

Linear Systems Analysis

in the Time Domain III

- Transient Response -



System Response with Additional Poles

Underdamped System with additional pole at $-\alpha$

$$p_1, p_2 = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

$$C(s) = \frac{A}{s} + \frac{B(s + \zeta\omega_n) + C\omega_d}{(s + \zeta\omega_n)^2 + \omega_d^2} +$$

$$y_{step}(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \cos(\omega_d t - \phi) +$$



■ 극점 추가의 영향

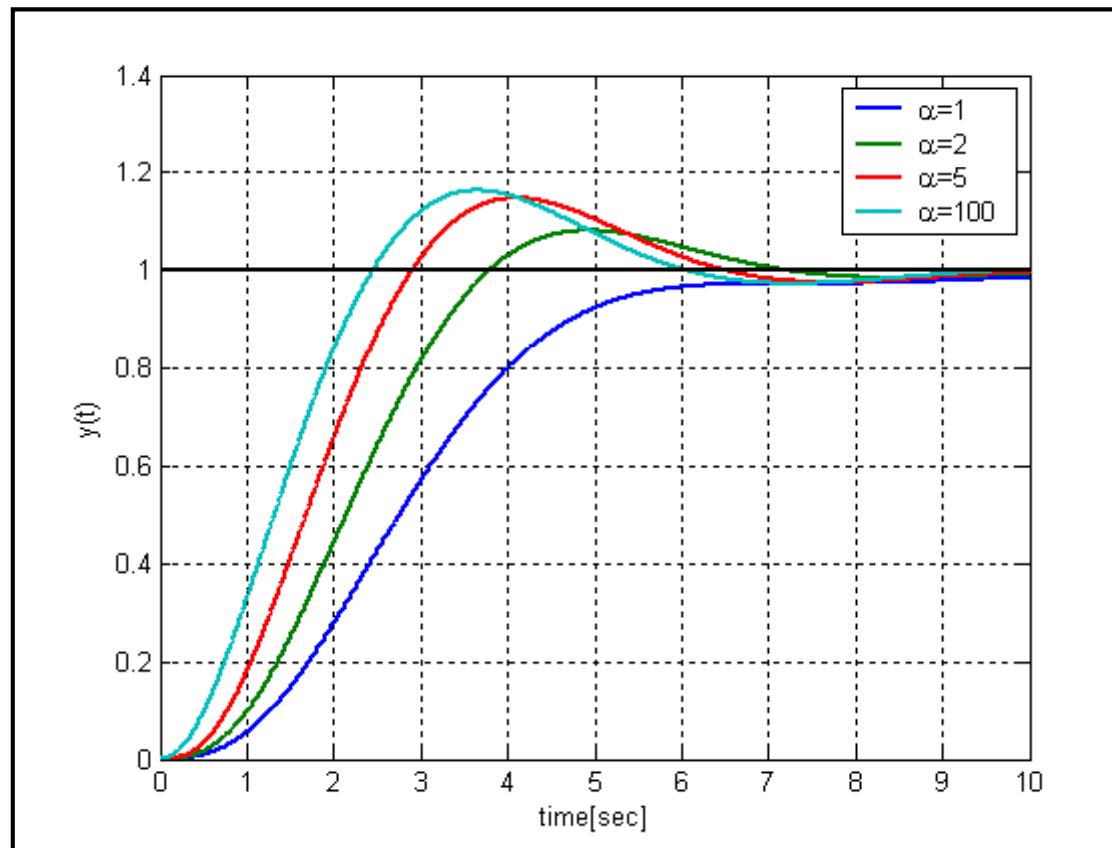
$$H(s) = \frac{1}{(s / \alpha\zeta + 1)[s^2 + 2\zeta s + 1]}, \quad \zeta = 0.5$$

- 목적 : 제어시스템을 설계할 때 극점이 과도응답에 미치는 영향을 알아본다.
- 표준 2차 계단응답에 대한 부가 극점의 효과를 고려



■ 극점 추가의 영향

■ Simulation Result $H(s) = \frac{1}{(s/\alpha\zeta + 1)[s^2 + 2\zeta s + 1]}, \quad \zeta = 0.5$



■ 영점 추가의 영향(2)

$$H(s) = \frac{s/\alpha\zeta + 1}{s^2 + 2\zeta s + 1}, \quad \zeta = 0.5, \quad \alpha = 1, -1$$

$$H(s) = H_0(s) + H_d(s) = \frac{1}{s^2 + 2\zeta s + 1} + \frac{1}{\alpha\zeta} \frac{s}{s^2 + 2\zeta s + 1}$$



