Mechanics and Design

Chapter 2

Stresses and Strains

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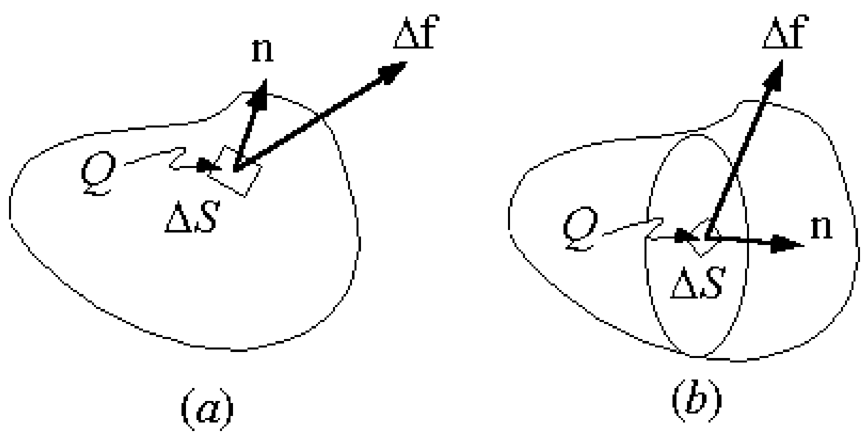
■ Traction or stress vector; stress components.

Consider a surface element, ΔS , of either the bounding surface of the body or the fictitious internal surface of the body as shown in Fig. 2.1.
Assume that ΔS contains the point .
The traction vector, $\mathbf{t}$, is defined by

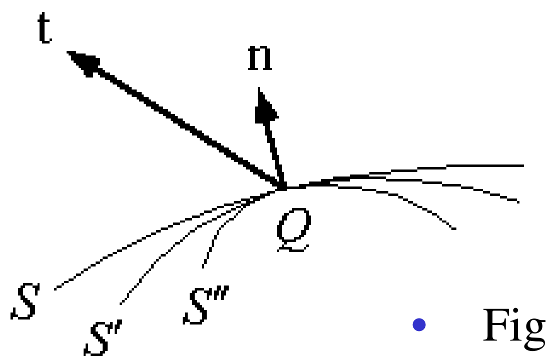
$$\mathbf{t} = \lim_{\Delta S \to 0} \left[\frac{\Delta \mathbf{f}}{\Delta S} \right] \qquad (2\text{-}1)$$



• Fig. 2.1 Definition of surface traction

- **Traction or stress vector; stress components.**

It is assumed that $\mathbf{\Delta f}$ and ΔS approach zero but the fraction, in general, approaches a finite limit.
An even stronger hypothesis is made about the limit approached at Q by the surface force per unit area.
First, consider several different surfaces passing through Q all having the same normal $\mathbf{n}$ at Q as shown in Fig. 2.2.



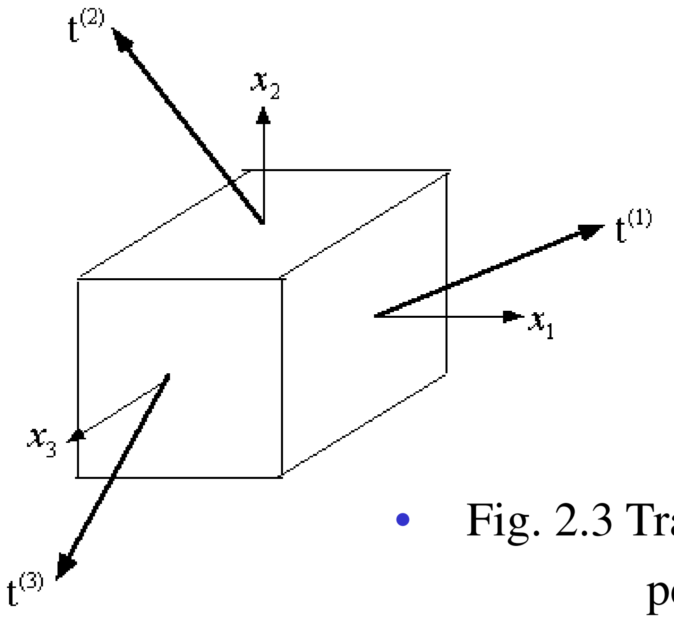
- Fig. 2.2 Traction vector $\mathbf{t}$ and vectors at Q

Then the tractions on S , S' and S'' are the same. That is, the traction is independent of the surface chosen so long as they all have the same normal.

