Introduction

When slightly disturbed from an equilibrium configuration, does a system tend to return to its equilibrium position or does it tend to depart even further?

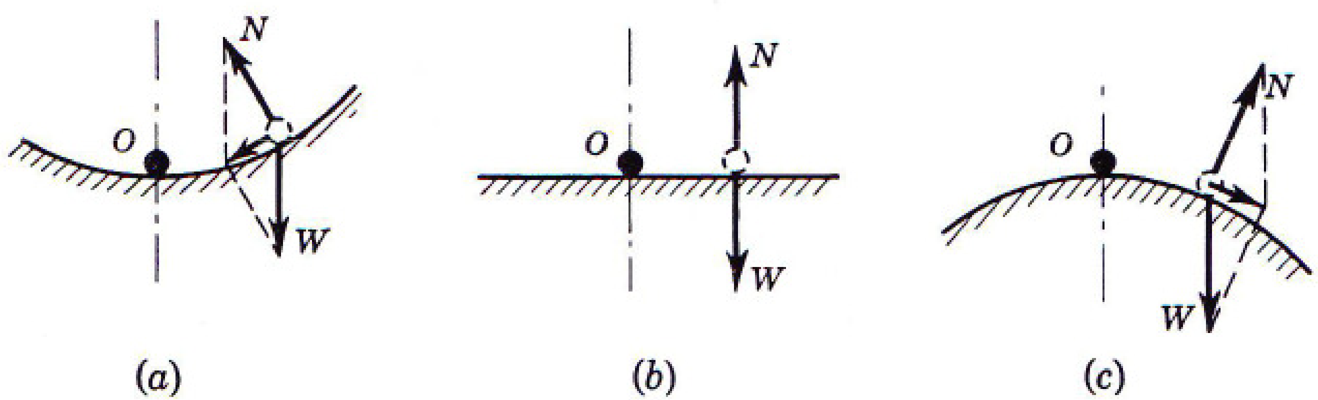


Fig. 9.1 Example of (*a*) stable, (*b*) neutral, and (*c*) unstable equilibrium.

Elastic Stability

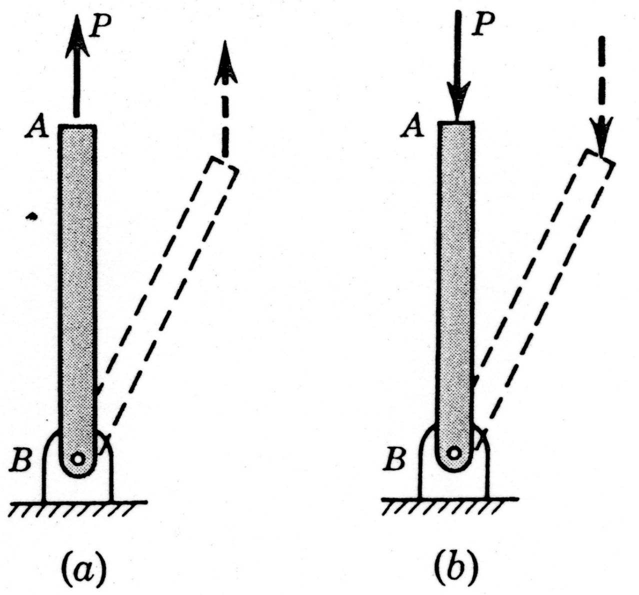
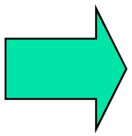


Fig. 9.2 Hinged bar is (*a*) stable for tensile load, (*b*) unstable for compressive load.



Make Stable

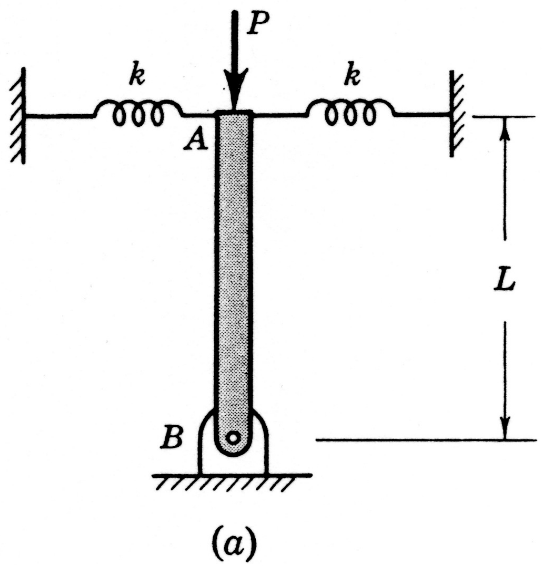
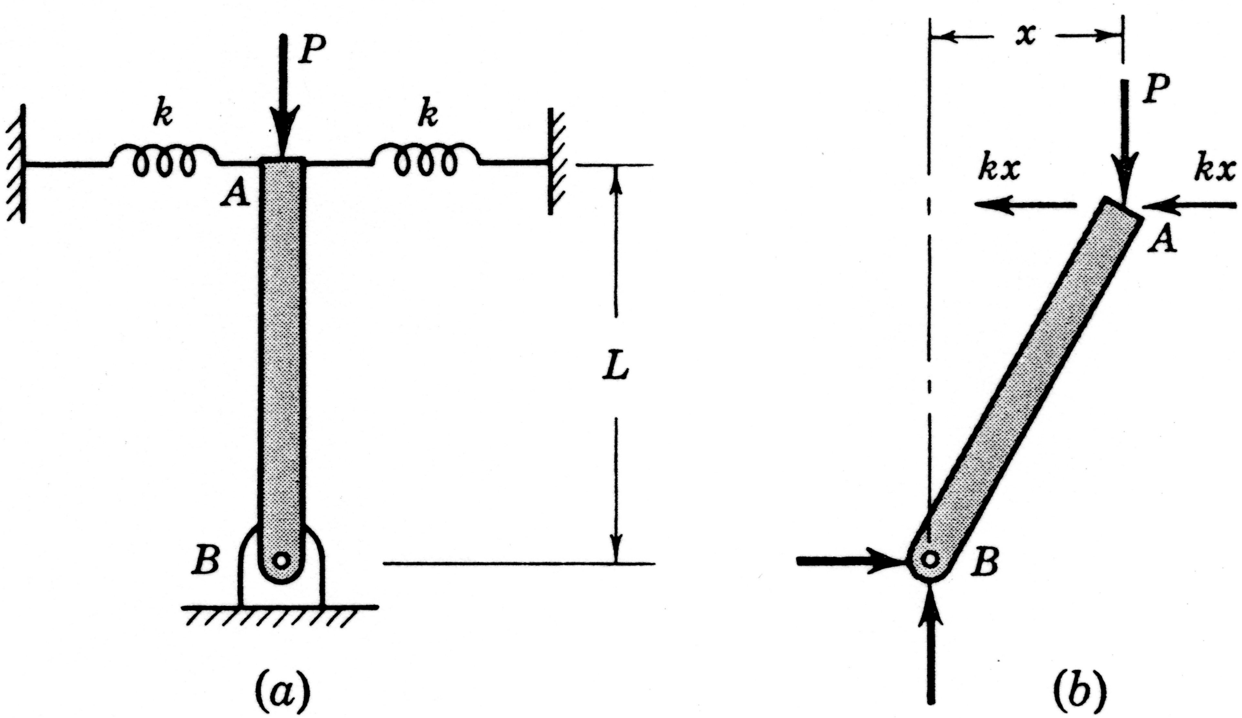


Fig. 9.3 (*a*)



