Chap. 2 Constitutive behavior of materials

- 3 types of relationships for the sol. of elasticity problems:
  1. equilibrium eqns. (Sec 1.1.2)
  2. strain-displacement relationships (Sec 1.4.1)
  3. constitutive laws ... mechanical behavior of the material

i) Homogeneity and isotropy

- "homogenous material" ... physical properties are identical at each point
  "isotropic " " ... " in all direction

  Ex. mild steel, aluminum .. both homogeneous and isotropic

- composite material ... neither homogeneous nor isotropic
  → heterogeneous, anisotropic

- "scale dependent" ...
  1. at atomic level, Al is neither homogenous nor isotropic
     → assumption of homogeniety and isotropy only hold for a very large number of atoms
  2. high temperature turbine blade applications {poly-crystalline / single crystal} materials
     - single crystal ... regular lattice structures → homogenous, but anisotropic
     - poly-crystalline ... { crystals oriented in a specific dir. → "
       ex) forged metals
       crystals arranged at random orientations → {homogeneous, isotropic}
       ex) common structural metals (steel, Al) }
  3. composite material ... clearly anisotropic, but samples containing a very large number of fibers → reasonably assumed as homogeneous

ii) Material testing

- If deformation very small → linear stress-strain relationship
  large deformation → material is ductile or brittle
- tensile test ... strain $\epsilon_1 = \Delta l / l$, stress $\sigma_1 = N/A$ } → stress-strain diagram

2.1 Constitutive laws for isotropic materials

2.1.1 Homogeneous, isotropic, linearly elastic materials

- small deformations → linear stress-strain behavior

$$\sigma_1 = E\epsilon_1 \quad : \text{Hooke's law} \qquad (2.1)$$

E ← Young modulus or modulus of elasticity [Pa]

- elongation of a bar --- accompanied by a lateral contraction

$$\epsilon_1 = \frac{1}{E}\sigma_1 \text{ or } \sigma_2, \quad \epsilon_2 = -\frac{\nu}{E}\sigma_1 \text{ or } \sigma_2, \quad \epsilon_3 = -\frac{\nu}{E}\sigma_1 \text{ or } \sigma_2 \qquad (2.2)\ (2.3)$$

material isotropy

ν : Poisson's ratio, non-dimensional

i) Generalized Hooke's law

- deformation under 3 stress components ... sum of those obtained for each stress component

⇒ generalized Hooke's law

$$\epsilon_1 = \frac{1}{E}\left[\sigma_1 - \nu(\sigma_2 + \sigma_3)\right] \qquad (2.4a)$$

... extensional strains depend only on the direct stress, and not on the shear stress ← isotropic material

ii) Shear stress - shear strain relationships

- Pure shear state in a plane stress state (Sec. 1.3.5)
  - 2 principal stresses $\sigma_{p2} = -\sigma_{p1}$, $\sigma_{p3} = 0$

Eq. (2.4a), (2.4b) → $\epsilon_1 = \frac{1+\nu}{E}\sigma_{p1}$, $\epsilon_2 = -\frac{1+\nu}{E}\sigma_{p2}$, $\gamma_{12} = 0$ (2.5)

-- on faces oriented at a 45° angle w.r.t. the principal stress directions

$$\tau^*_{s_{12}} = \sigma_{p2} = -\sigma_{p1}, \quad \sigma^*_{s_1} = \sigma^*_{s_2} = 0 \qquad (2.6)$$

$*, s$: specially rotated axis with max. shear stress

$$\text{Eq. (1.94)} \rightarrow \theta_s = 45°, \quad \gamma^*_{s_{12}} = -(\epsilon_1 - \epsilon_2) = -\frac{2(1+\nu)}{E}\sigma_{p1}; \qquad (2.7)$$

$$\epsilon^*_{s_1} = \epsilon^*_{s_2} = 0$$

$$\text{Eq. (2.6), (2.7)} \rightarrow \gamma^*_{s_{12}} = -\frac{2(1+\nu)}{E}\sigma_{p2} = 2(1+\nu)\frac{\tau^*_{s_{12}}}{E} = G\,\gamma^*_{s_{12}}$$

$$\Rightarrow \quad G = \frac{E}{2(1+\nu)} \quad \text{"shear modulus"} \qquad (2.8)$$

... generalized Hooke's law for shear strains

$$\gamma_{23} = \tau_{23}/G, \quad \gamma_{13} = \tau_{13}/G, \quad \gamma_{12} = \tau_{12}/G \qquad (2.9)$$

iii) Matrix form of the constitutive laws

- compact matrix form of the generalized Hooke's law

$$\underline{\underline{\epsilon}} = \underline{\underline{S}}\;\underline{\underline{\sigma}} \qquad (2.10)$$

$$\underline{\epsilon} = \{\epsilon_1, \epsilon_2, \epsilon_3, \gamma_{23}, \gamma_{13}, \gamma_{12}\}^T \qquad (2.11a)$$

$$\underline{\sigma} = \{\sigma_1, \sigma_2, \sigma_3, \tau_{23}, \tau_{13}, \tau_{12}\}^T \qquad (2.11b)$$