■ Traction or stress vector; stress components.

Stress vectors on three coordinate plane

Let the traction vectors on planes perpendicular to the coordinate axes be $\mathbf{t}^{(1)}$, $\mathbf{t}^{(2)}$, and $\mathbf{t}^{(3)}$ as shown in Fig. 2.3.
Then the stress vector at that point on any other plane inclined arbitrarily to the coordinate axes can be expressed in terms of $\mathbf{t}^{(1)}$, $\mathbf{t}^{(2)}$, and $\mathbf{t}^{(3)}$.
Note that the vector $\mathbf{t}^{(1)}$ acts on the positive x_1 side of the element. The stress vector on the negative side will be denoted by $-\mathbf{t}^{(1)}$.



- Fig. 2.3 Traction vectors on three planes perpendicular to coordinate axes

Traction or stress vector; stress components.

Stress components

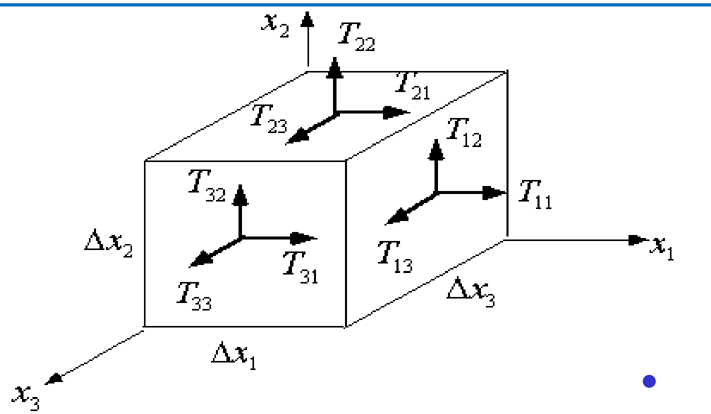
The traction vectors on planes perpendicular to the coordinate axes, x_1 , x_2 and x_3 are $\mathbf{t}^{(1)}$, $\mathbf{t}^{(2)}$, and $\mathbf{t}^{(3)}$.

The three vectors can be decomposed into the directions of coordinate axes as

$$\begin{aligned} \mathbf{t}^{(1)} &= T_{11}\mathbf{i} + T_{12}\mathbf{j} + T_{13}\mathbf{k} \\ \mathbf{t}^{(2)} &= T_{21}\mathbf{i} + T_{22}\mathbf{j} + T_{23}\mathbf{k} \qquad (2\text{-}2) \\ \mathbf{t}^{(3)} &= T_{31}\mathbf{i} + T_{32}\mathbf{j} + T_{33}\mathbf{k} \end{aligned}$$

The nine rectangular components T_{ij} are called the stress components. Here

> the first subscript represents the "plane" and
> the second subscript represents the "direction".

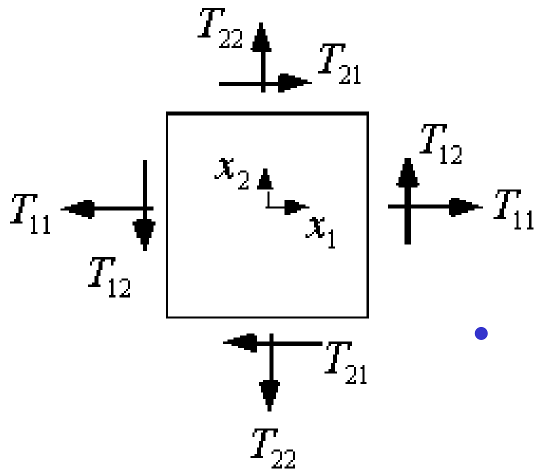


• Fig. 2.4 Stress components

- **Traction or stress vector; stress components.**

Sign convention

A stress component is positive when it acts in the positive direction of the coordinate axes, and on a plane whose outer normal points in one of the positive coordinate directions.