$$Px > 2kxL \quad \text{(unstable)}$$

$$Px < 2kxL \quad \text{(stable)} \qquad (9.1)$$

$$P = 2kL \quad \text{(critical load or buckling load)}$$

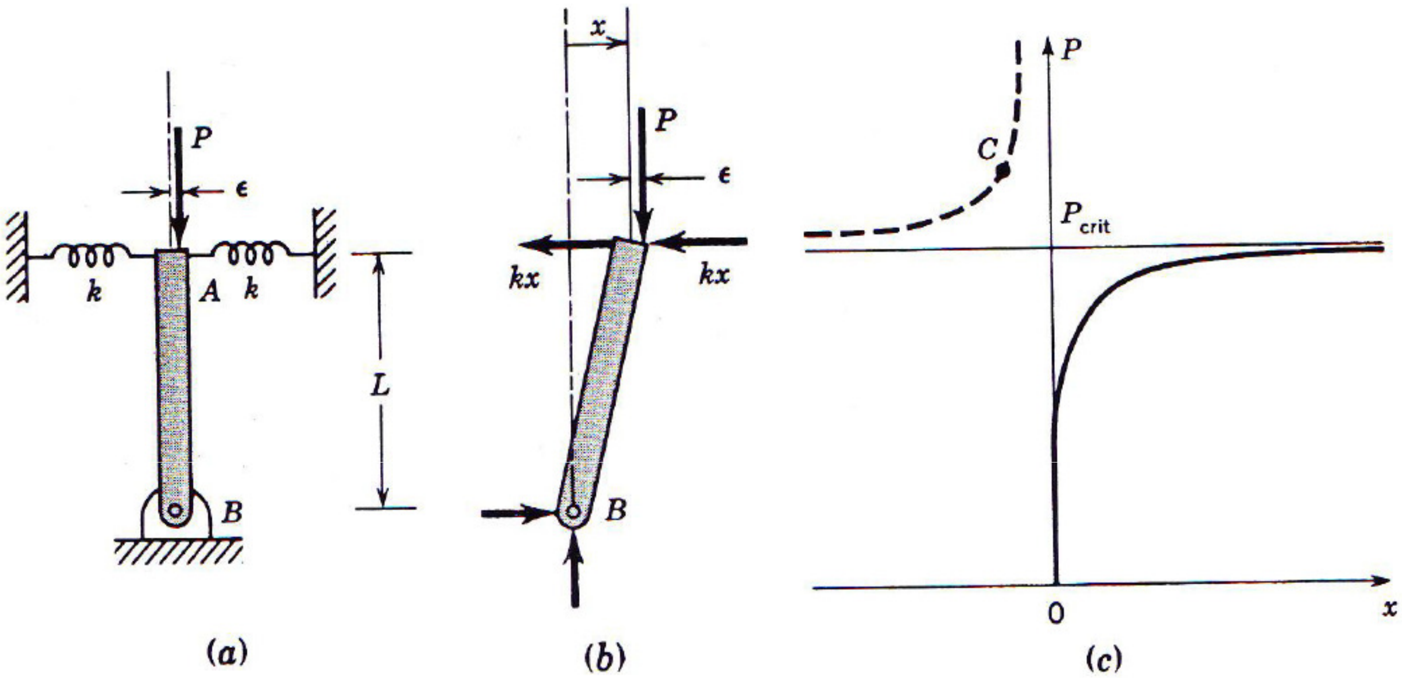


Fig. 9.4 Transverse displacement x due to load eccentricity ε.

$$P(x+\varepsilon) = 2kxL$$

$$x = \varepsilon \frac{P}{2kL - P} \qquad (9.2)$$

If P is not too close to the critical load (e.g., $P < \frac{1}{2} P_{crit}$) the equilibrium displacement (x) is small.

Examples of Instability

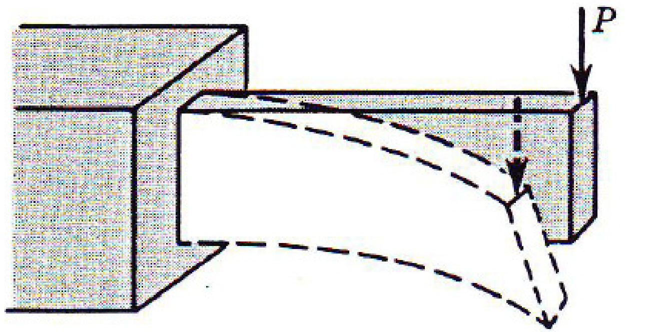


Fig. 9.6 Twist-bend buckling of a deep, narrow beam.

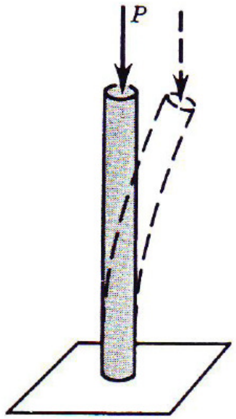


Fig. 9.7 Buckling of a column under a compressive load.

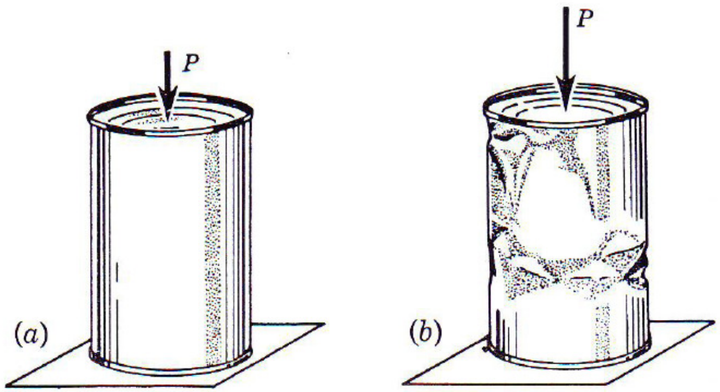


Fig. 9.8 Buckling and crumpling of the cylindrical walls of a can subjected to compressive force.

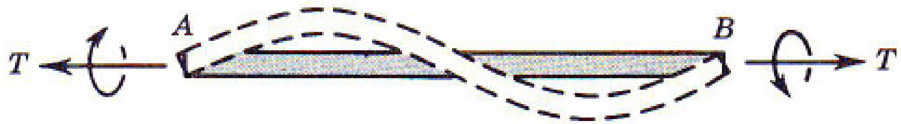


Fig. 9.9 Twist-bend buckling of a shaft in torsion.

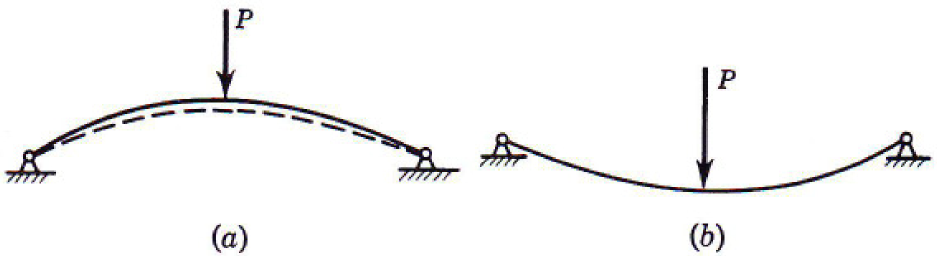


Fig. 9.10 "Snap-through" instability of a shallow curved member.

Elastic Stability of Flexible Columns

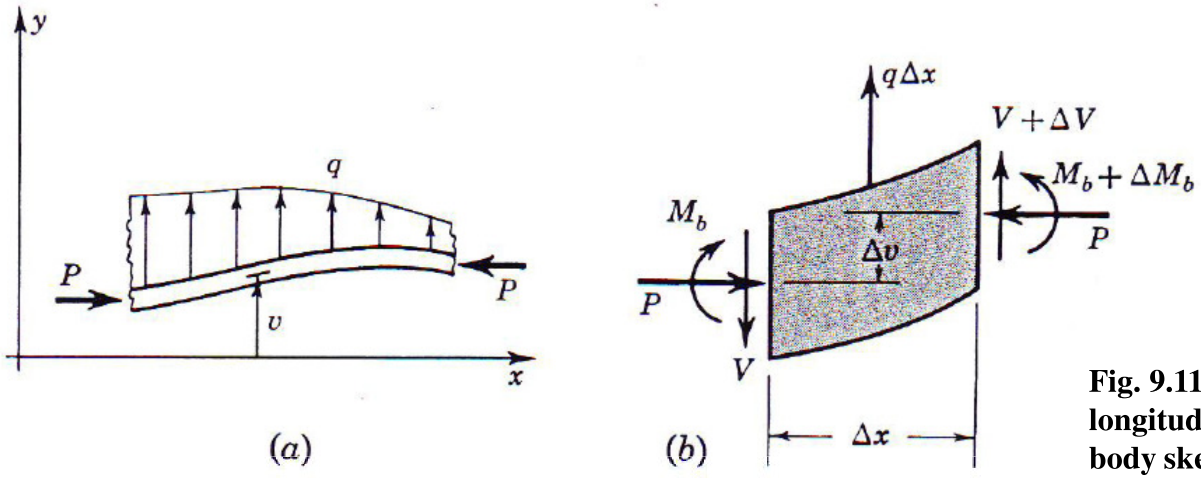


Fig. 9.11 (a) Beam subjected to longitudinal and transverse loads; (b) free-body sketch of element of beam.

$$(V + \Delta V) - V + q\Delta x = 0$$

$$(M_b + \Delta M_b) - M_b + V\frac{\Delta x}{2} + (V + \Delta V)\frac{\Delta x}{2} + P\Delta v = 0 \quad (9.3)$$

$$\frac{dV}{dx} + q = 0$$

$$\frac{dM_b}{dx} + V + P\frac{dv}{dx} = 0 \quad (9.4)$$

$$EI\frac{d^2v}{dx^2} = M_b \quad (9.5)$$

$$\frac{d^2}{dx^2}(EI\frac{d^2v}{dx^2}) + \frac{d}{dx}(P\frac{dv}{dx}) = q \quad (9.6)$$

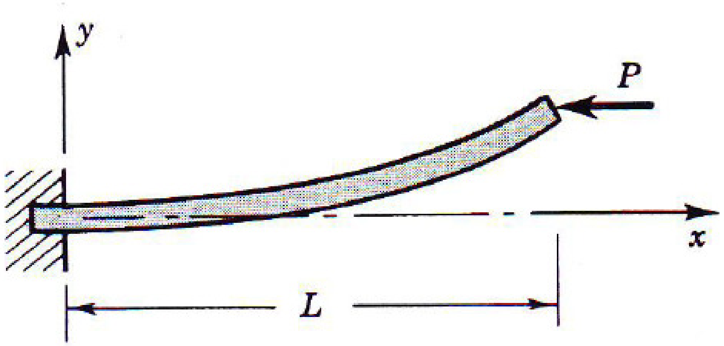


Fig. 9.12 Column in a state of neutral equilibrium in the bent position.