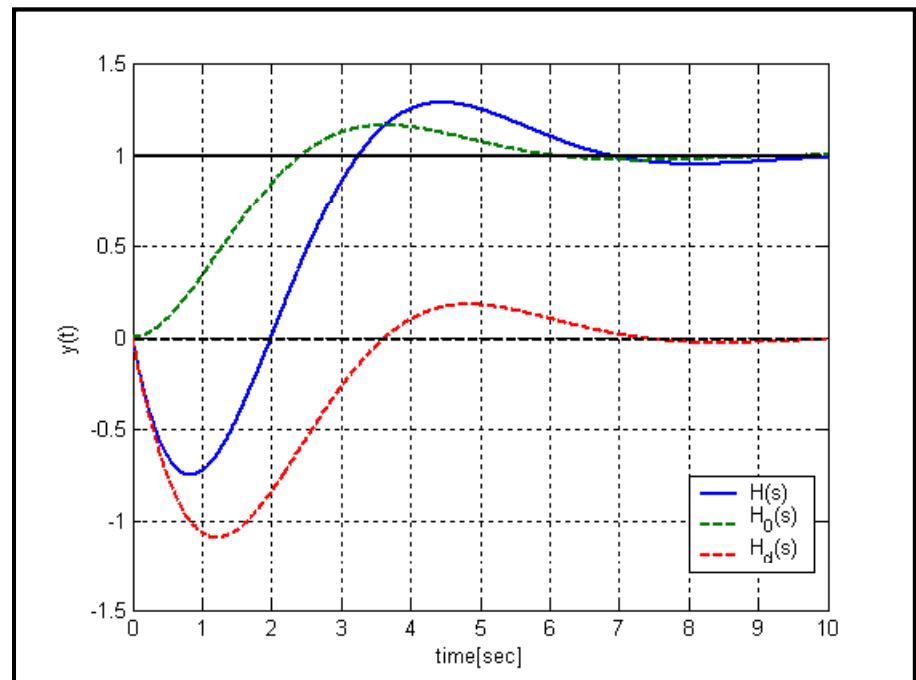
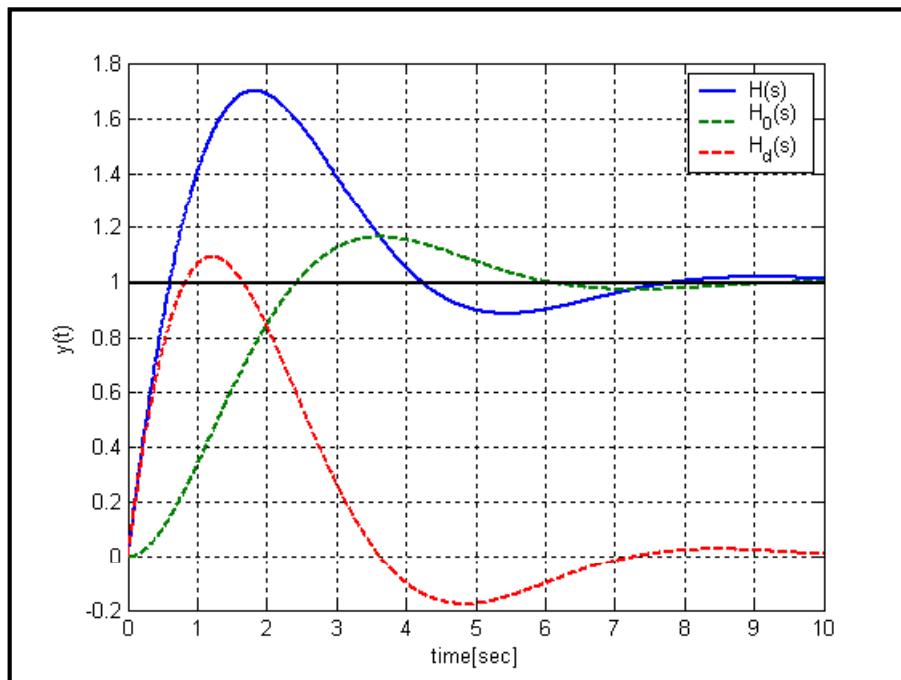
■ 영점 추가의 영향(2)