- Fig. 2.5 Sign convention of stress components

Stress state at a point

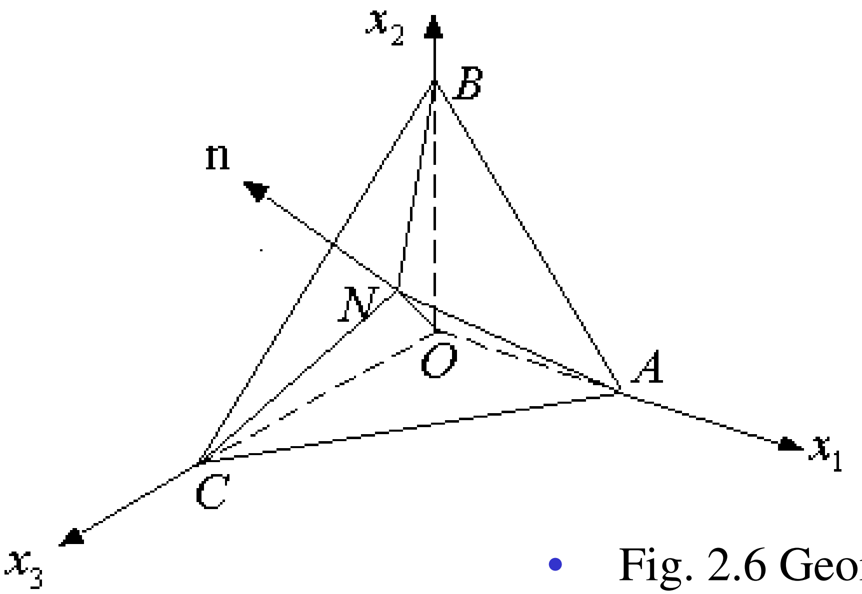
The stress state at a point $\boldsymbol{Q}$ is uniquely determined by the tensor $\mathbf{T}$ which is represented by

$$\mathbf{T} = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} \qquad (2\text{-}3)$$

- **Traction or stress vector; stress components.**

Traction vector on an arbitrary plane: The Cauchy tetrahedron

When the stress at a point ***O*** is given, then the traction on a surface passing the point ***Q*** is uniquely determined.
Consider a tetrahedron as shown in Fig. 2.6. The orientation of the oblique plane ABC is arbitrary. Let the surface normal of ΔABC be **n** and the line ON is perpendicular to ΔABC.
The components of the unit normal vector **n** are the direction cosine as



$$
\begin{aligned}
n_1 &= \cos(\angle AON) \\
n_2 &= \cos(\angle BON) \qquad \text{(2-4)} \\
n_3 &= \cos(\angle CON)
\end{aligned}
$$

- Fig. 2.6 Geometry of tetrahedron

Traction or stress vector; stress components.

If we let $\boldsymbol{ON} = h$, then

$$h = OA \cdot n_1 = OB \cdot n_2 = OC \cdot n_3 \qquad (2\text{-}5)$$

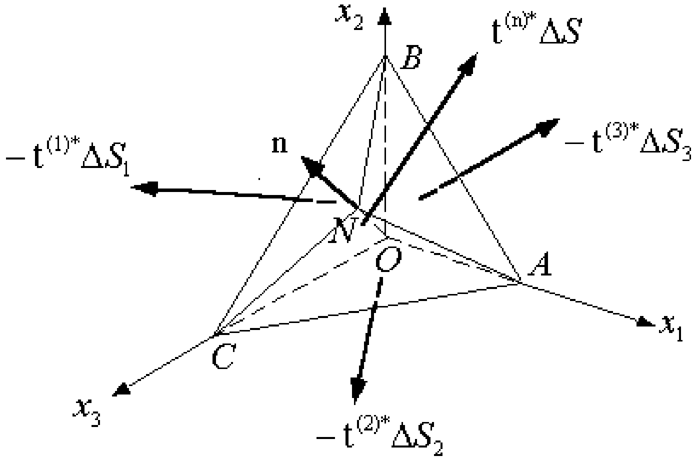
Let the area of ΔABC, ΔOBC, ΔOCA & ΔOAB be ΔS, ΔS_1, ΔS_2 & ΔS_3, respectively. Then the volume of the tetrahedron, ΔV, can be obtained by