$\underline{\underline{S}}$: 6x6 material compliance matrix

Eq(2.4)

$$\underline{\underline{S}} = \frac{1}{E}\left[\begin{array}{ccc|ccc} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 2(1+\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1+\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1+\nu) \end{array}\right] \qquad (2.12)$$

absence of coupling between $\left\{\begin{array}{l}\text{axial stresses} \\ \text{shear strains}\end{array}\right.$ and vice versa

Eq (2.9)

- stiffness form of the same laws

$$\underline{\underline{\sigma}} = \underline{\underline{C}}\;\underline{\underline{\epsilon}} \qquad (2.13)$$

$$\underline{\underline{C}} = \left[\begin{array}{cc} \cdot & \\ & \cdot \end{array}\right] \qquad (2.14)$$

iv) Plane stress state

$$\underset{=}{\epsilon} = \{\epsilon_1, \epsilon_2, \gamma_{12}\}^T \qquad (2.15)$$

$$\underset{=}{\sigma} = \{\sigma_1, \sigma_2, \tau_{12}\}^T$$

$$\underset{=}{C} = \frac{E}{(1-\nu^2)}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \qquad (2.16)$$

ϵ_3 does not vanish due to Poisson's ratio effect, $\epsilon_3 = -\nu(\sigma_1+\sigma_2)$

v) Plane strain state

same $\underset{=}{\epsilon}$, $\underset{=}{\sigma}$

$$\underset{=}{C} = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \qquad (2.17)$$

σ_3 does not vanish due to Poisson's ratio effect, $\sigma_3 = \nu E \dfrac{(\epsilon_1+\epsilon_2)}{[(1+\nu)(1-2\nu)]}$

vi) The bulk modulus

-- volumetric strain --- Eq. (1.75)

$$e = \epsilon_1 + \epsilon_2 + \epsilon_3 = \frac{1-2\nu}{E}(\sigma_1+\sigma_2+\sigma_3) = \frac{1-2\nu}{E}\underbrace{I_1}_{1^{st}\text{ stress invariant}} \qquad (2.18)$$

-- hydrostatic pressure, $\sigma_1 = \sigma_2 = \sigma_3 = p$

$$\rightarrow \quad p = \kappa e, \qquad (2.19)$$

$$\kappa = \frac{E}{3(1-2\nu)} \; : \text{"bulk modulus"} \qquad (2.20)$$

When $\nu \rightarrow \frac{1}{2}$, $\kappa \rightarrow \infty$ ... "incompressible material" (ex: rubber)

2.1.2 Thermal effects

∘ Under a change in temperature, homogeneous isotropic materials will expand in all directions → "thermal strain"

$$\epsilon^t = \underbrace{\alpha}_{\text{C.T.E.}} \Delta T \qquad (2.21)$$

① thermal strains are purely extensional, do not induce shear strains.

② " do not generate internal stresses

... Unconfined material sample simply expands subject to a temp. change, but remains unstressed

- Total strains ··· mechanical strains + thermal strains

$$\epsilon_1 = \frac{1}{E}[\sigma_1 - \nu(\sigma_2+\sigma_3)] + \alpha\Delta T \qquad (2.22a)$$

$$\cdots$$

But shear stress - shear strain relationships unchanged

- constrained material ··· a bar constrained at its two ends by rigid walls

$$\epsilon_1 = \frac{1}{E}[\sigma_1] + \alpha\Delta T = 0 \rightarrow \sigma_1 = -E\alpha T$$

··· temp. change → compressive stress ("thermal stress")

2.1.4 Ductile materials

- Fig. 2.5 ··· mild steel
  - 0 → A ··· Hooke's law, slope = Young's modulus
  - A ··· limit of proportionality, $\sigma_e \simeq \sigma_y$ ("yield stress")
  - B → C ··· "plastic flow" ($\epsilon_1 = 5 \sim 10\%$)
  - C → E ··· increasing stress, σ_f : max.