■ Simulation Result

$$H(s) = \frac{s/\alpha\zeta + 1}{s^2 + 2\zeta s + 1}, \quad H_0(s) = \frac{1}{s^2 + 2\zeta s + 1}, \quad H_d(s) = \frac{1}{\alpha\zeta} \frac{s}{s^2 + 2\zeta s + 1}$$

$$\zeta = 0.5, \quad \alpha = 1$$

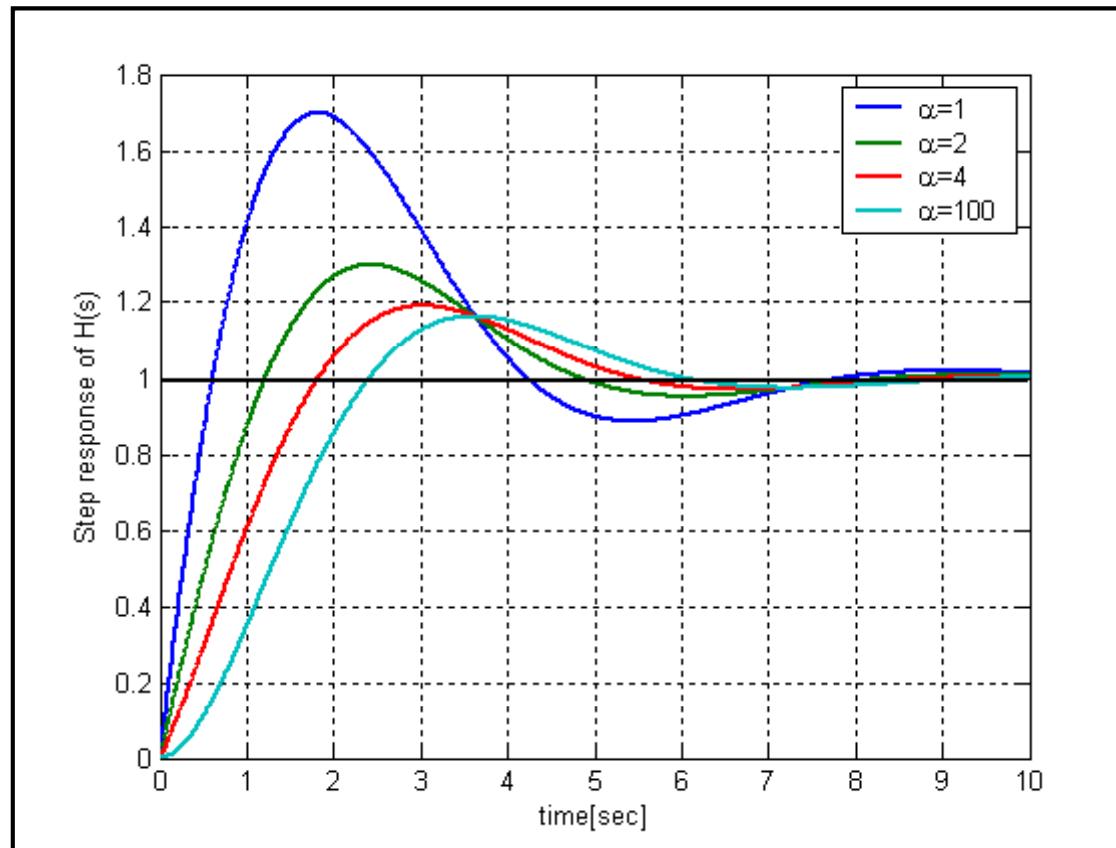
$$\zeta = 0.5, \quad \alpha = -1$$



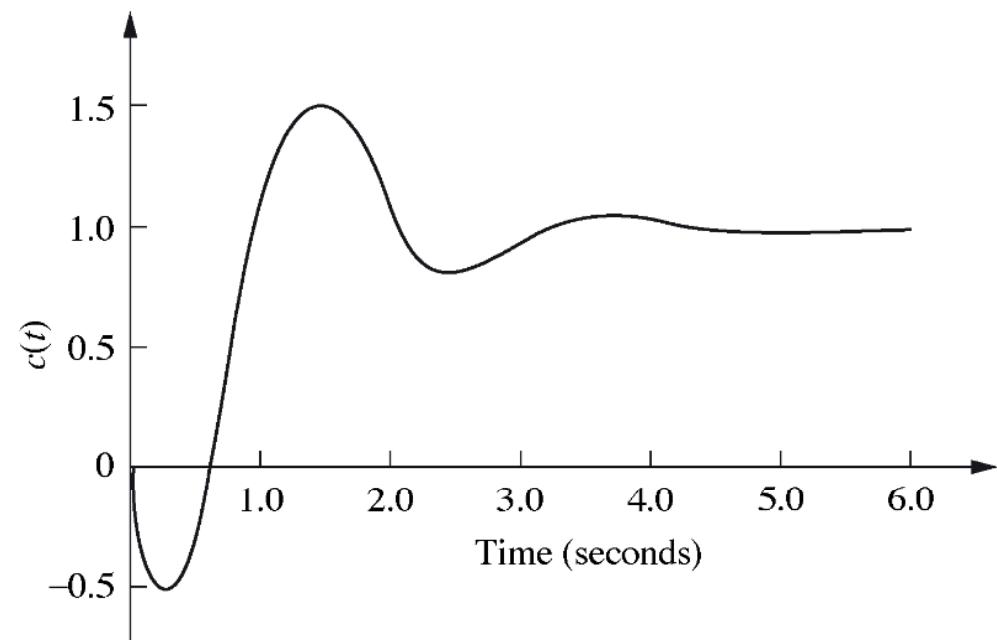
■ 영점 추가의 영향(1)

