

Mathematical Modeling of Dynamic Systems in State Space II



System without Input Derivatives

ex1) Consider a system defined by $\ddot{y} + 6\dot{y} + 11y + 6u = 0$

Choose state variables,

$$x_1 = y$$

$$x_2 = \dot{y}$$

$$x_3 = \ddot{y}$$

→ Phase variables (each subsequent state variable is defined to be the derivative of the previous state variable.)

Then we obtain, $\dot{x}_1 = x_2$,

$$\dot{x}_3 = \ddot{y} = -6x_1 - 11x_2 - 6x_3 + 6u$$

$$\therefore \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix} u, \quad y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$



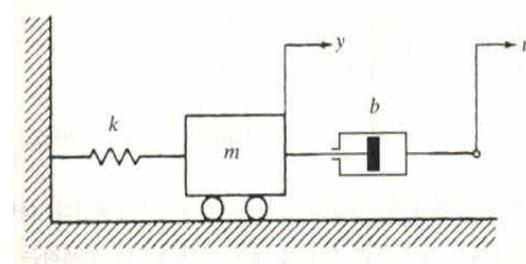
System with Input Derivates

ex2) Consider a mechanical system,

$$m\ddot{y} = -ky - b(\dot{y} - \dot{u}), \quad \ddot{y} = -\frac{k}{m}y - \frac{b}{m}\dot{y} + \frac{b}{m}\dot{u}$$

Choose state variables, $x_1 = y, x_2 = \dot{y}$

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -\frac{k}{m}x_1 - \frac{b}{m}x_2 + \frac{b}{m}\dot{u}$$



The right side includes \dot{u} term. To explain the reason we should not include differentiation of u , assume $u = \delta(t)$ (unit impulse function)

$$x_2 = -\frac{k}{m} \int y dt - \frac{b}{m}y + \frac{k}{m}\delta(t)$$

x_2 includes $(k/m)\delta(t)$ term. It means $x_2(0) = \infty$ and cannot be accepted as a state variable.

That's why the standard form is $\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu}$ $\mathbf{y} = \mathbf{Cx} + \mathbf{Du}$

System with Input Derivatives

Method 1: Choose a state variable that includes u

To eliminate \dot{u} term,

$$\ddot{y} = -\frac{k}{m}y - \frac{b}{m}\dot{y} + \frac{b}{m}\dot{u} \rightarrow \ddot{y} - \frac{b}{m}\dot{u} = -\frac{k}{m}y - \frac{b}{m}\dot{y}$$

$$\frac{d}{dt}\left(\dot{y} - \frac{b}{m}u\right) = -\frac{k}{m}y - \frac{b}{m}\left(\dot{y} - \frac{b}{m}u\right) - \left(\frac{b}{m}\right)^2 u$$

So we choose state variables as, $x_1 = y, x_2 = \dot{y} - \frac{b}{m}u$

$$\begin{aligned}\dot{x}_2 &= \ddot{y} - \frac{b}{m}\dot{u} = \left(-\frac{k}{m}y - \frac{b}{m}\dot{y} + \frac{b}{m}\dot{u}\right) - \frac{b}{m}\dot{u} = -\frac{k}{m}x_1 - \frac{b}{m}\left(x_2 + \frac{b}{m}u\right) \\ &= -\frac{k}{m}x_1 - \frac{b}{m}x_2 - \left(\frac{b}{m}\right)^2 u \quad \rightarrow \quad \dot{u} \text{ term has been eliminated.}\end{aligned}$$

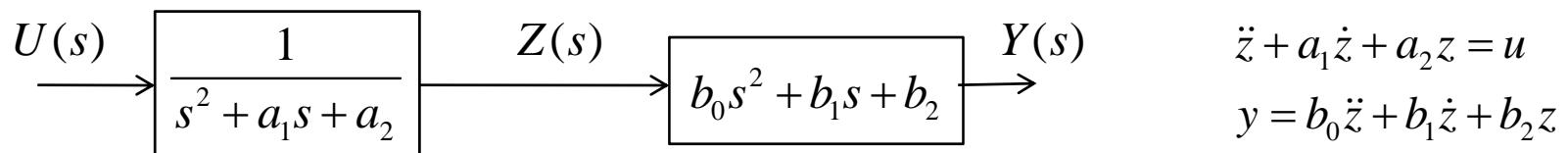
$$\therefore \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \frac{b}{m} \\ -\left(\frac{b}{m}\right)^2 \end{bmatrix} u$$