Boundary conditions

$$\left.\begin{aligned} v &= 0 \\ \frac{dv}{dx} &= 0 \end{aligned}\right\} \quad at\ x = 0 \qquad \left.\begin{aligned} M_b &= 0 \\ V &= 0 \end{aligned}\right\} \quad at\ \ x = L \qquad (9.7)$$

$$\left.\begin{aligned} M_b &= EI\frac{d^2v}{dx^2} = 0 \\ -V &= \frac{d}{dx}(EI\frac{d^2v}{dx^2}) + P\frac{dv}{dx} = 0 \end{aligned}\right\} at \quad x = L \qquad (9.8)$$

When *EI* and P are constants, the governing equation (9.6) is

$$EI\frac{d^4v}{dx^4} + P\frac{d^2v}{dx^2} = 0 \qquad (9.9)$$

A solution to (9.9) for arbitrary values of the four constants is

$$v = c_1 + c_2 x + c_3 \sin\sqrt{\frac{P}{EI}}x + c_4 \cos\sqrt{\frac{P}{EI}}x \tag{9.10}$$

Substituting (9.10) into the four boundary conditions of (9.7) and (9.8)

$$\begin{aligned}
c_1 \qquad\qquad\qquad\qquad + c_4 \qquad\qquad\qquad &= 0 \\
c_2 + c_3\sqrt{\frac{P}{EI}} \qquad\qquad\qquad\qquad &= 0 \\
- c_3 \frac{P}{EI}\sin\sqrt{\frac{P}{EI}}L - c_4 \frac{P}{EI}\cos\sqrt{\frac{P}{EI}}L &= 0 \\
c_2 P \qquad\qquad\qquad\qquad\qquad &= 0
\end{aligned} \tag{9.11}$$

This is an eigenvalue problem.

$c_2 = c_3 = 0 \quad and \quad c_4 = -c_1$

Then the third equation becomes simply

$$c_1 \frac{P}{EI} \cos\sqrt{\frac{P}{EI}} L = 0 \qquad (9.12)$$

This can be satisfied by having a value of P such that

$$\cos\sqrt{\frac{P}{EI}} L = 0 \qquad (9.13)$$

The smallest value of P meeting this condition is

$$P = \frac{\pi^2}{4} \frac{EI}{L^2} \qquad \text{(Critical load)} \qquad (9.14)$$

Substituting back into (9.10), the corresponding deflection curve is

$$v = c_1 \left(1 - \cos \frac{\pi}{2} \frac{x}{L} \right) \qquad (9.15)$$

For smaller value of P the straight column is stable.

For larger value of P the straight column is no longer stable. → Buckling

Another insight into column buckling:
imperfection in either the column or the loading

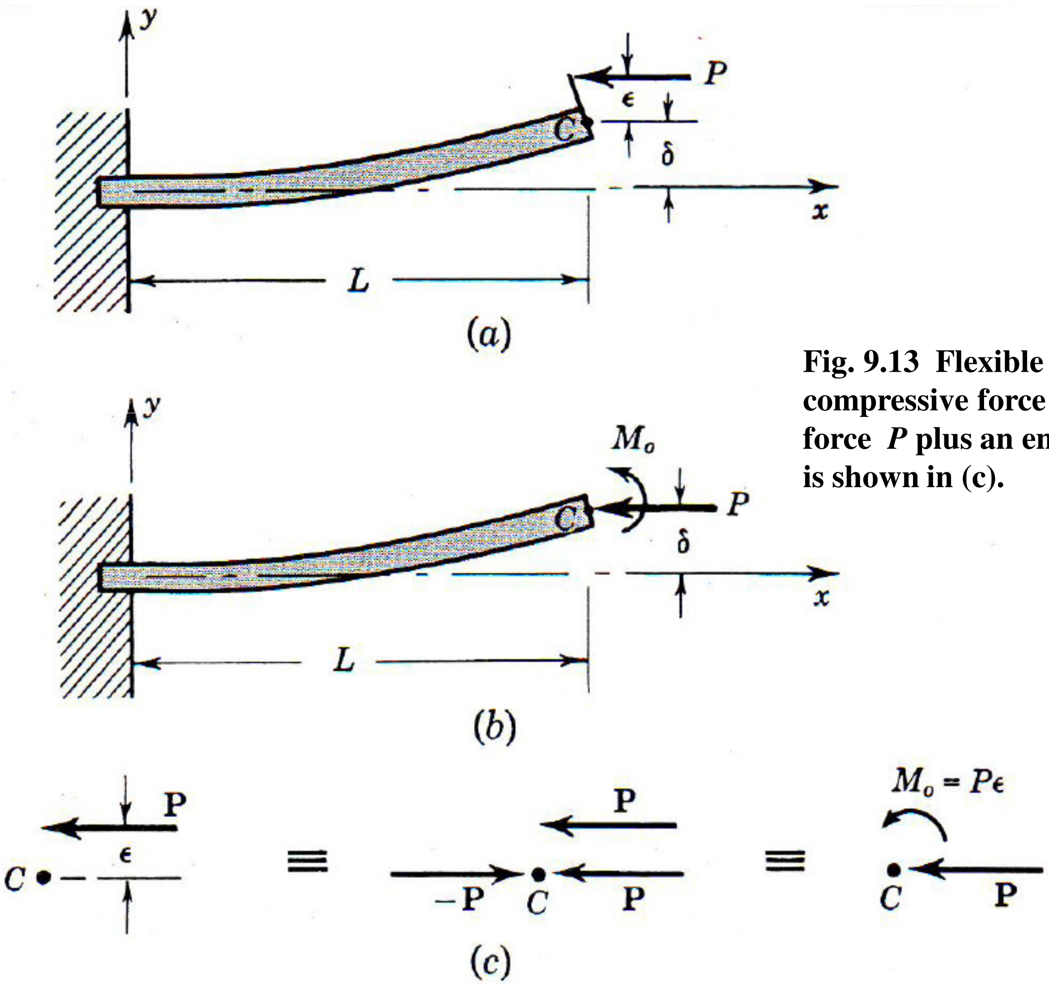


Fig. 9.13 Flexible column held in equilibrium by (a) a longitudinal compressive force P with eccentricity ϵ and (b) the same compressive force P plus an end moment M_0. The equivalence of the two loadings is shown in (c).

Boundary conditions:

$$\left.\begin{aligned} v &= 0 \\ \frac{dv}{dx} &= 0 \end{aligned}\right\} \quad at\ x = 0 \qquad \left.\begin{aligned} M_b &= M_0 \\ V &= 0 \end{aligned}\right\} \quad at\ x = L$$

$$\begin{aligned} c_1 + c_4 &= 0 \\ c_2 + c_3\sqrt{\frac{P}{EI}} &= 0 \\ -c_3\frac{P}{EI}\sin\sqrt{\frac{P}{EI}}L - c_4\frac{P}{EI}\cos\sqrt{\frac{P}{EI}}L &= \frac{M_0}{EI} \\ c_2 P &= 0 \end{aligned} \qquad (9.16)$$

$$\Rightarrow \quad v = \frac{M_0}{P}\frac{1-\cos\sqrt{P/EI}\,x}{\cos\sqrt{P/EI}\,L}$$

$x = L$

$$\Rightarrow \quad \delta = \frac{M_0}{P}\left(\sec\sqrt{\frac{P}{EI}}L - 1\right) \quad (9.17)$$

which reduces to

$$\delta = \varepsilon\left(\sec\sqrt{\frac{P}{EI}}L - 1\right) \quad (9.18)$$

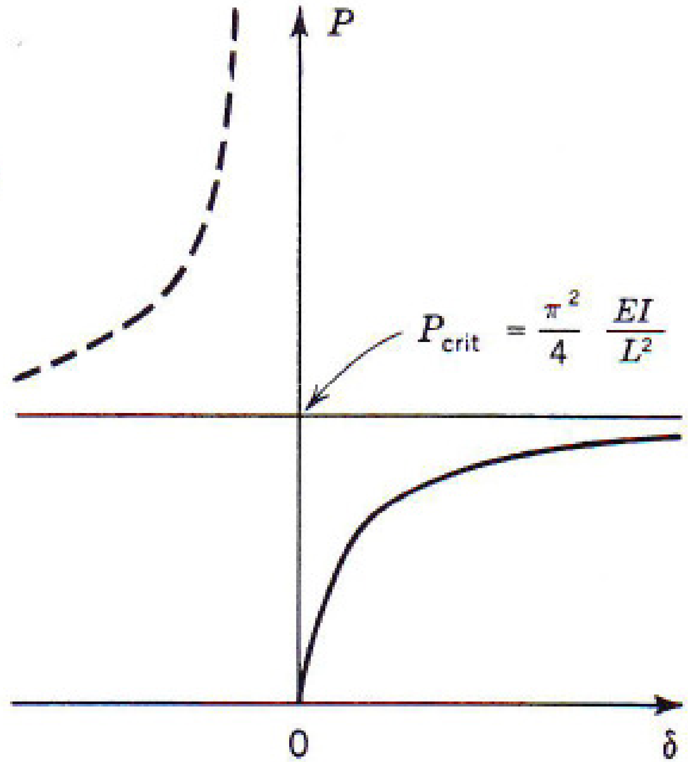


Fig. 9.14 Relation between compressive force P and transverse deflection δ due to eccentricity ϵ.