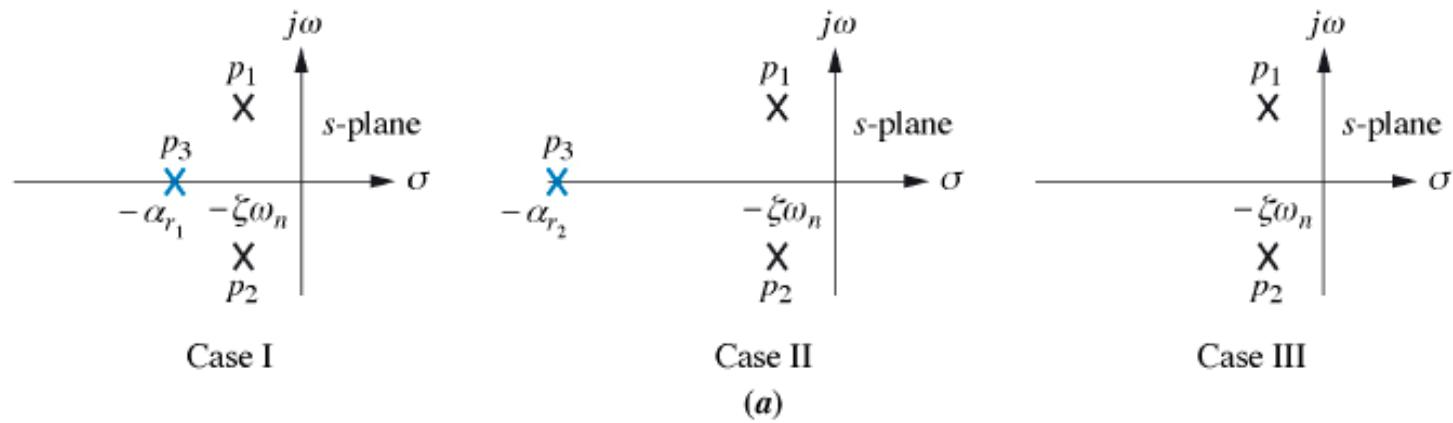
▪ Simulation Result

$$H(s) = \frac{s/\alpha\zeta + 1}{s^2 + 2\zeta s + 1}, \quad \zeta = 0.5$$