$$\Delta V = \frac{1}{3} h \cdot \Delta S = \frac{1}{3} OA \cdot \Delta S_1 = \frac{1}{3} OB \cdot \Delta S_2 = \frac{1}{3} OC \cdot \Delta S_3 \qquad (2\text{-}6)$$

From this we get,

$$\begin{aligned} \Delta S_1 &= \Delta S \cdot \frac{h}{OA} = \Delta S \cdot n_1 \\ \Delta S_2 &= \Delta S \cdot \frac{h}{OB} = \Delta S \cdot n_2 \\ \Delta S_3 &= \Delta S \cdot \frac{h}{OC} = \Delta S \cdot n_3 \end{aligned} \qquad (2\text{-}7)$$

Traction or stress vector; stress components.



- Fig. 2.7 Forces on tetrahedron

Now consider the balance of the force on $OABC$ as shown in Fig. 2.7. The equation expressing the equilibrium for the tetrahedron becomes

$$\mathbf{t}^{(\mathbf{n})*}\Delta S + \rho^{*}\mathbf{b}^{*}\Delta V - \mathbf{t}^{(1)*}\Delta S_1 - \mathbf{t}^{(2)*}\Delta S_2 - \mathbf{t}^{(3)*}\Delta S_3 = 0 \qquad \text{(2-8)}$$

Here the subscript * indicates the average quantity. Substituting for ΔV ,ΔS_1, ΔS_2 and ΔS_3, and dividing through by ΔS , we get

$$\mathbf{t}^{(\mathbf{n})*} + \frac{1}{3}h\rho^{*}\mathbf{b}^{*} = \mathbf{t}^{(1)*}n_1 + \mathbf{t}^{(2)*}n_2 + \mathbf{t}^{(3)*}n_3 \qquad \text{(2-9)}$$

- **Traction or stress vector; stress components.**

Now let h approaches zero, then the term containing the body force approaches zero, while the vectors in the other terms approach the vectors at the point O. The result is in the limit

$$\mathbf{t}^{(\mathbf{n})} = \mathbf{t}^{(1)} n_1 + \mathbf{t}^{(2)} n_2 + \mathbf{t}^{(3)} n_3 = \mathbf{t}^{(k)} n_k \qquad (2\text{-}10)$$

This important equation permits us to determine the traction $\mathbf{t}^{(n)}$ at a point acting on an arbitrary plane through the point, when we know the tractions on only three mutually perpendicular planes through the point.

The equation (2-10) is a vector equation, and it can be rewritten by

$$\begin{aligned} t_1^{(n)} &= t_1^{(1)} n_1 + t_1^{(2)} n_2 + t_1^{(3)} n_3 \\ t_2^{(n)} &= t_2^{(1)} n_1 + t_2^{(2)} n_2 + t_2^{(3)} n_3 \\ t_3^{(n)} &= t_3^{(1)} n_1 + t_3^{(2)} n_2 + t_3^{(3)} n_3 \end{aligned} \qquad (2\text{-}11)$$

Comparing these with eq. (2-2), we get

$$\begin{aligned} t_1^{(n)} &= T_{11} n_1 + T_{21} n_2 + T_{31} n_3 = T_{k1} n_k \\ t_2^{(n)} &= T_{12} n_1 + T_{22} n_2 + T_{32} n_3 = T_{k2} n_k \\ t_3^{(n)} &= T_{13} n_1 + T_{23} n_2 + T_{33} n_3 = T_{k3} n_k \end{aligned} \qquad (2\text{-}12)$$