System with Input Derivatives

Method 2: Include the input derivatives in the output equation

Consider a second-order system, $\ddot{y} + a_1\dot{y} + a_2y = b_0\ddot{u} + b_1\dot{u} + b_2u$

$$\frac{Y(s)}{U(s)} = \frac{b_0s^2 + b_1s + b_2}{s^2 + a_1s + a_2} \rightarrow \boxed{\frac{Z(s)}{U(s)} = \frac{1}{s^2 + a_1s + a_2}}, \quad \boxed{\frac{Y(s)}{Z(s)} = b_0s^2 + b_1s + b_2}$$



let, $x_1 = z, \quad x_2 = \dot{z}$

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -a_2x_1 - a_1x_2 + u\end{aligned}$$

$$b_0\ddot{z} + b_1\dot{z} + b_2z = b_0(-a_2x_1 - a_1x_2 + u) + b_1x_2 + b_2x_1 = y$$

$$\therefore \dot{x}_1 = x_2, \quad \dot{x}_2 = -a_2x_1 - a_1x_2 + u$$

$$y = (b_2 - a_2b_0)x_1 + (b_1 - a_1b_0)x_2 + b_0u$$

System with Input Derivates

Method 2: Include the input derivates in the output equation

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \quad y = [b_2 - a_2 b_0 \quad \vdots \quad b_1 - a_1 b_0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + b_0 u$$

N-th order differential equation,

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} \dot{y} + a_n y = b_0 u^{(n)} + b_1 u^{(n-1)} + \cdots + b_{n-1} \dot{u} + b_n u$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u,$$

$$y = [b_n - a_n b_0 \quad \vdots \quad b_{n-1} - a_{n-1} b_0 \quad \vdots \quad \cdots \quad \vdots \quad b_1 - a_1 b_0] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + b_0 u$$



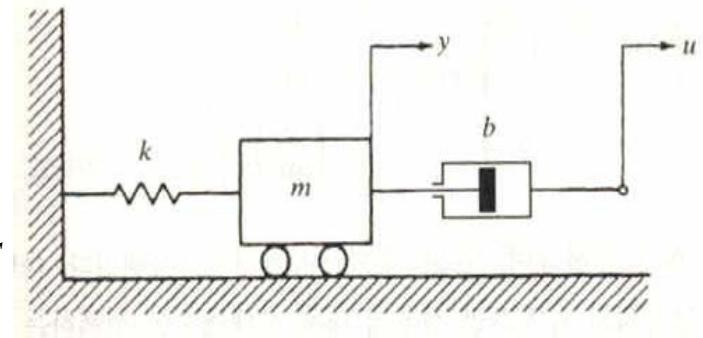
System with Input Derivatives

ex2) Consider this mechanical system again,

$$m\ddot{y} = -ky - b(\dot{y} - \dot{u}), \quad m\ddot{y} + b\dot{y} + ky = b\dot{u}$$

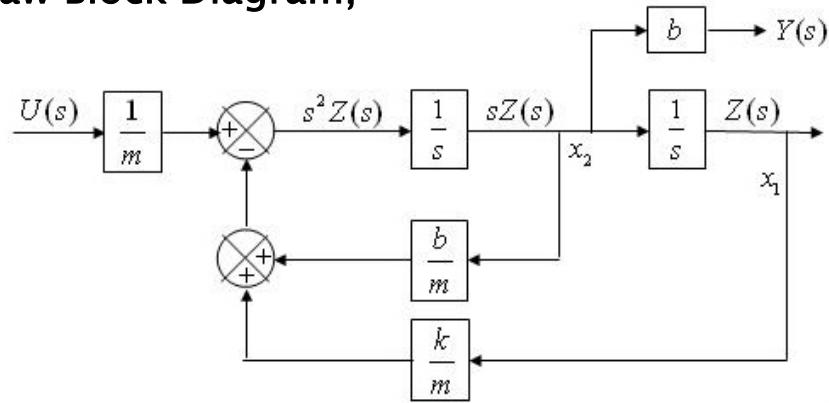
$$\frac{Y(s)}{U(s)} = \frac{bs}{ms^2 + bs + k}, \quad \frac{Z(s)}{U(s)} = \frac{1}{ms^2 + bs + k}, \quad \frac{Y(s)}{Z(s)} = bs$$

$$(ms^2 + bs + k)Z(s) = U(s), \quad bsZ(s) = Y(s)$$



$$s^2 Z(s) = \frac{1}{m} U(s) - \frac{b}{m} s Z(s) - \frac{k}{m} Z(s)$$

Draw Block Diagram,



State variables,

$$x_1 = z, \quad x_2 = \dot{z}$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\frac{k}{m} x_1 - \frac{b}{m} x_2 + \frac{1}{m} u$$