    "necking" ··· x-s area decrease

    E ··· "failure stress" σ_f
- large deformations before failure ··· B → E

  When unloading, will follow DG // AO, with a permanent deformation OG

  " reloading, " GD, and further DEF

  ↳ higher yield stress at D ← "strain hardening"
- shear behavior ··· similar (Fig. 2.6)
- Idealization ··· Fig. 2.7, "elastic-perfectly plastic", mild steel, annealed Al.
- Fig. 2.8 ··· Al, Cu, no plastic flow regime

  specific permanent deformation defined for σ_y

  ex) $\epsilon = 0.2\%$ for Al

2.1.5 Brittle materials

- very little deformation beyond the elastic limit ... Fig 2.9

  ex) glass, concrete, stone, wood, uni-dir. composites or ceramic

2.2 Allowable stress

Factors influencing the design

① strength of the structure ← focus of the present section

② elastic deformation "

③ dynamic characteristics " ... natural frequencies and resonance

④ stability characteristics " ... buckling

⑤ time dependent deformations associated with creep ... turbine engine design

- Numerous uncertainties which decrease service loads

  ① actual magnitude of the applied service loads

  ② strength of materials ... statistical

  ③ manufacturing variability

  ④ corrosion, wear, chemically aggressive environment

  ⑤ predicted stresses might be very different from their actual values

- load factor $= \dfrac{\text{failure load}}{\text{service load}} > 1$, as large as 10

- factor of safety → allowable stress $= \dfrac{\text{yield stress}}{\text{safety factor}}$, or $\sigma_{allow} = \dfrac{\sigma_y}{n}$ (2.26)

  ... adequate for ductile materials.

  - for brittle materials, allowable stress $= \dfrac{\text{ultimate stress}}{\text{safety factor}}$, or $\sigma_{allow} = \dfrac{\sigma_f}{n}$ (2.27)

2.3 Yielding under combined loading

- Proper yield criterion under multiple stress components acting

  - isotropic material ... no directional dependency of the yield criterion

  state of stress { 6 stress components defining the stress tensor
  3 principal stresses, $\sigma_{p1}, \sigma_{p2}, \sigma_{p3}$ and the corresponding 3 orientations

no dir. dependency → only the magnitudes of the principal stress should appear

2.3.1 Tresca's criterion

- $|\sigma_{p1} - \sigma_{p2}| \le \sigma_y, \quad |\sigma_{p2} - \sigma_{p3}| \le \sigma_y, \quad |\sigma_{p3} - \sigma_{p1}| \le \sigma_y$ (2.29)

σ_y : yield stress observed in a uniaxial test (Fig. 2.5)

- whenever, any one of Eq. (2.29) is violated, yielding develops.
- interpretation → $\tau_{23\,max} \le \frac{\sigma_y}{2}, \quad \tau_{13\,max} \le \frac{\sigma_y}{2}, \quad \tau_{12\,max} \le \frac{\sigma_y}{2}$

or, $\tau_{max} \le \frac{\sigma_y}{2}$

... the material reaches the yield condition when the max. shear stress = half the yield stress under a uniaxial stress state.

"max. shear stress criterion"

① Uniaxial state ... $\sigma_{p1} \le \sigma_y$

② Plane state of stress ... Eq. (2.31)

③ Pure shear state ... $\tau \le \sigma_y/2$

2.3.2 Von Mises' criterion

- $\sigma_{eq} = \frac{1}{\sqrt{2}} \sqrt{[(\sigma_{p1} - \sigma_{p2})^2 + (\sigma_{p2} - \sigma_{p3})^2 + (\sigma_{p3} - \sigma_{p1})^2]} \le \sigma_y$ (2.32)

"equivalent stress"

- Octahedral face (Example 1.3) ---> shear stress acting on octahedral face

$$3\tau_{oc}^2 = \frac{2}{3}\sigma_{eq}^2 \quad \rightarrow \quad \sigma_{eq} = \frac{3}{\sqrt{2}}\tau_{oc} \qquad (2.33)$$

... "the yield cond. is reached when the octahedral shear stress $= \frac{3}{\sqrt{2}}$ of the yield stress for a uniaxial stress state, σ_y

- σ_{eq} can be expressed in terms of the stress invariants

$$\sigma_{eq}^2 = I_1^2 - 3I_2 \qquad (2.34)$$

$$\rightarrow \sigma_{eq} = \sqrt{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_2\sigma_3 - \sigma_3\sigma_1 - \sigma_1\sigma_2 + 3(\tau_{23}^2 + \tau_{13}^2 + \tau_{12}^2)} \le \sigma_y \qquad (2.35)$$

① Uniaxial stress state ... $\sigma_{p1} \le \sigma_y$

② Plane state of stress ... $\sigma_{eq} = \sqrt{\sigma_1^2+\sigma_2^2-\sigma_1\sigma_2+3\tau_{12}^2} \le \sigma_y$ (2.26)

③ Pure shear state ... $\tau \le \frac{1}{\sqrt{3}}\sigma_y \simeq 0.577$ (60%), more accurate than that of Tresca's