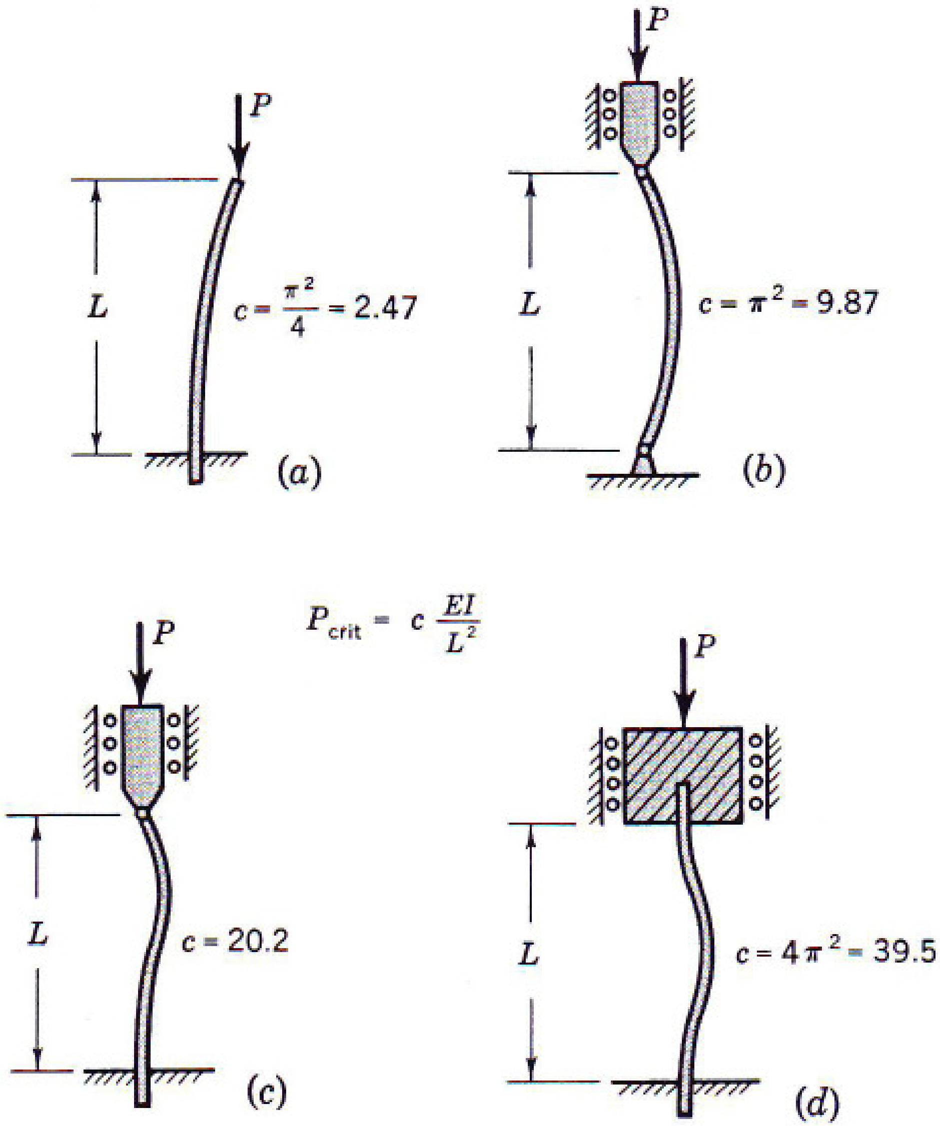
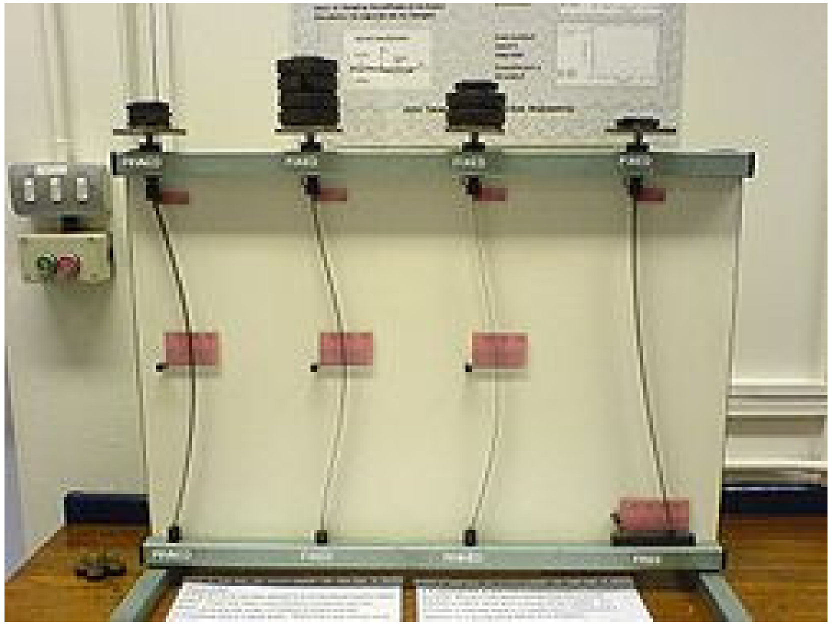


Fig. 9.15 Critical loads for (a) clamped-free, (b) hinged-hinged, (c) clamped-hinged, and (d) clamped-clamped columns. In each case the constant c shown is to be inserted in the formula $P_{crit} = cEI/L^2$.



Sun kink in rail tracks

Lateral-torsional buckling of an aluminium alloy plate girder designed and built by students atImperial College London.

Elastic Postbuckling Behavior

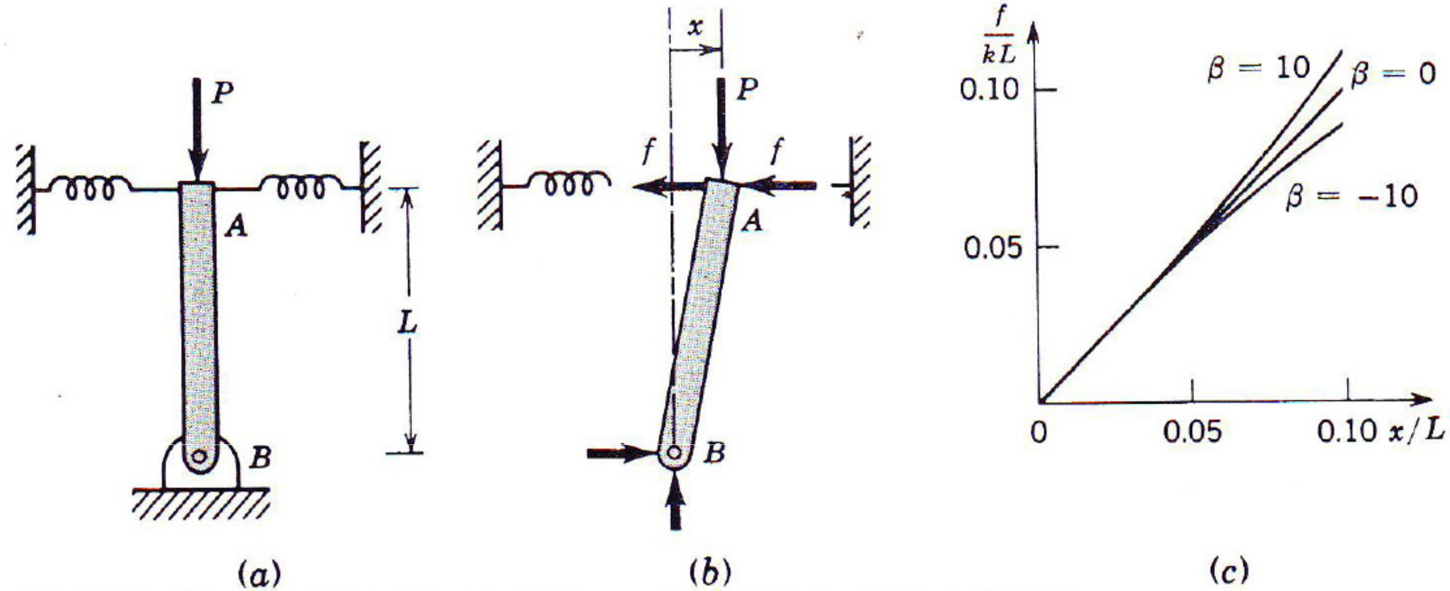


Fig. 9.16 Strut supported by nonlinear springs with $f = kx(1+\beta x^2/L^2)$.