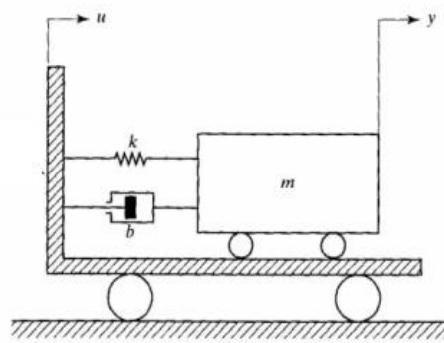
$$y = b x_2$$

System Matrices:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 0 & 1 \end{bmatrix} \quad \mathbf{D} = \begin{bmatrix} 0 \end{bmatrix}$$

System with Input Derivatives

ex) Consider a spring-mass-damper system



m

$$m \frac{d^2 y}{dt^2} = -b \left(\frac{dy}{dt} - \frac{du}{dt} \right) - k(y - u)$$

$$\text{or } m \frac{d^2 y}{dt^2} + b \frac{dy}{dt} + ky = b \frac{du}{dt} + ku$$

$$\text{Transfer Function} = \frac{Y(s)}{U(s)} = \frac{bs + k}{ms^2 + bs + k}$$

State Eq:

Output Eq:

State variables:

Rewrite
equations:

$$b_0 = 0, \quad b_1 = b, \quad b_2 = k \quad b_2 - a_2 b_0 = \frac{k}{m} - \frac{k}{m} + 0 = \frac{k}{m}$$

$$a_1 = \frac{b}{m}, \quad a_2 = \frac{k}{m} \quad b_1 - a_1 b_0 = \frac{b}{m} - \frac{b}{m} + 0 = \frac{b}{m}$$

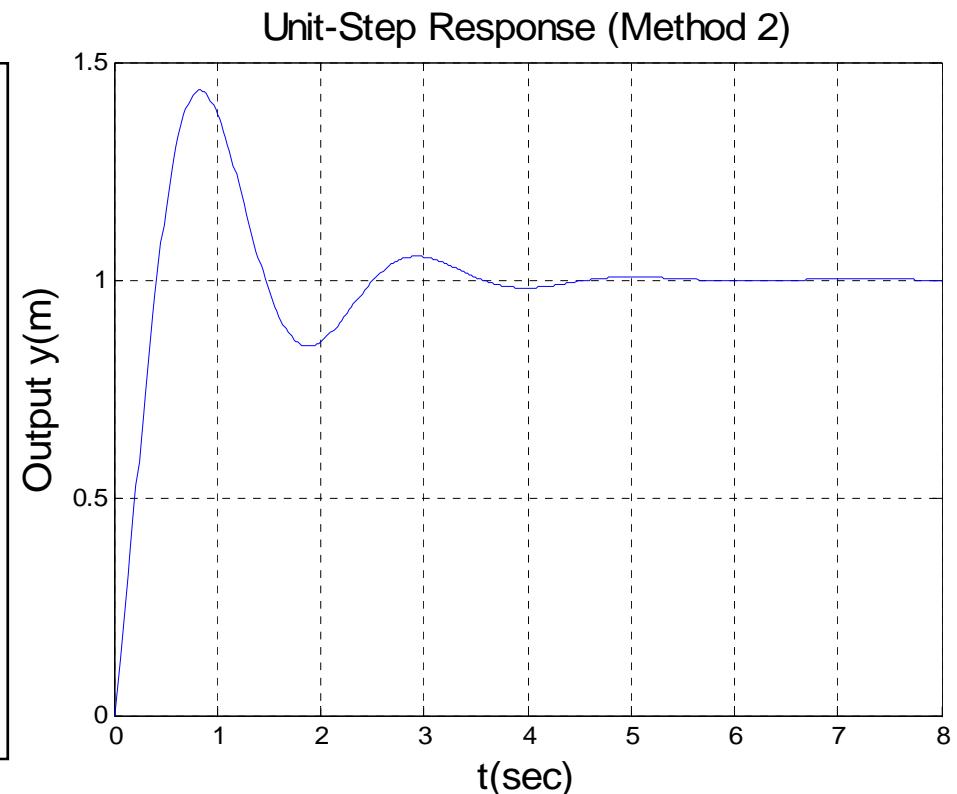
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \quad y = \begin{bmatrix} k & b \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Matlab Example