- **Traction or stress vector; stress components.**

Or for simplicity, we put

in indicial notation	$t_i^{(n)} = T_{ji} n_j$	
in matrix notation	$t^{(\mathrm{n})} = T^{\mathrm{T}} n$	(2-13)
in dyadic notation	$\mathbf{t}^{(\mathbf{n})} = \mathbf{n} \cdot \mathbf{T} = \mathbf{T}^T \cdot \mathbf{n}$	

From the derivation of this section,
It can be shown that the relation (2-13) also holds for fluid mechanics.

T_{ij} : Cauchy stress tensor.

This stress tensor is the linear vector function which associates with **n** the traction vector $\mathbf{t}^{(\mathrm{n})}$.

Coordinate transformation of stress tensors

As we discussed in the previous chapter, stress tensor follows the tensor coordinate transformation rule. That is, let x and $\bar{x}$ be the two coordinate systems and A be a transformation matrix as

$$v = A\bar{v} \qquad \text{or} \qquad \bar{v} = A^T v$$

Then the stress tensor T transforms to $\bar{T}$ as

$$\bar{T} = A^T T A$$

We may consider the stress tensor transformation in two dimensional case. Let the angle between x axis and $\bar{x}$ axis is θ . Then the transformation matrix A becomes

$$A = \begin{bmatrix} a_1^1 & a_2^1 \\ a_1^2 & a_2^2 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Coordinate transformation of stress tensors

The stress T transforms to $\bar{T}$ according to the following

$$\bar{T} = A^T T A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Evaluating the equation, we get

$$\bar{T}_{11} = T_{11}\cos^2\theta + 2T_{21}\sin\theta\cos\theta + T_{22}\cos\theta\sin\theta$$

$$\bar{T}_{12} = (T_{22} - T_{11})\sin\theta\cos\theta + T_{12}(\cos^2\theta - \sin^2\theta)$$

$$\bar{T}_{22} = T_{11}\sin^2\theta + T_{22}\cos^2\theta - 2T_{12}\sin\theta\cos\theta$$

By using double angle trigonometry, we can get

$$\bar{T}_{11}, \bar{T}_{22} = \frac{(T_{11} + T_{22})}{2} \pm \frac{(T_{11} - T_{22})}{2}\cos 2\theta \pm T_{12}\sin 2\theta$$

$$\bar{T}_{12} = \frac{(T_{11} - T_{22})}{2}\sin 2\theta + T_{12}\cos 2\theta$$

Coordinate transformation of stress tensors

Now we recognize the last two equations are the same as the ones we derived for Mohr circle, (eq 4-25) of Crandall's book.
For the two dimensional stress state, Mohr circle may be convenient because we recognize the stress transformation more intuitively.
However, for 3D stress state and computation, it is customary to use the tensor equation directly to calculate the stress components in transformed coordinate system.

Principal axes of stress, Principal stress, etc.

Characteristic of the principal stress

(1) When we consider the stress tensor **T** as a transformation, then there exist a line which is transformed onto itself by **T**.

(2) There are three planes where the traction of the plane is in the direction of the normal vector, i.e.

$$\mathbf{t}^{(\mathbf{n})} // \mathbf{n} \qquad \text{or} \qquad \mathbf{t}^{(\mathbf{n})} = \lambda \mathbf{n} \qquad \text{(2-14)}$$

Definitions

- Principal axes
- Principal plane
- Principal stress

Principal axes of stress, Principal stress, etc.

Determination of the principal stress

Let $\mathbf{T}$ be the stress at a point in some Cartesian coordinate system, $\mathbf{n}$ be a unit vector in one of the unknown directions and λ represent the principal component on the plane whose normal is $\mathbf{n}$.