$$f = kx\left(1+\beta\frac{x^2}{L^2}\right) \qquad (9.19)$$

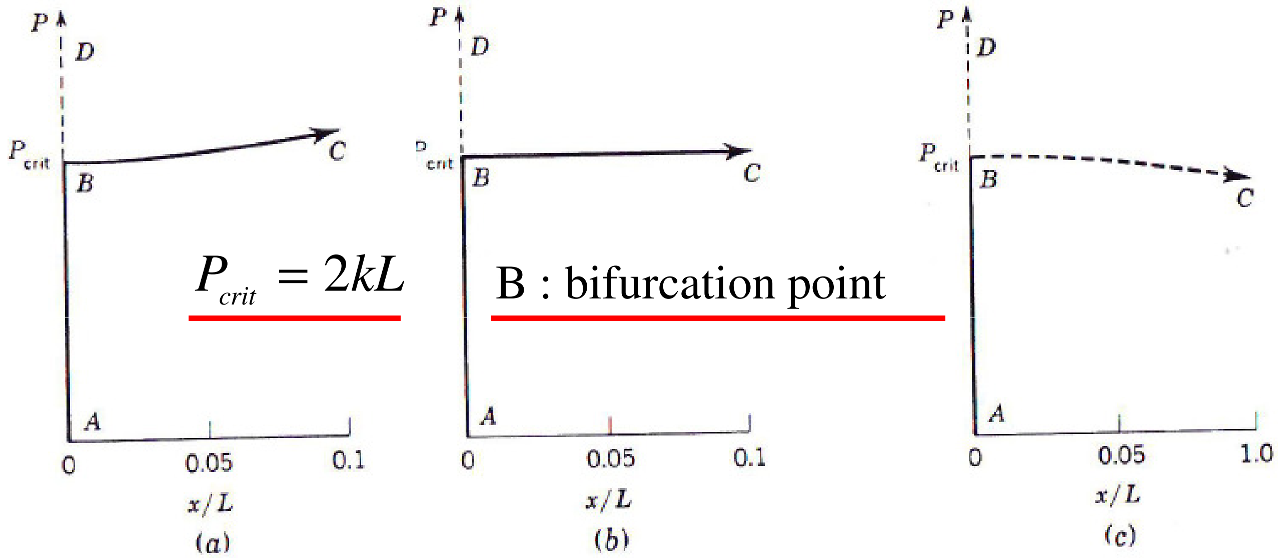
where β is a parameter which fixes the nature of the nonlinearity

$\beta > 0$: *stiffening spring*

$\beta < 0$: *softening spring*

From Fig. 9.16(b)

$$Px - 2kLx\left(1+\beta\frac{x^2}{L^2}\right) = 0 \quad \Rightarrow \quad x = 0 \quad or \quad P = 2kL\left(1+\beta\frac{x^2}{L^2}\right)$$



$P_{crit} = 2kL$

B : bifurcation point

Fig. 9.17 Ideal postbuckling curves for (a) β = 10, (b) β = 0, (c) β = -10.

In every case the branch BD represents **unstable** equilibrium positions.

The branch BC represents
- **stable** equilibrium positions. $for\ \beta > 0$
- **neutral** equilibrium positions. $for\ \beta = 0$
- **unstable** equilibrium positions. $for\ \beta < 0$

When the load is positioned slightly off-center:

$$P(x+\varepsilon) = 2kLx\left(1+\beta\frac{x^2}{L^2}\right) \qquad (9.20)$$

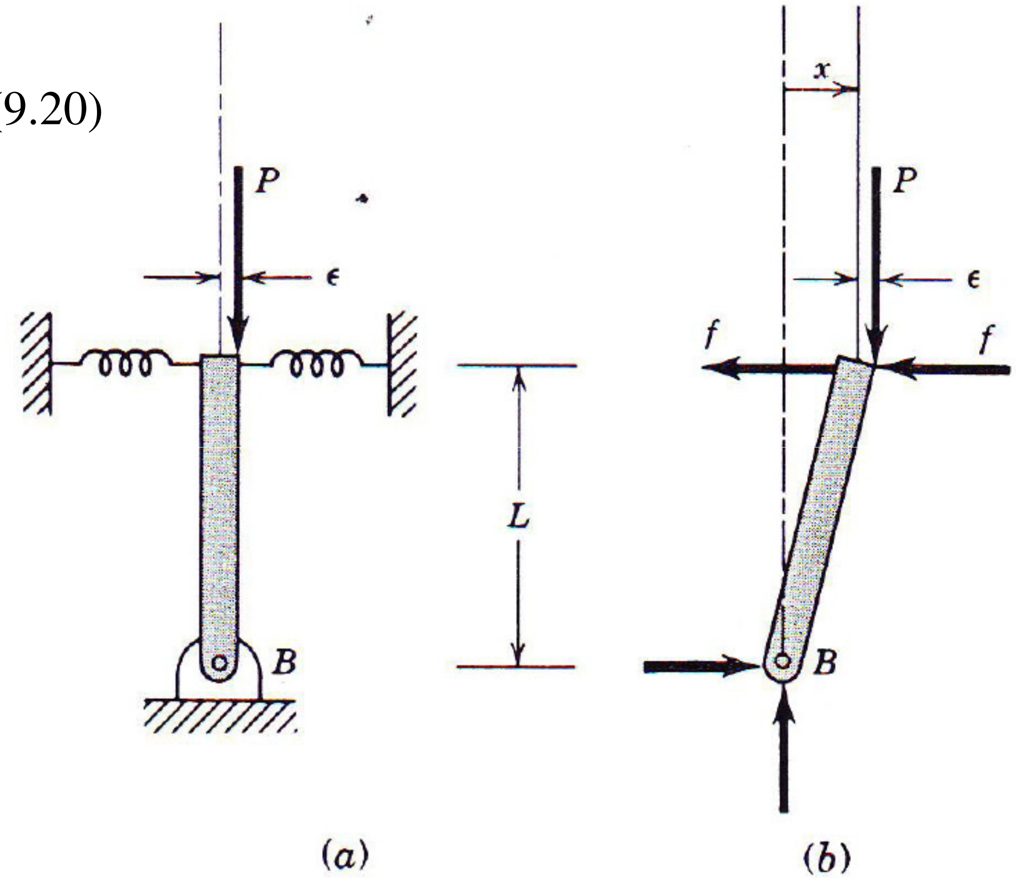


Fig. 9.18 Eccentric load on strut supported by nonlinear springs.

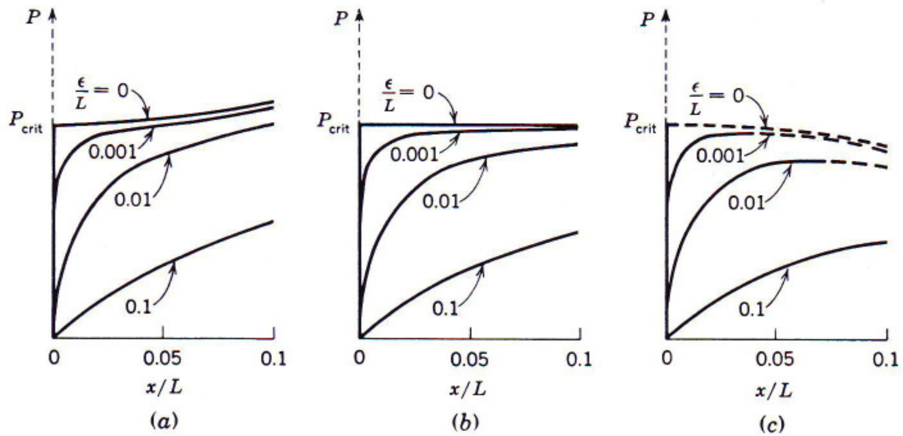


Fig. 9.19 Effect of imperfection parameter *ε/L* on postbuckling behavior for (a) β = 10, (b) β = 0, (c) β = -10.

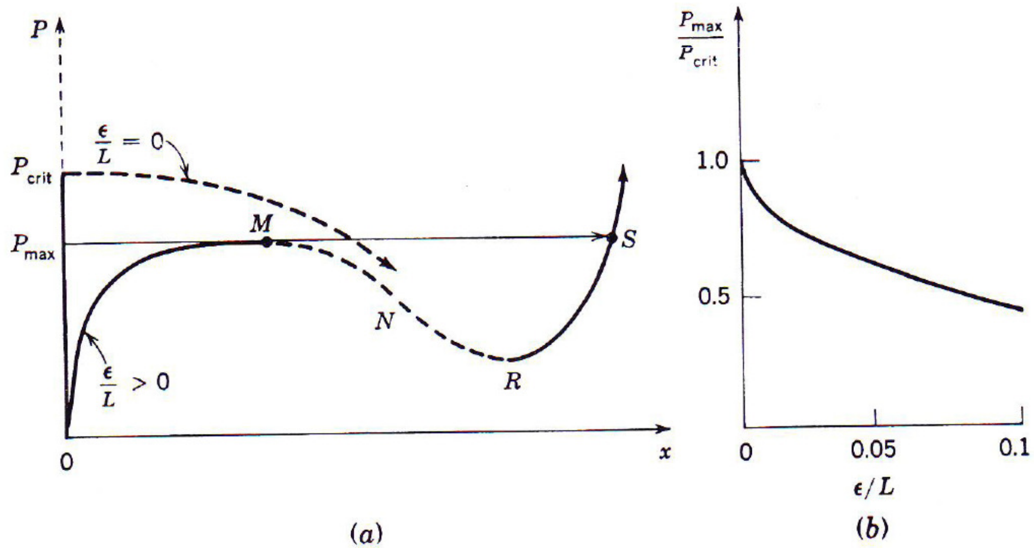


Fig. 9.20 Maximum load for softening nonlinearity (β = -10) depends on magnitude of imperfection.