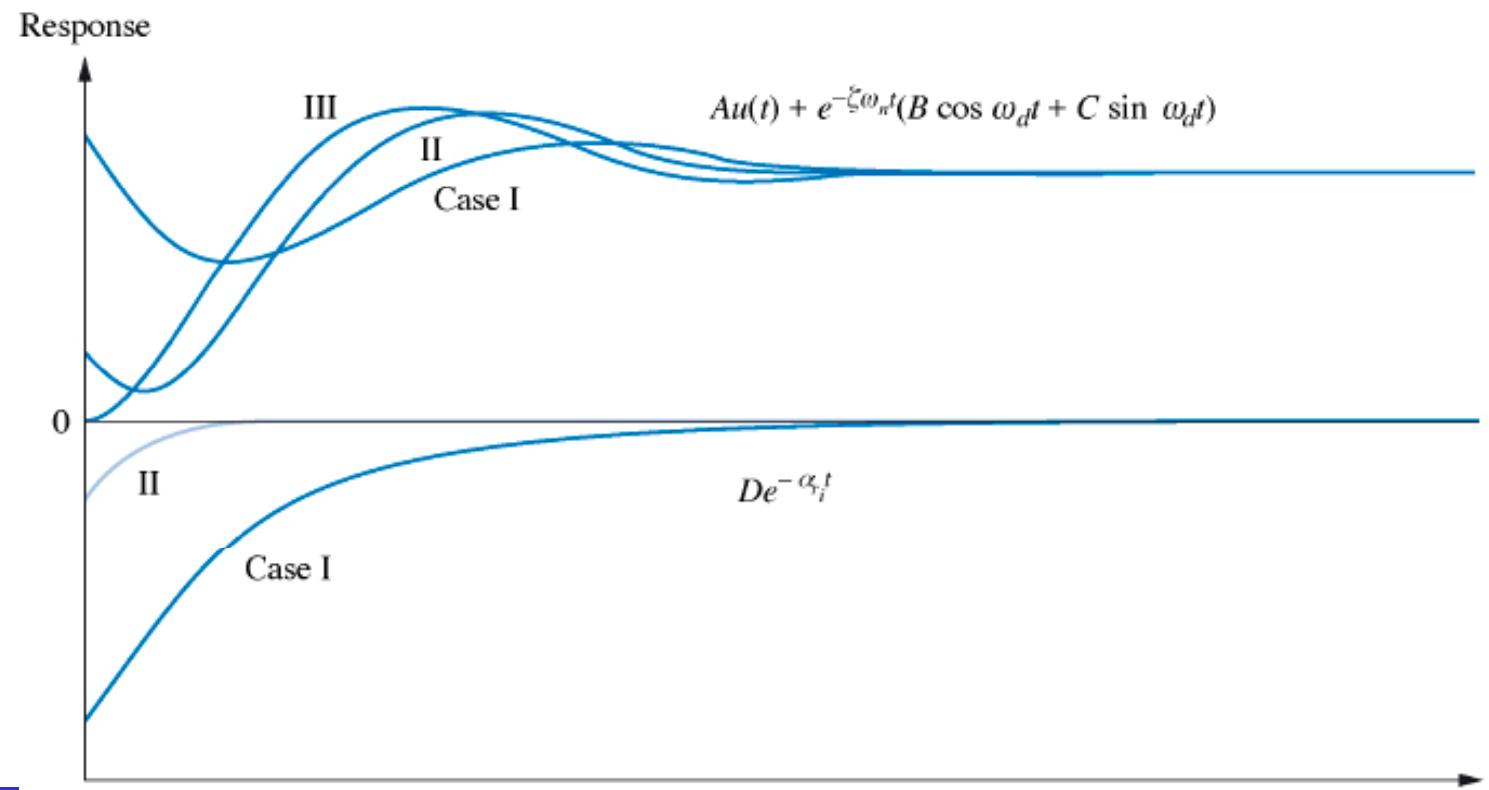


Non-minimum Phase System





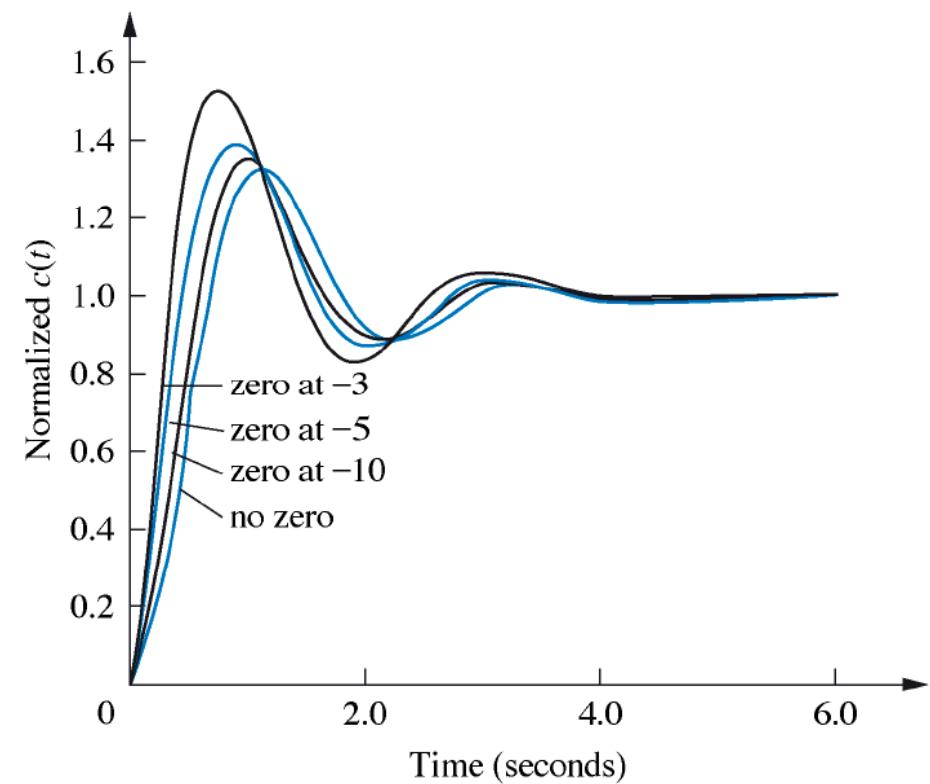
Case II
(a)