Then

$$\mathbf{t}^{(\mathbf{n})} = \lambda \mathbf{n} \quad \text{; that is,} \quad \mathbf{n} \cdot \mathbf{T} = \lambda \mathbf{n}$$

In indicial notation, we have

$$T_{rs} n_r = \lambda n_s = \lambda \delta_{rs} n_r$$

Rearranging, we have

$$(T_{rs} - \lambda \delta_{rs}) n_r = 0 \qquad (2\text{-}15)$$

The three direction cosines cannot be all zero, since

$$n_r n_r = n_1^2 + n_2^2 + n_3^2 = 1$$

A system of linear homogeneous equations, such as eq. (2-15) has solutions which are not all zero if and only if the determinant

$$\left| T_{rs} - \lambda \delta_{rs} \right| = 0 \qquad (2\text{-}16)$$

The equation (2-16) represents third order polynominal equation w.r.t. λ and it has 3 real roots for λ since $\mathbf{T}$ represents a real symmetric matrix.

Principal axes of stress, Principal stress, etc.

Determination of principal direction

When we get $\lambda = \lambda_1, \lambda_2, \lambda_3$, we can substitute the λ_i into the system of three algebraic equations

$$\begin{bmatrix} T_{11}-\lambda & T_{12} & T_{13} \\ T_{21} & T_{22}-\lambda & T_{23} \\ T_{31} & T_{32} & T_{33}-\lambda \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = 0 \qquad (2\text{-}17)$$

From these, we get the ratio of $n_1 : n_2 : n_3$. Since $|\mathrm{n}| = \sqrt{n_i n_i} = 1$, we can determine $(n_1 \quad n_2 \quad n_3)$ uniquely.

There can be three different cases;

(1) 3 distinctive roots
(2) Two of the roots are the same (cylindrical)
(3) All three roots are the same (spherical)

When the two of the principal stresses, say σ_1 and σ_2, are not equal, the corresponding principal directions $\mathbf{n}^{(1)}$ and $\mathbf{n}^{(2)}$ are perpendicular.

- **Principal axes of stress, Principal stress, etc.**

Proof>

Recall eq. (2-15)

$$(T_{rs} - \lambda \delta_{rs}) n_r = 0 \qquad (2\text{-}15)$$

By substituting λ_i and $\mathbf{n}^{(i)}$, we get

$$(T_{rs} - \lambda_1 \delta_{rs}) n_r^{(1)} = 0 \qquad (2\text{-}18)$$

$$(T_{rs} - \lambda_2 \delta_{rs}) n_r^{(2)} = 0$$

Note that $(T_{rs} - \lambda_1 \delta_{rs}) n_r^{(1)}$ represent a vector. From eq. (2-18), we can have

$$(T_{rs} - \lambda_1 \delta_{rs}) n_r^{(1)} n_s^{(2)} = 0 \qquad (a)$$

$$(T_{rs} - \lambda_2 \delta_{rs}) n_r^{(2)} n_s^{(1)} = 0 \qquad (b)$$

By subtracting (b) from (a), we get

$$T_{rs} n_r^{(1)} n_s^{(2)} - T_{rs} n_r^{(2)} n_s^{(1)} + \lambda_2 n_r^{(2)} n_r^{(1)} - \lambda_1 n_r^{(1)} n_r^{(2)} = 0 \qquad (c)$$

The first two terms of eq. (c) become

$$T_{rs} n_r^{(1)} n_s^{(2)} - T_{rs} n_r^{(2)} n_s^{(1)} = T_{rs} n_r^{(1)} n_s^{(2)} - T_{sr} n_r^{(1)} n_s^{(2)}$$

$$= (T_{rs} - T_{sr}) n_r^{(1)} n_s^{(2)} = 0$$

■ Principal axes of stress, Principal stress, etc.

because **T** is symmetric. The remaining terms of eq. (c) become

$$(\lambda_2 - \lambda_1) n_r^{(1)} n_r^{(2)} = 0 \qquad \text{(d)}$$

Since $\lambda_1 \neq \lambda_2$ and $\left|\mathbf{n}^{(1)}\right| = \left|\mathbf{n}^{(2)}\right| = 1$, eq. (d) implies

$$n_r^{(1)} n_r^{(2)} = \mathbf{n}^{(1)} \cdot \mathbf{n}^{(2)} = 0$$

Therefore, $\mathbf{n}^{(1)}$ and $\mathbf{n}^{(2)}$ are perpendicular to each other.