If, $m=10\text{kg}$, $b=20\text{N}\cdot\text{s/m}$, $k=100\text{N/m}$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -10 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \quad y = [10 \quad 2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

```
t=0:0.02:8;
A=[0 1;-10 -2];
B=[0;1];
C=[10 2];
D=[0];
sys=ss(A,B,C,D);
[y,t]=step(sys,t);
plot(t,y)
grid
title('Unit-Step Response (Method 2)','FontSize',15)
xlabel('t(sec)','FontSize',15)
ylabel('Output y(m)','FontSize',15)
```



Transformation of Mathematical Models with MATLAB

$$\frac{Y(s)}{U(s)} = \frac{\text{numerator polynomial in } s}{\text{denominator polynomial in } s} = \frac{\text{num}}{\text{den}}$$

MATLAB command, $[A, B, C, D] = \text{tf2ss}(\text{num}, \text{den})$ gives a state space representation.

ex) Consider, $\frac{Y(s)}{U(s)} = \frac{s}{s^3 + 14s^2 + 56s + 160}$

One of many possible state-space representations is,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -14 & -56 & 160 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = [0 \ 1 \ 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + [0]u$$

```
>> num=[0 0 1 0];
>> den=[1 14 56 160];
>> [A,B,C,D]=tf2ss(num,den)
A =
-14 -56 -160
1 0 0
0 1 0
B =
1
0
0
C =
0 1 0
D =
0
```



Transformation of a State-Space Models into Another One

- $\dot{x} = Ax + Bu$ can be written as,
 $y = Cx + Du$

$$P\dot{\hat{x}} = AP\hat{x} + Bu \quad \text{or} \quad \dot{\hat{x}} = P^{-1}AP\hat{x} + P^{-1}Bu$$
$$y = CP\hat{x} + Du \quad \quad \quad y = CP\hat{x} + Du$$

– Since infinitely many $n \times n$ matrices can be a transformation matrix P , there are infinitely many state-space models for a given system.

- Eigenvalues of an $n \times n$ matrix A are the roots of the characteristic equation.

$$|\lambda I - A| = 0$$

The eigenvalues are also called the characteristic roots.

ex) $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}$, $|\lambda I - A| = \begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 6 & 11 & \lambda + 6 \end{vmatrix} = \lambda^3 + 6\lambda^2 + 11\lambda + 6$

$$= (\lambda + 1)(\lambda + 2)(\lambda + 3) = 0$$



Diagonalization of State Matrix A

Consider an $n \times n$ state matrix A : $A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix}$

If matrix A has distinct eigenvalues and the state vector x is transformed into another state vector z by use of a transformation matrix P ,

$$x = Pz, \text{ where } P = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_n^2 \\ \vdots & \vdots & & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \end{bmatrix} \quad P^{-1}AP = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

$P^{-1}AP$ is a canonical matrix and each column of P is an eigenvector of matrix A



Jordan Canonical Form

If matrix A involves multiple eigenvalues, diagonalization is not possible but matrix A can be transformed into a Jordan Canonical Form.

Consider the 3×3 matrix A : $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix}$

Assume that matrix A has eigenvalues $\lambda_1, \lambda_2, \lambda_3$ where $\lambda_1 = \lambda_2 \neq \lambda_3$

$$(\lambda I - A)v_1 = 0 \quad A : 3 \times 3 \text{ matrix}, \quad \lambda_1, \lambda_2, \lambda_3 \quad v_1, v_2, v_3$$

Case 1. $\text{rank}(\lambda_1 I - A) = 1$ can determine two eigenvectors v_1, v_2 .

$$Av_1 = \lambda_1 v_1, \quad Av_2 = \lambda_1 v_2, \quad Av_3 = \lambda_3 v_3$$

$$A \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}, \quad P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

Jordan Canonical Form

Case 2. $\text{rank}(\lambda_1 \mathbf{I} - \mathbf{A}) = 2$ can determine one eigenvector v_1 .

$$\mathbf{A}v_1 = \lambda_1 v_1, \quad \mathbf{A}v_3 = \lambda_3 v_3$$

Find v_2 such that $|\mathbf{A} - \lambda_1 \mathbf{I}| v_2 = v_1 \quad \mathbf{A}v_2 = v_1 + \lambda_1 v_2$

$$\mathbf{A} \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

$$x = S z \quad \text{where} \quad S = \begin{bmatrix} 1 & 0 & 1 \\ \lambda_1 & 1 & \lambda_3 \\ \lambda_1^2 & 2\lambda_1 & \lambda_3^2 \end{bmatrix} \quad \text{will yield} \quad S^{-1} \mathbf{A} S = \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = J$$

This is in the Jordan Canonical Form.