(b)

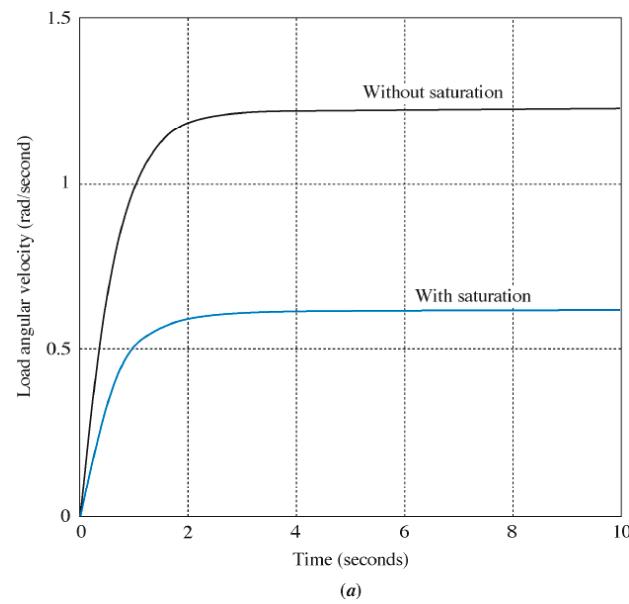


System Response with Zeros

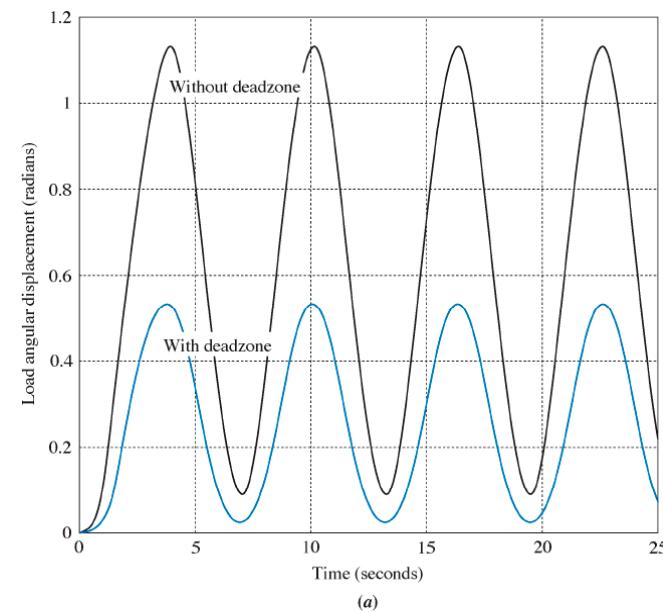


Effects of Nonlinearities

Saturation

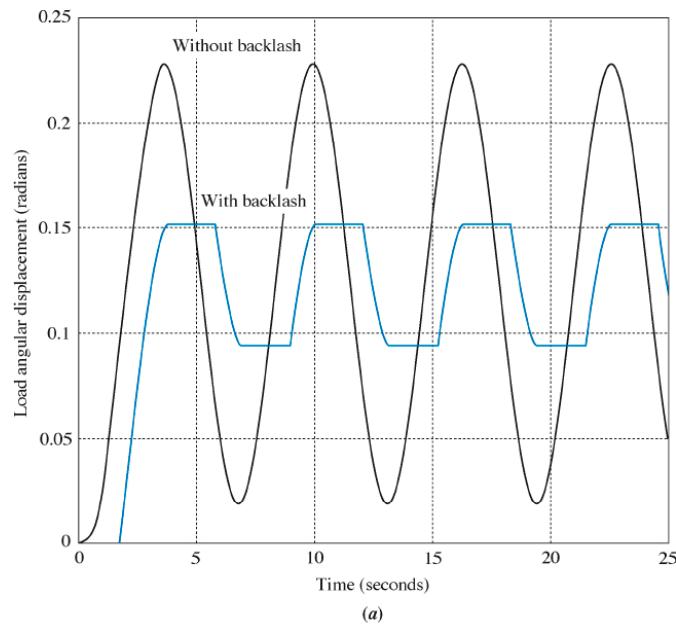


Deadzone



Effects of Nonlinearities

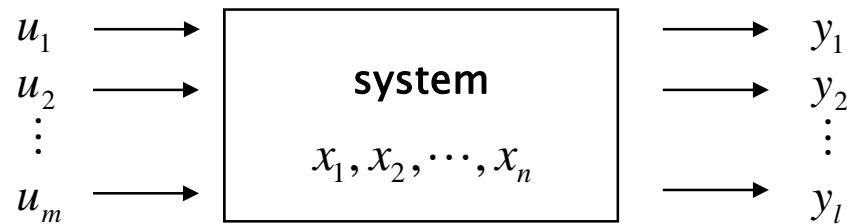
Backlash



(a)



Solution of Linear (Time Invariant) State Equation



$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t) + Du(t)$$

$$x(t) = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^T$$



Basic Matrix Linear Algebra -Homogeneous Solution

Scalar Function $u(t) = 0$

i) $\dot{x} = ax$

$$x(t) = Ce^{at} \quad t = 0, \quad x(0) = C$$

$$x(t) = x(0)e^{at}$$

$$x(t) = e^{a(t-t_0)}x(t_0), \quad t = t_0, \quad x(t_0)$$

ii) $e^{at} = \exp(at) = 1 + \frac{at}{1!} + \frac{(at)^2}{2!} + \dots + \frac{(at)^k}{k!} + \dots$

Homogeneous Solution

i) $\dot{x} = Ax \quad A : n \times n, \quad x : n \times 1$

$$x(t) = \exp[A(t - t_0)]x(t_0), \quad \frac{d}{dt}(e^{At}) = A e^{At}$$

ii) How to evaluate e^{At}



State Transition Matrix

$$\begin{aligned} \mathbf{x}(t) &= e^{\mathbf{A}(t-t_0)} \mathbf{x}(t_0) \\ &= \Phi(t-t_0) \mathbf{x}(t_0) \end{aligned}$$

$\Phi(t-t_0) = e^{A(t-t_0)} = \exp[A(t-t_0)]$: State transition matrix (STM)
: fundamental matrix of the system

Properties of STM

1. $\Phi(t_2 - t_1)\Phi(t_1 - t_0) = \Phi(t_2 - t_0)$ for any t_0, t_1, t_2

2. $\Phi(0) = I$