2.3.3 Comparing Tresca's and von Mises' criteria

- Plane stress problem, $\sigma_{p3} = 0$
  - Tresca's criterion ... 3 inequalities

$$\left|\frac{\sigma_{p1}}{\sigma_y}\right| < 1, \quad \left|\frac{\sigma_{p2}}{\sigma_y}\right| < 1, \quad \left|\frac{\sigma_{p2}}{\sigma_y} - \frac{\sigma_{p1}}{\sigma_y}\right| < 1$$

... slightly more conservative

→ irregular hexagon enclosed by 6 dashed line segments (Fig. 2.10)

  - von Mises' criterion ... oblique ellipse (Fig. 2.10)

$$\left(\frac{\sigma_{p1}}{\sigma_y}\right)^2 + \left(\frac{\sigma_{p2}}{\sigma_y}\right)^2 - \left(\frac{\sigma_{p1}}{\sigma_y}\right)\left(\frac{\sigma_{p2}}{\sigma_y}\right) = 1$$

... often preferred since a single analytic expression

- Table 2.1 ... 3 radial lines OA, OB, OC in Fig. 2.10
  → max. discrepancy between 2 criteria ... 15.5 %

2.4 Material selection for structural performance

Table 2.2 ... $\left\{\begin{array}{l}\text{ultimate stress}\\ \text{modulus of elasticity}\\ \text{density}\end{array}\right\}$ of $\left\{\begin{array}{l}\text{Al}\\ \text{Ti}\\ \text{steel}\end{array}\right.$ ← far superior, but heavier

Table 2.3 .. fibers

- 3 categories of structural design $\left\{\begin{array}{l}\text{strength design}\\ \text{stiffness ''}\\ \text{buckling ''}\end{array}\right.$

2.4.1 Strength design

- For a given mass and geometry, the max. load it can carry

$$P_{max} \propto \frac{\sigma_{ult}}{\rho} \quad (2.38)$$

← material performance index

2.4.2 Stiffness design

- cantilevered, thin-walled beam of length L (Fig. 2.11), natural freq.

$$\omega \propto \frac{h}{L^2}\left[\frac{E}{\rho}\right]^{1/2} \quad (2.40)$$

material performance index ... $\sqrt{\frac{E}{\rho}}$

2.4.3 Buckling design

- critical load that will cause the plate to buckle

$$P_{cr} \propto \frac{M^3}{b^4 L^3} \frac{E}{\rho^3}$$

index ... $\frac{E}{\rho^3}$

- Table 2.4, 2.5 ... performance indices for metals and fibers

strength design ... steel is the best

stiffness 〃 -- 3 equally well

strength and buckling ... Al >> steel and Ti

remarkably high performance indices of fibers → potential use in structural applications

2.5 Composite materials

2.5.1 Basic characteristics

- embedding fiber aligned in a single direction, in a matrix material

- matrix material ... thermostat polymeric material, ex) epoxy

- "rule of mixture" ... strength

$$S_c = V_f S_f + V_m S_m \qquad (2.45)$$

S : strength, V : volume fraction, $V_f + V_m = 1$

Ex) graphite fiber ($V_f = 0.6$) embedded in an epoxy matrix ($V_m = 0.4$)

$$S_c = 1{,}700 \times 0.6 + \underbrace{50 \times 0.4}_{\text{contributes little}} = 1{,}040 \text{ (MPa)}$$

- stiffness ... assuming that perfectly bonded together

$$\epsilon_m = \epsilon_f = \epsilon_c \qquad (2.47)$$

- Average stress σ_c

$$P = A_c \sigma_c = A_f \sigma_f + A_m \sigma_m \qquad (2.48)$$

Dividing by A_c

$$\sigma_c = \frac{A_f}{A_c} \sigma_f + \frac{A_m}{A_c} \sigma_m = V_f \sigma_f + V_m \sigma_m \qquad (2.49)$$

- fiber, matrix ··· linearly elastic, isotropic

$$\sigma_f = E_f \epsilon_f, \qquad \sigma_m = E_m \epsilon_m \qquad (2.50)$$

- modulus of elasticity for the composite, E_c

$$\sigma_c = E_c \epsilon_c \qquad (2.51)$$

Eq. (2.50), (2.51) → (2.49) : $E_c = V_f E_f + V_m E_m$ (2.52)

ex) graphite-epoxy : $E_c = \underbrace{250 \times 0.6 + 3.5 \times 0.4}_{\text{contributes little}} = 150$ GPa

- what is the role of the matrix material?
  ① keep all the fibers together
  ② diffuse the stresses among the otherwise isolated fibers

2.5.2 stress diffusion in composites

∘ Fig. 2.12 ··· single broken fiber of length 2L
  → matrix material adjacent to the broken fiber will transfer stress from the surrounding material to the broken fiber ··· "stress diffusion process"