Extension of Euler's Formula To Columns with Other End Conditions

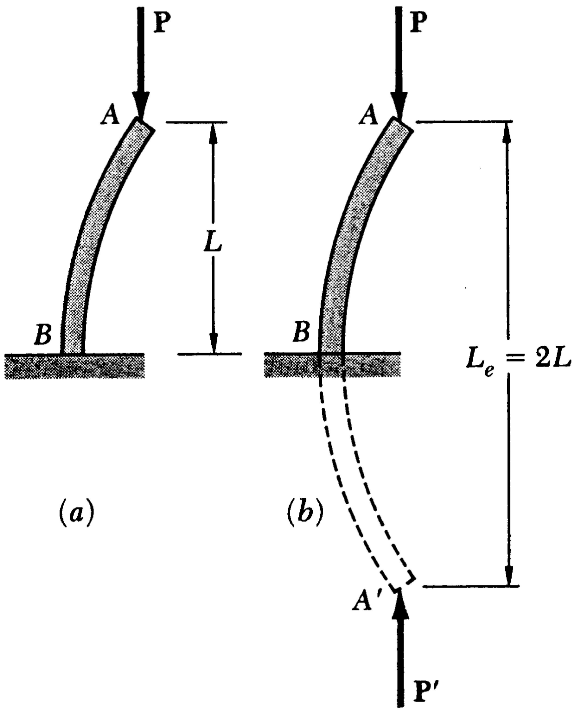


Fig. 11.9

Free end A and Fixed end B

Behaves as the upper half of a pin-connected column.

- Effective length: $L_e = 2L$
- Critical Load:

$$P_{cr} = \frac{\pi^2}{4}\frac{EI}{L} = \frac{\pi^2 EI}{L_e^2} \qquad (11\text{-}11')$$

- Critical Stress:

$$\sigma_{cr} = \frac{\pi^2 E}{(L_e / r)^2} \qquad (11\text{-}13')$$

L_e / r : Effective slenderness ratio

Extension of Euler's Formula To Columns with Other End Conditions (continued)

Two Fixed ends A and B

- The shear at C and the horizontal components of the reaction at A and B are 0.
- Restraints upon AC and CB are identical.
- Portion AC and BC: symmetric about its midpoint D and E.

  → D and E are points of inflection (M=0)
- Portion DE must behave as a pin- ended column.

  → The effective length is: $L_e = L/2$

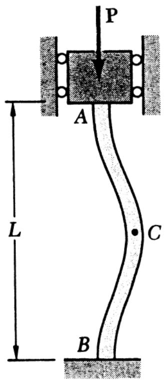


Fig. 11.10

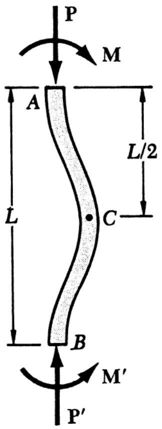


Fig. 11.11

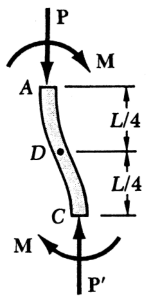


Fig. 11.12

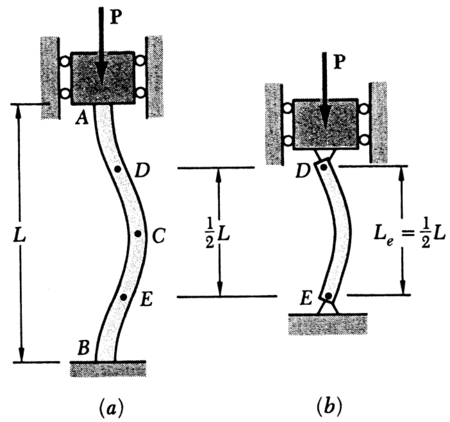


Fig. 11.13

Extension of Euler's Formula To Columns with Other End Conditions (continued)

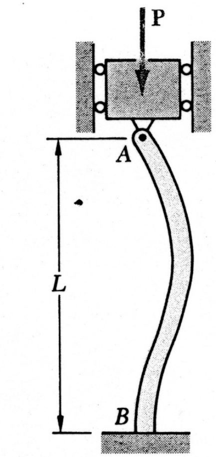


Fig. 11.14

One Pin- Connected end A and One Fixed end B

- Differential equation of the elastic curve:

Portion AQ: $M = -Py - Vx$

$$\frac{d^2 y}{dx^2} = \frac{M}{EI} = -\frac{Py}{EI} - \frac{Vx}{EI} \quad , \quad \frac{d^2 y}{dx^2} + p^2 y = -\frac{Vx}{EI}$$

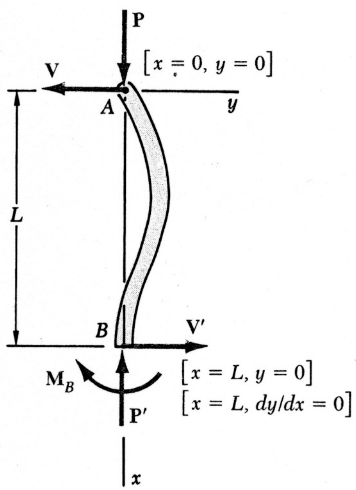


Fig. 11.15

,where $p^2 = \dfrac{P}{EI}$ (11- 6)

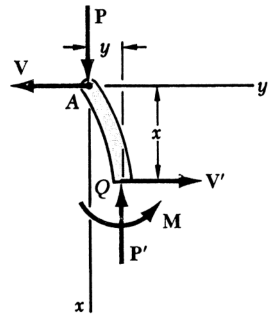


Fig. 11.16

Particular solution is: $y = -\dfrac{V}{p^2 EI}x = -\dfrac{V}{P}x$

General solution is: $y = A\sin px + B\cos px - \dfrac{V}{P}x$

Extension of Euler's Formula To Columns with Other End Conditions (continued)

One Pin- Connected end A and One Fixed end B (continued)

BC 1: $y(0) = o \quad \rightarrow \quad A\sin pL = \dfrac{V}{P}L$ (11- 17)

$$\frac{dy}{dx} = Ap\cos px - \frac{V}{P}$$

BC 2: $y(L) = o, \quad dy/dx\big|_{x=L} = o$

$$\frac{dy}{dx} = Ap\cos pL - \frac{V}{P} = 0 \quad , \quad Ap\cos pL = \frac{V}{P} \qquad (11\text{- }18)$$

(11- 17, 18): $\tan pL = pL \rightarrow \quad pL = 4.4934$ (11- 19, 20)

(11- 6): $P_{cr} = \dfrac{20.19EI}{L^2}$ (11- 21)

(11- 11, 21'): $\dfrac{\pi^2 EI}{L_e^2} = \dfrac{20.19EI}{L^2} \quad , \quad L_e = 0.699L \approx 0.7L$

Extension of Euler's Formula To Columns with Other End Conditions (continued)