Principal axes of stress, Principal stress, etc.

Invariants

The principal stresses are physical quantities. Their value does not depend on the choice of the coordinate system. Therefore, the principal stresses are invariants of the stress state.

That is, they are invariant w.r.t. the rotation of the coordinate axes.

The determinant in the characteristic equation becomes

$$\begin{vmatrix} T_{11}-\lambda & T_{12} & T_{13} \\ T_{21} & T_{22}-\lambda & T_{23} \\ T_{31} & T_{32} & T_{33}-\lambda \end{vmatrix} = 0$$

Evaluating the determinant, we get

$$\lambda^3 - \mathbf{I}_T\lambda^2 - \mathbf{II}_T\lambda - \mathbf{III}_T = 0 \qquad (2\text{-}20)$$

Principal axes of stress, Principal stress, etc.

where

$$\mathbf{I}_T = T_{11} + T_{22} + T_{33} = T_{kk} = tr(\mathbf{T})$$

$$\mathbf{II}_T = -(T_{11}T_{22} + T_{22}T_{33} + T_{33}T_{11}) + T_{23}^2 + T_{31}^2 + T_{12}^2$$

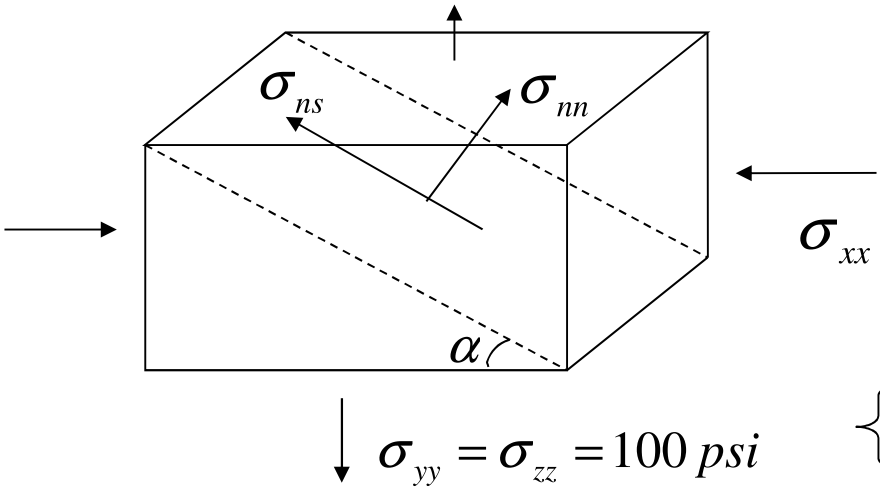
$$= \frac{1}{2}(T_{ij}T_{ij} - T_{ii}T_{jj})$$

$$\mathbf{III}_T = \det(\mathbf{T}) = \begin{vmatrix} T_{11}-\lambda & T_{12} & T_{13} \\ T_{21} & T_{22}-\lambda & T_{23} \\ T_{31} & T_{32} & T_{33}-\lambda \end{vmatrix}$$

$$= \frac{1}{6} e_{ijk} e_{pqr} T_{ip} T_{jq} T_{kr}$$

Since the roots of the cubic equation are invariants, the coefficients should be invariants.

- **Example 1 > Determine the normal and shear stress at the interface**



$$\sigma_{xx} = \sigma_{11} = -500\ psi$$

$$\sigma_{yy} = \sigma_{zz} = 100\ psi$$

Other stress components are zero.
Stress distribution is uniform.