∘ Fig. 2.13 ··· simplified model
  Assumptions ① matrix carries shear stresses only
  ② axial stress in the fiber is uniformly distributed
  ③ existence of individual fibers ignored in the remaining composite
  ④ perfectly bonded together

- strain-displacement relationship

$$\epsilon_f = \frac{du_f}{dx_1}, \qquad \epsilon_a = \frac{du_a}{dx_1}, \qquad \gamma_m = \frac{u_a - u_f}{r_m - r_f} \qquad (2.54)$$

- axial force equilibrium of a differential element of fiber (Fig. 2.14)

$$\frac{d\sigma_f}{dx_1} + \frac{2}{r_f}\tau_m = 0 \qquad (2.55)$$

- overall equilibrium of an entire model (Fig. 2.13)

$$\sigma_a = \frac{\sigma_0}{1 - \frac{r_m^2}{r_a^2}} - \frac{r_f^2}{r_a^2}\frac{\sigma_f}{1 - \frac{r_m^2}{r_a^2}} \simeq \sigma_0 \qquad (2.56)$$

$\frac{r_f}{r_a} \ll 1$ → 2nd term negligible ; $\frac{r_m}{r_a} \ll 1$

- constitutive laws for fiber, composite, and matrix

$$\sigma_f = E_f \epsilon_f , \quad \sigma_a = E_a \epsilon_a , \quad \tau_m = G_m \gamma_m \qquad (2.57)$$

- Eq. (2.57c), (2.54c) → Eq. (2.55)

$$\frac{d\sigma_f}{dx_1} + \frac{2G_m}{r_f(r_m - r_f)} (u_a - u_f) = 0$$

- Differentiate w.r.t. x_1, and substituting Eqs. (2.54a), (2.54b), (2.57a), (2.57b)

$$\frac{d^2\sigma_f}{dx_1^2} + \frac{2G_m}{r_f(r_m - r_f)} \left(\frac{\sigma_a}{E_a} - \frac{\sigma_f}{E_f}\right) = 0$$

- Since $\sigma_a \simeq \sigma_0$ (Eq. 2.56),

$$\frac{d^2\sigma_f}{dx_1^2} - \frac{2}{r_f(r_m - r_f)} \frac{G_m}{E_f} \sigma_f = - \frac{2}{r_f(r_m - r_f)} \frac{G_m}{E_f} \frac{E_f}{E_a} \sigma_0$$

-- Non-dimensional variable $\eta = (L - x_1)/(2r_f)$ (Fig. 2.13)

- Then, the governing eqn.

$$\sigma_f'' - \lambda^2 \sigma_f = -\lambda^2 \frac{E_f}{E_a} \sigma_0$$

$(.)'$: derivative w.r.t. η, $\lambda^2 = 8 \frac{G_m}{E_f} \frac{r_f}{r_m} \frac{1}{1 - r_f/r_m}$

- $\frac{E_f}{E_a} = \frac{E_f}{V_f E_f + V_m E_m} \simeq \frac{E_f}{V_f E_f} = \frac{1}{V_f}$ since $E_m \ll E_f$

- governing eqn.

$$\sigma_f'' - \lambda^2 \sigma_f = -\lambda^2 \frac{\sigma_0}{V_f} \qquad (2.58)$$

$$\text{where } \lambda^2 = 8 \frac{G_m}{E_f} \frac{\sqrt{V_f}}{1 - \sqrt{V_f}} \qquad (2.59)$$

B.C. : $\sigma_f = 0$ at $\eta = 0$ (broken fiber)

$\sigma_f' = 0$ at $\eta = L/2r_f$ (symmetry)

- Sol.

$$\frac{\sigma_f}{\sigma_0} = \frac{1}{V_f} \left(1 - \frac{\cosh \lambda (L/2r_f - \eta)}{\cosh(\lambda L/2r_f)}\right) \simeq \frac{1}{V_f} (1 - e^{-\lambda \eta}) \qquad (2.60)$$

Since $\sigma_0 = V_f \sigma_{f\infty} + (1 - V_f) \sigma_{m\infty} \simeq V_f \sigma_{f\infty}$,

$$\text{Eq. (2.60)} \rightarrow \frac{\sigma_f}{\sigma_{f\infty}} = 1 - e^{-\lambda \eta} \qquad (2.61)$$

--- fiber axial stress distribution near the fiber break → Fig. 2.15

- "ineffective length δ" ... the distance where the fiber stress reaches 95% of its far field value

$$0.95 = 1 - \exp(-\lambda\delta/d_f)$$

$$\rightarrow \frac{\delta}{d_f} \simeq \left[\frac{E_f}{G_m}\frac{1-\sqrt{V_f}}{\sqrt{V_f}}\right]^{1/2} \qquad (2.62)$$