Effective length of column for various end conditions

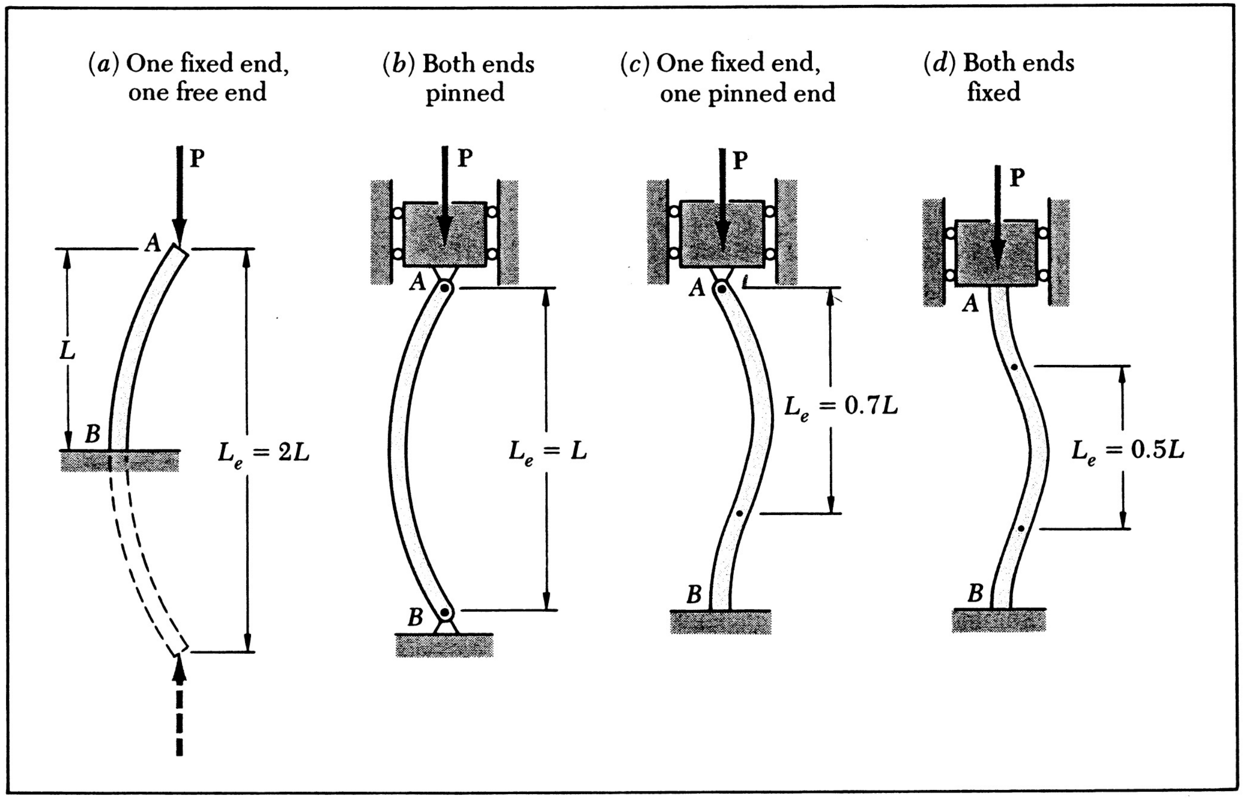
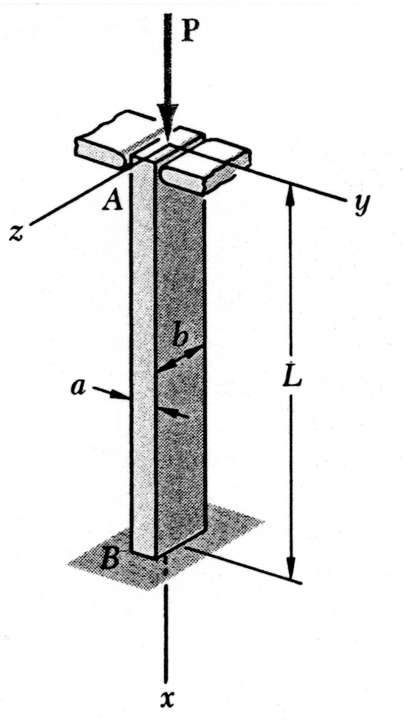


Fig. 11.17 Effective length of column for various end conditions.

Extension of Euler's Formula To Columns with Other End Conditions (continued)

Sample Problem 11.1



An aluminum column of length L and rectangular cross section has a fixed end B and supports a centric load at A. Two smooth and rounded fixed plates restrain end A from moving in one of the vertical planes of symmetry of the column, but allow it to move in the other plane. (a) Determine the ratio a/b of the two sides of the cross section corresponding to the most efficient design against buckling. (b) Design the most efficient cross section for the column, knowing that L=500 mm, E=70 GPa, P=20 kN, and that a factor safety of 2.5 is required.

Extension of Euler's Formula To Columns with Other End Conditions (continued)

SAMPLE PROBLEM 11.1

Buckling in x, y plane

Effective length with respect to buckling in this plane: $L_e = 0.7L$

Radius of gyration: $r_z = I_z / A = (1/12)ba^3 / ab = a/\sqrt{12}$

Effective slenderness ratio: $L_e / r_z = (0.7L)/(a/\sqrt{12})$ (1)

Buckling in x, z plane

Effective length with respect to buckling in this plane: $L_e = 2L$

Radius of gyration: $r_y = I_y / A = (1/12)ab^3 / ab = b/\sqrt{12}$

Effective slenderness ratio: $L_e / r_y = (2L)/(b/\sqrt{12})$ (2)

(a) Most effective design.

→The critical stresses corresponding to the possible modes of buckling are equal.

$$\sigma_{cr} = \frac{\pi^2 E}{(L_e/r)^2} \quad \rightarrow \quad \frac{0.7L}{a/\sqrt{12}} = \frac{2L}{b/\sqrt{12}}; \qquad \frac{a}{b} = 0.35$$

Extension of Euler's Formula To Columns with Other End Conditions (continued)

Sample Problem11.1

(b) Design for given data.

$$P_{cr} = (F.S.)P = (2.5)(20\text{kN}) = 50\text{kN}$$

$$\sigma_{cr} = \frac{P_{cr}}{A} = \frac{50 \times 10^3\,\text{N}}{0.35b^2} \qquad (A = ab = (0.35b)b)$$

$L = 0.5\text{m}$; eqn(2) → $L_e / r_y = 3.464 / b$

$$\sigma_{cr} = \frac{\pi^2 E}{(L_e / r)^2} = \frac{50 \times 10^3\,\text{N}}{0.35b^2} = \frac{\pi^2 (70 \times 10^9\,\text{Pa})}{(3.464 / b)^2}$$

$$b = 39.7\text{mm} \qquad a = 0.35b = 13.9\text{mm}$$

Eccentric Loading: The Secant Formula

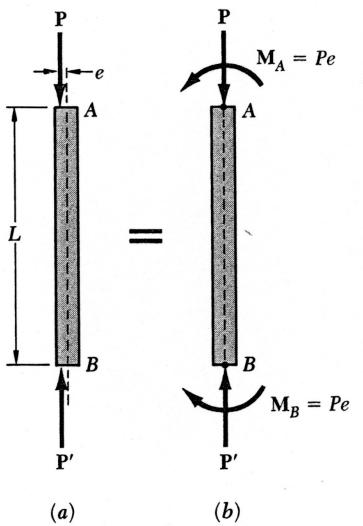


Fig. 11.18

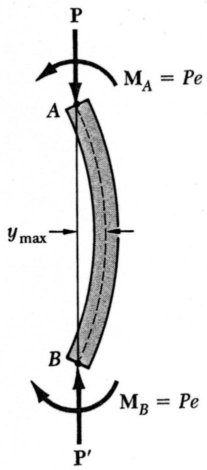


Fig. 11.19

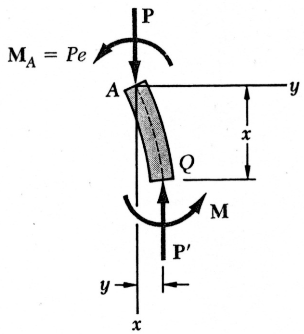


Fig. 11.20

Portion AQ:

Bending moment at Q is

$$M = -Py - M_A = -Py - Pe \qquad (11\text{-}22)$$

$$\frac{d^2y}{dx^2} = \frac{M}{EI} = -\frac{P}{EI}y - \frac{Pe}{EI}$$

$$\frac{d^2y}{dx^2} + p^2 y = -p^2 e \qquad (11\text{-}23)$$

where, $p^2 = \dfrac{P}{EI}$

General solution of (11- 23):

$$y = A\sin px + B\cos px - e \qquad (11\text{-}24)$$

Eccentric Loading: The Secant Formula (continued)

Boundary conditions:

$$y(0) = 0 \qquad B = e$$

$$y(L) = 0 \qquad A \sin pL = e(1 - \cos pL) \qquad (11\text{-}25)$$

$$A = e \tan \frac{pL}{2}$$

(since $\sin pL = 2 \sin \frac{pL}{2} \cos \frac{pL}{2}$ and $1 - \cos pL = 2 \sin^2 \frac{pL}{2}$)

$$y = e\left(\tan \frac{pL}{2} \sin px + \cos px - 1 \right) \qquad (11\text{-}26)$$

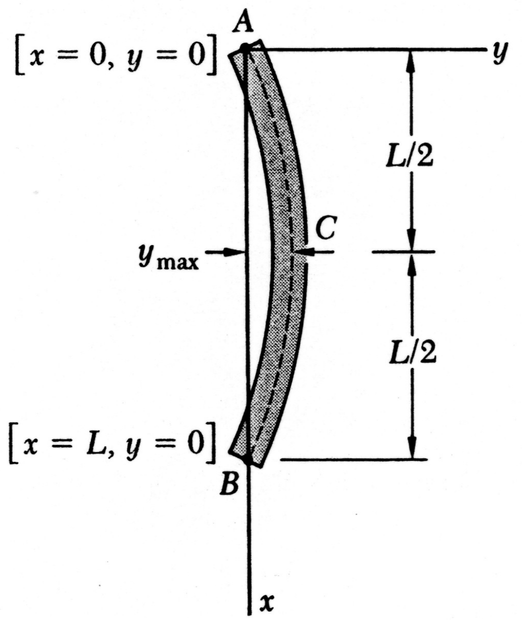


Fig. 11.21

Eccentric Loading: The Secant Formula (continued)

The value of the maximum deflection is obtained by setting $x = L/2$.

$$y_{max} = e\left(\tan\frac{pL}{2}\sin\frac{pL}{2}+\cos\frac{pL}{2}-1\right)$$

$$= e\left(\frac{\tan\frac{pL}{2}\cos^2\frac{pL}{2}}{\cos\frac{pL}{2}}-1\right)$$

$$y_{max} = e\left(\sec\frac{pL}{2}-1\right) \qquad (11\text{-}27)$$

$$y_{max} = e\left[\sec\left(\sqrt{\frac{P}{EI}}\frac{L}{2}\right)-1\right] \qquad \left(p^2 = \frac{P}{EI}\right) \qquad (11\text{-}28)$$