3. $\Phi(t)\Phi(t) = \Phi^2(t) = \Phi(2t)$ $\left(\because (e^{\mathbf{A}t})^2 = e^{\mathbf{A} \cdot 2t} = \Phi(2t) \right)$

$$\Phi^g(t) = \Phi(gt)$$

4. $\Phi^{-1}(t) = \Phi(-t)$

5. $\Phi(t)$ is nonsingular for all finite values of t (inverse exists)



Complete Solution of the State Equation

$$\dot{\mathbf{x}} = \mathbf{A} \mathbf{x} + \mathbf{B} u, \quad \dot{\mathbf{x}} - \mathbf{A} \mathbf{x} = \mathbf{B} u,$$

integrating factor e^{-At}

$$e^{-At} \dot{\mathbf{x}} - \mathbf{A} e^{-At} \mathbf{x} = e^{-At} \mathbf{B} u, \quad \frac{d}{dt} [e^{-At} \mathbf{x}] = e^{-At} \mathbf{B} u(t)$$

$$\Rightarrow e^{-At} (\mathbf{x}(t) - \mathbf{x}(0)) = \int_0^t e^{-A\tau} \mathbf{B} u(\tau) d\tau$$

$$\Rightarrow \mathbf{x}(t) = e^{At} \mathbf{x}(0) + \int_0^t e^{A(t-\tau)} \mathbf{B} u(\tau) d\tau$$

$$= \Phi(t) \mathbf{x}(0) + \int_0^t \Phi(t-\tau) \mathbf{B} u(\tau) d\tau$$



Complete Solution of the State Equation

$$\text{for } [t, t_0], \quad e^{-At} \mathbf{x}(t) - e^{-At_0} \mathbf{x}(t_0) = \int_{t_0}^t e^{-A\tau} \mathbf{B} u(\tau) d\tau$$

$$\Rightarrow \mathbf{x}(t) = \Phi(t - t_0) \mathbf{x}(t_0) + \int_{t_0}^t \Phi(t - \tau) \mathbf{B} u(\tau) d\tau$$

$$\Rightarrow \mathbf{x}(t) = \underbrace{\Phi(t) \mathbf{x}(0)}_{\begin{array}{l} \text{– Zero-input} \\ \text{response} \end{array}} + \underbrace{\int_0^t \Phi(t - \tau) \mathbf{B} u(\tau) d\tau}_{\begin{array}{l} \text{– Zero-state} \\ \text{response} \end{array}}$$