... length of fiber, near a fiber break, that does not carry axial stress at full capacity

$\Rightarrow$ matrix material transfers the load from the surrounding material to the broken fiber very rapidly ("shear lag").

shear stress in the matrix is effectively transferring the load to the fiber $\rightarrow$ Fig 2.16, $\frac{\tau_m}{\sigma_{f\infty}} = \frac{\lambda}{4}e^{-\lambda\eta}$ (2.63)

- zone affected by a fiber break $\rightarrow$ about 2δ in length

ex) graphite of dia. 10 microns $\rightarrow$ zone of only 200 microns in length

2.6 Constitutive laws for anisotropic materials

- Unidirectional composite materials ... fiber dir., dominated by that of fiber; transverse to fiber, dominated by that of matrix
- linear relationship between the stress and strain

$$\underline{\sigma} = \underline{\underline{C}}\,\underline{\epsilon} \quad ; \quad \underline{\epsilon} = \underline{\underline{S}}\,\underline{\sigma} \qquad (2.64)$$

6x6 stiffness ($\underline{\underline{C}}$) ; 6x6 compliance ($\underline{\underline{S}}$)

$$\underline{\underline{S}} = \underline{\underline{C}}^{-1} \qquad (2.65)$$

- strain energy $A = \frac{1}{2}\underline{\epsilon}^T\underline{\sigma} = \frac{1}{2}\underline{\epsilon}^T\underline{\underline{C}}\,\underline{\epsilon} = \frac{1}{2}\underline{\sigma}^T\underline{\underline{S}}\,\underline{\sigma}$ (2.66)

$\rightarrow$ both $\underline{\underline{C}}$ and $\underline{\underline{S}}$ are symm. and positive definite

- Due to symmetry, $6\times6 = 36$ independent consts $\rightarrow 21$ (2.67)

... "anisotropic" or "triclinic" material

- plane of symmetry ... $(\bar{i}_1, \bar{i}_2)$ plane of symmetry

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16} \\ & C_{22} & C_{23} & 0 & 0 & C_{26} \\ & & C_{33} & 0 & 0 & C_{36} \\ & & & C_{44} & C_{45} & 0 \\ & & & & C_{55} & 0 \\ & & & & & C_{66} \end{bmatrix} \qquad (2.68)$$

if $C_{14} \neq 0$, ϵ_1 would give rise to τ_{23} → violate the symmetry of response

⇒ $21-8 = 13$ independent consts "monoclinic" material

- 2 mutually orthogonal planes of symmetry ⋯ $(\bar{\imath}_1, \bar{\imath}_2)$, $(\bar{\imath}_2, \bar{\imath}_3)$

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ & C_{22} & C_{23} & 0 & 0 & 0 \\ & & C_{33} & 0 & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & & & & C_{55} & 0 \\ & & & & & C_{66} \end{bmatrix} \quad (2.69)$$

⇒ $21 - 12 = 9$ independent consts, "orthotropic" material

- laminated composite material ⋯ { 2 orthogonal planes of symmetry : $(\bar{\imath}_1, \bar{\imath}_2)$ $(\bar{\imath}_2, \bar{\imath}_3)$; 1 plane of isotropy : $(\bar{\imath}_2, \bar{\imath}_3)$ }

$$\begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ & C_{22} & C_{23} & 0 & 0 & 0 \\ & & C_{22} & 0 & 0 & 0 \\ & & & \frac{C_{22}-C_{23}}{2} & 0 & 0 \\ & & & & C_{55} & 0 \\ & & & & & C_{55} \end{bmatrix} \quad (2.70)$$

⇒ 5 constants, "transversely isotropic"

- Isotropic

$$\begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ & C_{11} & C_{12} & 0 & 0 & 0 \\ & & C_{11} & 0 & 0 & 0 \\ & & & \frac{C_{11}-C_{12}}{2} & 0 & 0 \\ & & & & \frac{C_{11}-C_{12}}{2} & 0 \\ & & & & & \frac{C_{11}-C_{12}}{2} \end{bmatrix} \quad (2.71)$$

⇒ 2 constants

- { Not clear about C_{11}, C_{12}
  "Engineering const.s" ⋯ Young's modulus, Poisson's ratio }