From the figure we can determine

$$\mathbf{n} = \sin\alpha\mathbf{e}_1 + \cos\alpha\mathbf{e}_2$$

We use eq (2-19), $T^{(n)} = T\mathbf{n}$ to find out surface traction. That is,

$$T^{(n)} = \begin{bmatrix} -500 & 0 & \\ 0 & 100 & \\ & & 100 \end{bmatrix} \begin{bmatrix} \sin\alpha \\ \cos\alpha \\ 0 \end{bmatrix} = \begin{bmatrix} -500\sin\alpha \\ 100\cos\alpha \\ 0 \end{bmatrix}$$

- **Example 1 > Determine the normal and shear stress at the interface**

Therefore, the traction at the surface whose normal is **n** given by

$$T^{(n)} = -500\sin\alpha\mathbf{e}_1 + 100\cos\alpha\mathbf{e}_2$$

Therefore, the surface traction in **n** direction is given by

$$T_{nn} = T^{(n)}\cdot\mathbf{n} = (-500\sin\alpha\mathbf{e}_1 + 100\cos\alpha\mathbf{e}_2)\cdot(\sin\alpha\mathbf{e}_1 + \cos\alpha\mathbf{e}_2)$$

$$= (-500\sin^2\alpha + 100\cos^2\alpha)$$

There may be different ways to obtain shear component of the traction. One way may be the vector subtraction.

Then the shear stress at the interface becomes

$$T_{ns} = T^{(n)} - (-500\sin^2\alpha + 100\cos^2\alpha)(\sin\alpha\mathbf{e}_1 + \cos\alpha\mathbf{e}_2)$$

$$= -500\sin\alpha\mathbf{e}_1 + 100\cos\alpha\mathbf{e}_2 - (-500\sin^2\alpha + 100\cos^2\alpha)(\sin\alpha\mathbf{e}_1 + \cos\alpha\mathbf{e}_2)$$

$$= (-500\sin\alpha + 500\sin^3\alpha - 100\cos^2\alpha\sin\alpha)\mathbf{e}_1$$

$$+ (100\cos\alpha + 500\sin^2\alpha\cos\alpha + 500\sin^2\alpha\cos\alpha)\mathbf{e}_2$$

$$= -600\sin\alpha\cos^2\alpha\mathbf{e}_1 + 600\cos\alpha\sin^2\alpha\mathbf{e}_2$$

➡ The magnitude of T_{ns} becomes $T_{ns} = |T_{ns}| = 600\sin\alpha\cos\alpha$

Example 2

$$\text{Let,}\quad T = \begin{bmatrix} 3000 & -1000 & 0 \\ -1000 & 2000 & 2000 \\ 0 & 2000 & 2000 \end{bmatrix}$$

Then determine the normal traction on the surface whose normal is

$$\mathbf{n} = 0.6\mathbf{e}_2 + 0.8\mathbf{e}_3$$

Again we use $T^{(n)} = T \cdot \mathbf{n}$

$$T_{nn} = T^{(n)} = T \cdot \mathbf{n} = \begin{bmatrix} 3000 & -1000 & 0 \\ -1000 & 2000 & 2000 \\ 0 & 2000 & 2000 \end{bmatrix} \begin{bmatrix} 0 \\ 0.6 \\ 0.8 \end{bmatrix}$$

$$= \begin{bmatrix} -600 \\ 2800 \\ 2800 \end{bmatrix} = -600\mathbf{e}_1 + 2800\mathbf{e}_2 + 2800\mathbf{e}_3$$

$$|T_{nn}| = 4050 \quad \text{and} \quad \mathbf{s} = \mathbf{e}_1$$

Then $\quad T_{ns} = T^{(n)} \cdot \mathbf{s} = (-600\mathbf{e}_1 + 2800\mathbf{e}_2 + 2800\mathbf{e}_3) \cdot \mathbf{s} = -600$

The generalized Hooke's law

$$\sigma_{ij}=C^{ijkl}e_{kl}$$

In as much as $\sigma_{ij} = \sigma_{ji}$ and $e_{ij} = e_{ji}$, $\quad C^{ijkl} = C^{ijlk}$ and $C^{ijkl} = C^{jikl}$

According to these symmetry properties, the maximum number of the independent elastic constants is 36.