Canonical Forms

- Diagonal (or Jordan) Canonical Form (Partial Fraction Expansion)

Case 3. Complex Roots

$$AP = P\Lambda$$

Note : Complex Roots, Complex State x

$$\dot{x} = \Lambda x + bu$$

$$y = Cx$$

→ Complex case의 diagonalization 방법 이용

$$\Lambda K = KJ$$

$$\Lambda = \begin{bmatrix} \sigma + j\omega & 0 \\ 0 & \sigma - j\omega \end{bmatrix}$$

$$K = \begin{bmatrix} \frac{1}{2} & -\frac{j}{2} \\ \frac{1}{2} & \frac{j}{2} \end{bmatrix}$$

$$K^{-1} = \frac{2}{j} \begin{bmatrix} \frac{j}{2} & \frac{j}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$J = K^{-1}\Lambda K$$

$$= K^{-1}P^{-1}APK$$

$$J = \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix}$$

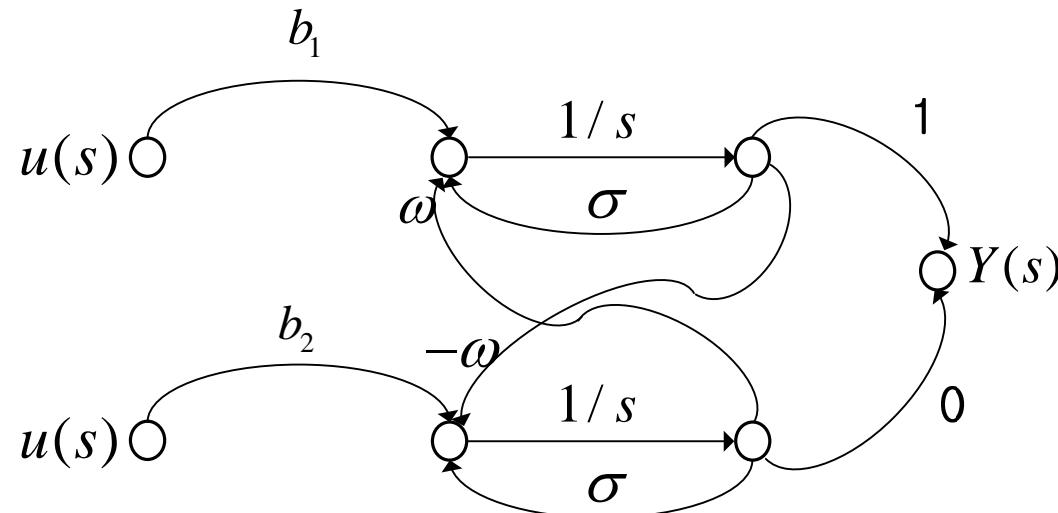


Canonical Forms

- Diagonal (or Jordan) Canonical Form (Partial Fraction Expansion)

Case 3. Complex Roots

Ex) $\dot{z} = \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix} z + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u$
 $y = [1 \ 0] z$



Canonical Forms

- Diagonal (or Jordan) Canonical Form (Partial Fraction Expansion)

Case 3. Complex Roots

Step 1 $\dot{x} = Ax + Bu$

Step 2 let $x = P\xi$
 $\dot{\xi} = \underbrace{P^{-1}AP}_{\Lambda} \xi + P^{-1}Bu$: diagonal

Step 3 let $\xi = Kz$
 $\dot{z} = \underbrace{K^{-1}\Lambda K}_J z + K^{-1}P^{-1}Bu$
 $= \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix} z + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u$

$$\begin{cases} x = P\xi = PKz \\ \dot{z} = \underbrace{K^{-1}\Lambda K}_J z + K^{-1}P^{-1}bu \\ y = CPKz \end{cases}$$

