + \frac{3}{s + 1}$$

```
>> num=[1 8 16 9 6];
>> den=[1 6 11 6];
>> [r,p,k]=residue(num,den)
r=
-6.0000
-4.0000
3.0000
p=
-3.0000
-2.0000
-1.0000
k=
1           2
```

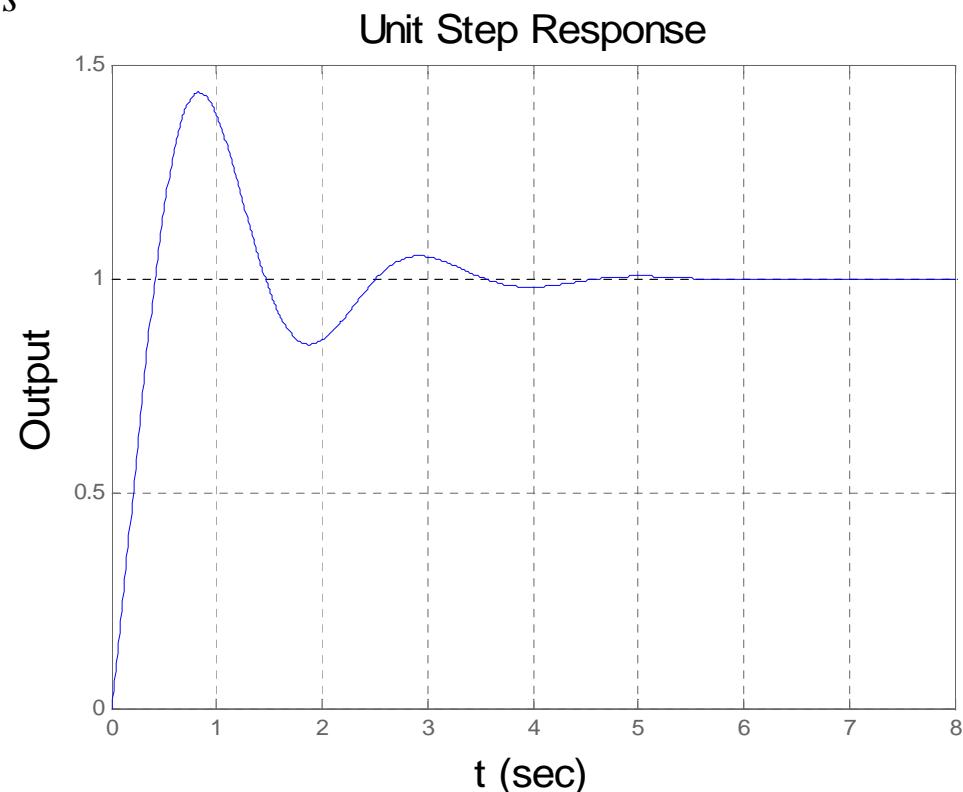


# Transient Response Analysis with MATLAB

$$\frac{Y(s)}{U(s)} = \frac{20s + 100}{10s^2 + 20s + 100} = \frac{2s + 10}{s^2 + 2s + 10}, \quad U(s) = \frac{1}{s}$$

$$\therefore Y(s) = \frac{2s + 10}{s^2 + 2s + 10} \cdot \frac{1}{s} = \frac{2s + 10}{s^3 + 2s^2 + 10s}$$

```
t=0:0.01:8;
num=[2 10];
den=[1 2 10];
sys=tf(num,den);
step(sys,t)
grid
title('Unit Step Response','FontSize',15')
xlabel('t(sec)','FontSize',15')
ylabel('Output','FontSize',15')
```



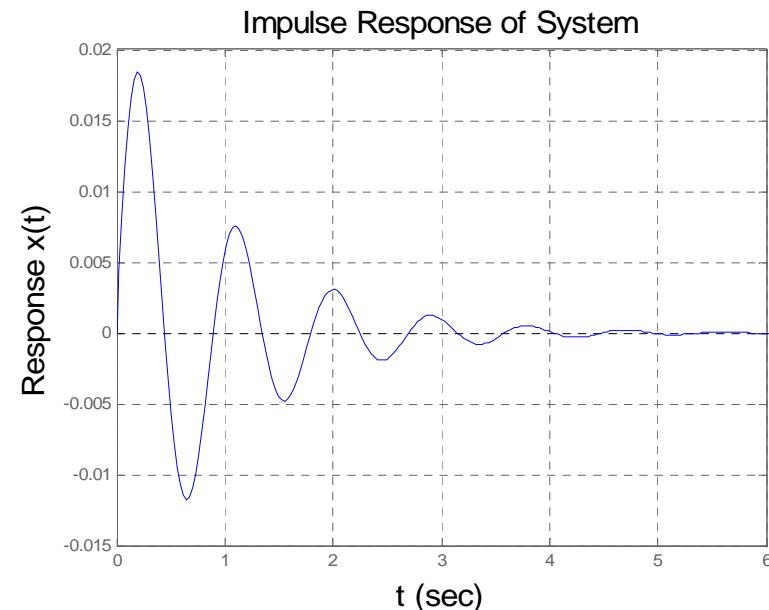
# Impulse Input

ex)

$$M = 50\text{kg}, \quad m = 0.01\text{kg}, \quad b = 100\text{Ns/m}, \quad k = 2500\text{N/m}, \quad v(0-) = 800\text{m/s}$$

$$X(s) = \frac{1}{50.01s^2 + 100s + 2500} \frac{50.01 \times 0.01 \times 800}{50.02} = \frac{7.9984}{50.01s^2 + 100s + 2500}$$

```
num=[7.9984];
den=[50.01 100 2500];
sys=tf(num,den);
impulse(sys)
grid
title('Impulse Response of System','Fontsize',15')
xlabel('t(sec)','Fontsize',15')
ylabel('Response x(t)','Fontsize',15')
```



# Ramp Response

$$\frac{Y(s)}{U(s)} = \frac{2s + 10}{s^2 + 2s + 10}$$

$$M=10\text{kg}, \quad b=20\text{Ns/m}, \quad k=100\text{N/m}$$

$u(t)$ : Unit ramp input,  $u = \alpha t$ ,  $\alpha = 1$

```
num=[2 10];
den=[1 2 10];
sys=tf(num,den);
t=0:0.01:4;
u=t;
lsim(sys,u,t)
grid
title('Unit-Ramp Response','Fontsize',15)
xlabel('t')
ylabel('Output y(t) and Input u(t)=t','Fontsize',15)
text(0.8,0.4,'y','Fontsize',12)
text(0.4,0.8,'u','Fontsize',12)
legend('y')
```