– free response – forced response

$$\text{change of variable } \tau, \quad \text{let } \beta = t - \tau, \quad \tau = 0 \rightarrow \beta = t$$
$$\tau = t \rightarrow \beta = 0$$

$$d\beta = -d\tau$$

$$\Rightarrow \mathbf{x}(t) = \Phi(t) \mathbf{x}(0) + \int_0^t \Phi(\beta) \mathbf{B} u(\beta) d\beta$$



Laplace Transformation Solution of State Equations

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu}$$

$$\mathbf{y} = \mathbf{Cx} + \mathbf{Du}$$

Laplace transformation →

$$s \mathbf{X}(s) - x_0 = \mathbf{A} \mathbf{X}(s) + \mathbf{B} \mathbf{U}(s),$$

$$\mathbf{X}(s) = (s \mathbf{I} - \mathbf{A})^{-1} x_0 + (s \mathbf{I} - \mathbf{A})^{-1} \mathbf{B} \mathbf{U}(s)$$

$$\mathbf{X}(s) = \frac{\text{adj}(s \mathbf{I} - \mathbf{A})}{\det(s \mathbf{I} - \mathbf{A})} [x(0) + \mathbf{B} \mathbf{U}(s)]$$

$$\mathbf{Y}(s) = \mathbf{C} \mathbf{X}(s) + \mathbf{D} \mathbf{U}(s)$$

Transfer function form →

$$\frac{Y(s)}{U(s)} = C \frac{\mathbf{X}(s)}{U(s)} + D$$

$$\frac{Y(s)}{U(s)} = C \left[\frac{\text{adj}(s \mathbf{I} - \mathbf{A})}{\det(s \mathbf{I} - \mathbf{A})} \right] \mathbf{B} + D$$

$$= \frac{C \text{adj}(s \mathbf{I} - \mathbf{A}) \mathbf{B} + D \det(s \mathbf{I} - \mathbf{A})}{\det(s \mathbf{I} - \mathbf{A})}$$

System Poles →

Roots of the denominator / Solution of $\det(s \mathbf{I} - \mathbf{A}) = 0$

Eigenvalues of A matrix!!



Time Domain Solution of State Eqns. using Laplace Transform

$$\dot{x} = Ax + Bu \quad x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau \\ = \Phi(t)x(0) + \int_0^t \Phi(t-\tau)Bu(\tau)d\tau$$

$$X(s) = (sI - A)^{-1}x_0 + (sI - A)^{-1}Bu(s)$$

$$\therefore e^{At} = \mathcal{L}^{-1}\left[(sI - A)^{-1}\right] = \mathcal{L}^{-1}\left[\frac{\text{adj}(sI - A)}{\det(sI - A)}\right] = \Phi(t)$$



Using Laplace transformation table

$$\mathcal{L}[t] = \frac{1}{s^2}, \quad \mathcal{L}\left[\frac{1}{(n-1)!} t^{n-1}\right] = \frac{1}{s^n}, \quad \mathcal{L}[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} F(s)$$

$$\Rightarrow \mathcal{L}^{-1}[(sI - A)^{-1}] = \mathcal{L}^{-1}\left[\frac{1}{s} I + \frac{1}{s^2} A + \frac{1}{s^3} A^2 + \frac{1}{s^4} A^3 \dots\right]$$

$$= I + At + \frac{1}{2!} A^2 t^2 + \frac{1}{3!} A^3 t^3 \dots$$

$$\therefore e^{At} = I + At + \frac{1}{2!} A^2 t^2 + \frac{1}{3!} A^3 t^3 + \dots$$



Example1. Find the state transition matrix and solve for $x(t)$. $u(t)$ is a unit step.

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -8 & -6 \end{bmatrix}x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u(t) \quad x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$



Example2. Find the state transition matrix using $(sI - A)^{-1}$

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -8 & -6 \end{bmatrix}x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u(t) \quad x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