→ experimental determination and physical interpretation

2.6.1 Constitutive laws for a lamina in the fiber aligned triad

- thin sheet of composite material made of unidirectional fibers
  - $\bar{i}_1^*$: fiber direction, $\bar{i}_2^*$: transverse direction, $\bar{i}_3^*$: perpendicular to the plane of thin sheet } → "fiber aligned triad"
  - → can be assumed as a homogeneous, transversely isotropic material
- plane stress state ··· constitutive laws in compliance form

$$\begin{Bmatrix} \epsilon_1^* \\ \epsilon_2^* \\ \gamma_{12}^* \end{Bmatrix} = \begin{bmatrix} 1/E_1^* & -\nu_{21}^*/E_2^* & 0 \\ -\nu_{12}^*/E_1^* & 1/E_2^* & 0 \\ 0 & 0 & 1/G_{12}^* \end{bmatrix} \begin{Bmatrix} \sigma_1^* \\ \sigma_2^* \\ \tau_{12}^* \end{Bmatrix} \qquad (2.72)$$

- E_1^*, E_2^*, ν_{12}^*, G_{12}^* : engineering const.s
- symm. → $\nu_{12}^*/E_1^* = \nu_{21}^*/E_2^*$ ⇒ one of 5 const.s is not an independent quantity
- Single test of a known stress σ_1^*, then $\sigma_2^* = \tau_{12}^* = 0$ (Fig. 2.17
  - ① of Eq. (2.72) → $\epsilon_1^* = \sigma_1^*/E_1^*$, E_1^* can be determined
  - ② of " → $\epsilon_2^* = -\nu_{12}^*\sigma_1^*/E_1^*$, ν_{12}^* " .
- 2nd test " σ_2^*, then $\sigma_1^* = \tau_{12}^* = 0$ (Fig. 2.17 ②)
  - $\epsilon_2^* = \sigma_2^*/E_2^*$, E_2^* can be obtained.
- last test of a known τ_{12}^*, then $\sigma_1^* = \sigma_2^* = 0$ (Fig. 2.17 ③)
  - ③ of Eq. (2.72) → $\gamma_{12}^* = \tau_{12}^*/G_{12}^*$, G_{12}^* can be obtained.
- stiffness matrix ··· by inverting Eq. (2.72)

$$\begin{Bmatrix} \sigma_1^* \\ \sigma_2^* \\ \tau_{12}^* \end{Bmatrix} = \begin{bmatrix} \dfrac{E_1^*}{1-\nu_{12}^{*2}E_2^*/E_1^*} & \dfrac{\nu_{12}^*E_2^*}{1-\nu_{12}^{*2}E_2^*/E_1^*} & 0 \\ \dfrac{\nu_{12}^*E_2^*}{''} & \dfrac{E_2^*}{''} & 0 \\ 0 & 0 & G_{12}^* \end{bmatrix} \begin{Bmatrix} \epsilon_1^* \\ \epsilon_2^* \\ \gamma_{12}^* \end{Bmatrix} \qquad (2.73)$$

2.6.2 Constitutive laws for a lamina in an arbitrary triad

- Fig. 2.18 ··· lamina of a direction that might not coincide with that of fibers
  - counterclockwise θ orientation of fibers w.r.t. ref. direction
  - ← formulae for stresses and strains in a rotated axis system

i) Rotation of the stiffness matrix

- constitutive laws for a lamina in the fiber aligned triad

$$\underline{\sigma}^* = \underline{\underline{C}}^* \underline{\epsilon}^*$$

- introducing the rotation formulae, Eqs (1.47), (1.91)

$$\begin{bmatrix} m^2 & n^2 & 2mn \\ n^2 & m^2 & -2mn \\ -mn & mn & m^2-n^2 \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} = \underline{\underline{C}}^* \begin{bmatrix} m^2 & n^2 & mn \\ n^2 & m^2 & -mn \\ -2mn & 2mn & -n^2 \end{bmatrix} \begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \gamma_{12} \end{Bmatrix}$$

where $m = \cos\theta$, $n = \sin\theta$

- multiplying from the left by the inverse of the rotation matrix for the stress,

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} = \underbrace{\begin{bmatrix} m^2 & n^2 & -2mn \\ n^2 & m^2 & 2mn \\ mn & -mn & m^2-n^2 \end{bmatrix} \underline{\underline{C}}^* \begin{bmatrix} m^2 & n^2 & mn \\ n^2 & m^2 & -mn \\ -2mn & -2mn & m^2-n^2 \end{bmatrix}}_{\underline{\underline{C}}} \begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \gamma_{12} \end{Bmatrix}$$

(2.79)

- More compact manner of the relationship

$$\underline{C}(\theta) = \underline{\underline{\chi}}(\theta)\, \underline{\alpha} \qquad (2.86)$$

where

$$\underline{C} = \{ C_{11}, C_{22}, C_{12}, C_{66}, C_{16}, C_{26} \}^T \qquad (2.84)$$

$$\underline{\underline{\chi}}(\theta) = [\quad \cdots \quad] \qquad (2.83)$$

$$\underline{\alpha} = \{ \alpha_1, \alpha_2, \alpha_3, \alpha_4 \}^T \text{ "material invariants"} \qquad (2.85)$$

with Eq. (2.82)

· Fig. 2.19 ... C_{11}, C_{12} in terms of θ, sharp decline → high directionality

" 2.20 C_{66} very high near $\theta = 45°$

" 2.21 --- C_{16}, $C_{26} \neq 0$, coupling between extension and shearing

$= 0$ in $\underline{\underline{C}}^*$ ← response of the system must be symm., precluding extension-shear couple

ii) Rotation of the compliance matrix

$$\underline{\underline{S}} = \begin{bmatrix} m^2 & n^2 & -mn \\ n^2 & m^2 & mn \\ 2mn & -2mn & m^2-n^2 \end{bmatrix} \underline{\underline{S}}^* \begin{bmatrix} m^2 & n^2 & 2mn \\ n^2 & m^2 & -2mn \\ -mn & mn & m^2-n^2 \end{bmatrix} \qquad (2.88)$$

$$= \begin{bmatrix} S_{11} & S_{12} & S_{16} \\ S_{12} & S_{22} & S_{26} \\ S_{16} & S_{26} & S_{66} \end{bmatrix} = \begin{bmatrix} 1/E_1 & -\nu_{21}/E_2 & \nu_{61}/G_{12} \\ -\nu_{12}/E_1 & 1/E_2 & \nu_{62}/G_{12} \\ \nu_{16}/E_1 & \nu_{26}/E_2 & 1/G_{12} \end{bmatrix}$$

- E_1, E_2, ν_{12}, G_{12}, ν_{16}, ν_{26} ... engineering constants in the arbitrary triad
- $\underline{S}$ must be symmetric
- alternative expression for engineering const.s ... Eq. (2.92)
  ... various tests to determine the engineering consts (Fig. 2.22), similar to those in Sec. 2.6.1, but currently stress is applied at θ.
- Fig 2.19 -- E_1 shows precipitous drop w.r.t. θ

◦ Difference between C_{11} and E_1
- $E_1 = 1/S_{11}$, $1/S_{11} \neq C_{11}$ since the inverse of a matrix is not simply the inverse of its items

  { Fig. 2.22 ... to measure E_1, σ_1 is applied, $\sigma_2 = \tau_{12} = 0$, $\epsilon_1 \to E_1$, $\epsilon_2 \to \nu_{12}$,
  " 2.23 ... " C_{11}, ϵ_1 " , $\epsilon_2 = \gamma_{12} = 0$ $\gamma_{12} \to \nu_{16}$ in Eq (2.??)

  but test is very difficult to perform since would have to be constrained to prevent any deformations except ϵ_1.
- Effect of these constraints → considerably stiffen the material
  ex) $C_{11} \gg E_1$ (Fig. 2.19)
  $C_{66} \gg G_{12}$ (" 2.20)

2.7 Strength of a transversely isotropic lamina

2.7.1. Strength of a lamina under simple loading conditions

◦ Fig. 2.26 ① ... σ_1^* applied in the fiber direction, and $\sigma_2^* = \tau_{12}^* = 0$
will provide σ_{1t}^{*f} and σ_{1c}^{*f} (not equal, generally)
" ② ... σ_2^* applied in the transverse dir., and $\sigma_1^* = \tau_{12}^* = 0$
will provide σ_{2t}^* and σ_{2c}^{*f}
" ③ ... shear stress τ_{12}^* applied, and $\sigma_1^* = \sigma_2^* = 0$
→ τ_{12}^{*f}, no dependence on sign

• Tests can be very difficult to perform in practice

2.7.2. Strength of a lamina under combined loading conditions

◦ Fig. 2.27 .. failure envelope, rather than performing a large number of experiments, apply a failure criterion
→ many different failure criteria, widely used

∘ { matrix failure ⋯ not always a catastrophic event
  fiber 〃 ⋯ completely eliminates load carrying capability

2.7.3. The Tsai-Wu failure criterion

∘ combined stresses applied

$$F_{11}^* \sigma_1^{*2} + 2F_{12}^* \sigma_1^* \sigma_2^* + F_{22}^* \sigma_2^{*2} + F_{66}^* \tau_{12}^{*2} + F_1^* \sigma_1^* + F_2^* \sigma_2^* = 1$$

$\square^*$: fiber aligned triad (2.93)

① test with a single stress component σ_1^* applied

$$F_{11}^* \sigma_{1t}^{*f2} + F_1^* \sigma_{1t}^{*f} = 1, \quad F_{11}^* \sigma_{1c}^{*f2} - F_1^* \sigma_{1c}^{*f} = 1$$

② σ_2^* only

③ τ_{12}^* only

→ Then, can find 5 coefficients in Eq. (2.93)