Eccentric Loading: The Secant Formula (continued)

y_{max} becomes infinite when

$$\sqrt{\frac{P}{EI}}\frac{L}{2} = \frac{\pi}{2} \qquad (11\text{-}29)$$

While the deflection does not actually become infinite, and P should not be allowed to reach the critical value which satisfies (11- 29).

$$P_{cr} = \frac{\pi^2 EI}{L^2} \qquad (11\text{-}30)$$

Solving (11- 30) for EI and substituting into (11- 28),

$$y_{max} = e\left[\sec\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right) - 1\right] \qquad (11\text{-}31)$$

Eccentric Loading: The Secant Formula (continued)

The maximum stress: $$\sigma_{max} = \frac{P}{A} + \frac{M_{max}c}{I} \qquad (11\text{-}32)$$

Portion AC: $M_{max} = Py_{max} + M_A = P(y_{max} + e)$

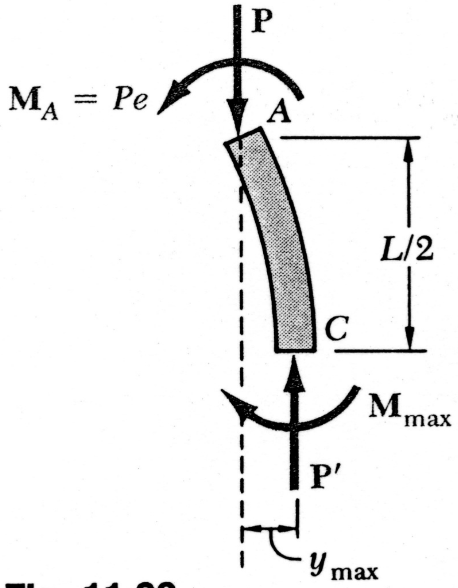


Fig. 11.22

$$\sigma_{max} = \frac{P}{A} + \frac{(y_{max}+e)c}{I} = \frac{P}{A}\left[1 + \frac{(y_{max}+e)c}{r^2}\right] \qquad (11\text{-}33)$$

Substituting y_{max}

$$\sigma_{max} = \frac{P}{A}\left[1 + \frac{ec}{r^2}\sec\left(\sqrt{\frac{P}{EI}}\frac{L}{2}\right)\right] \qquad (11\text{-}34)$$

$$= \frac{P}{A}\left[1 + \frac{ec}{r^2}\sec\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right)\right] \qquad (11\text{-}35)$$

Eccentric Loading: The Secant Formula (continued)

Since the maximum stress does not vary linearly with the load P, the principle of superposition does not apply to the determination of the stress due to the simultaneous application of several loads; the resultant load must first be computed, and (11- 34) or (11- 35) may be used to determine the corresponding stress. For the same reason, any given factor of safety should be applied to the load, and not to the stress.

The Secant Formula

(11- 34): Making $I = Ar^2$

$$\frac{P}{A} = \frac{\sigma_{max}}{1 + \frac{ec}{r^2}\sec\left(\frac{1}{2}\sqrt{\frac{P}{EA}}\frac{L_e}{r}\right)} \qquad (11\text{- }36)$$

Eccentric Loading: The Secant Formula (continued)

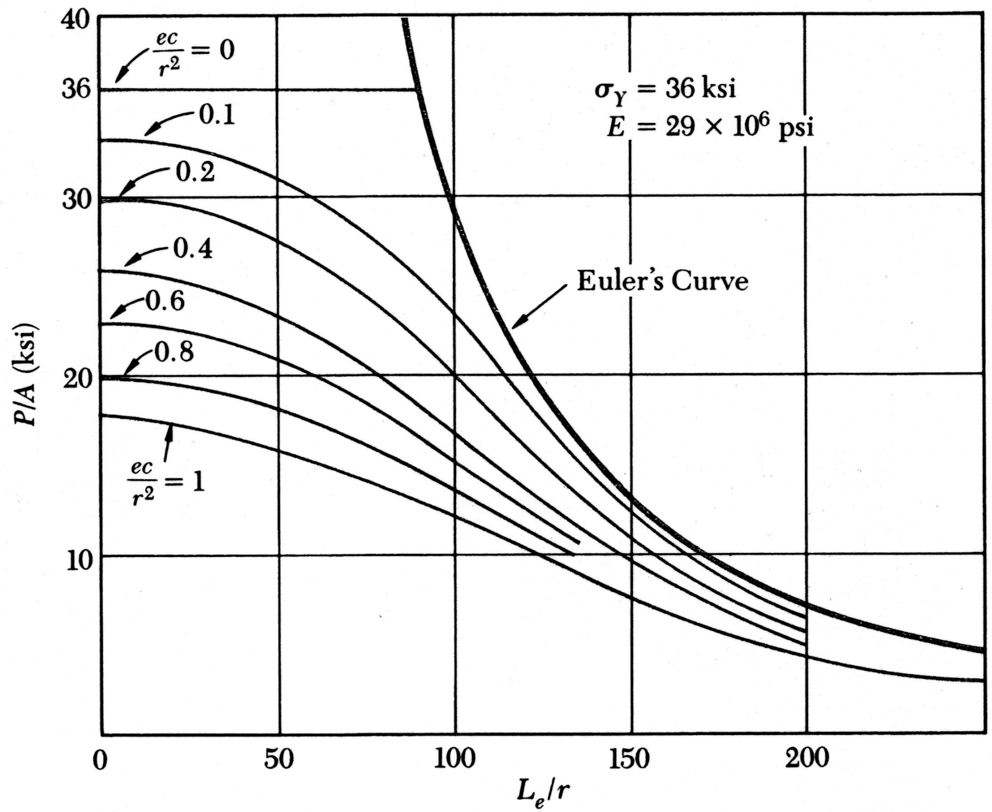


Fig. 11.23 Load per unit area, P/A, causing yield in column.

For a steel column $E = 29 \times 10^6$ psi $\sigma_Y = 36$ ksi

Eccentric Loading: The Secant Formula (continued)

For all small value of L_e / r^2, the secant is almost equal to 1:

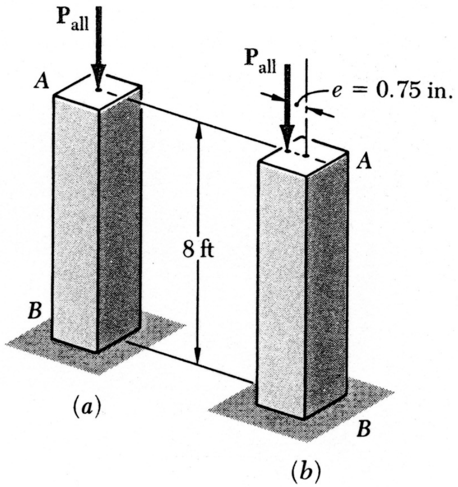
$$\frac{P}{A} = \frac{\sigma_{\max}}{1 + \frac{ec}{r^2}} \qquad (11\text{-}37)$$

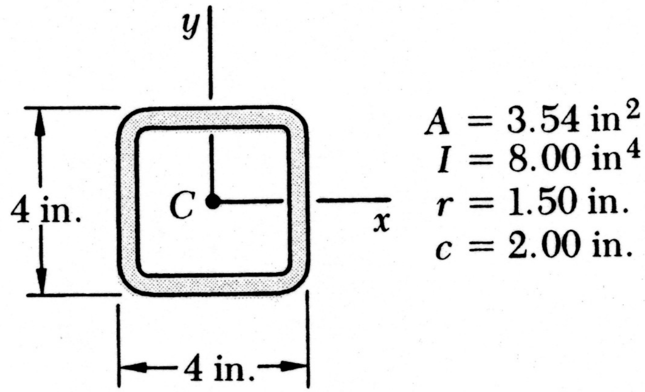
For large values of L_e / r^2, the curves corresponding to the various values of the ratio ec / r^2 get very close to Euler's curve defined by (11.13'), and thus that the effect of the eccentricity of the loading on the value of P / A becomes negligible.

Eccentric Loading: The Secant Formula (continued)

Sample Problem 11.2

The uniform column AB consists of an 8- ft section of structural tubing having the cross section shown. (a) Using Euler's formula and a factor of safety of two, determine the allowable centric load for the column and the corresponding normal stress. (b) Assuming that the allowable load, found in part a, is applied as shown at a point 0.75 in. from the geometric axis of he column, determine the horizontal deflection of the top of the column and the maximum normal stress in the column. Use $E = 29 \times 10^6$ psi.





Eccentric Loading: The Secant Formula (continued)

Effective Length

One end fixed and one end free: $L_e = 2(8\,\text{ft}) = 16\,\text{ft} = 192\,\text{in.}$

Critical Load

Using Euler's formula,

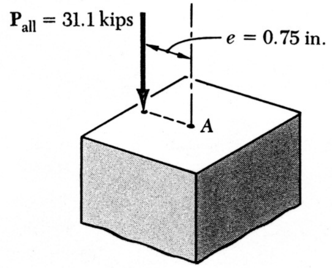
$$P_{cr} = \frac{\pi^2 EI}{L_e^2} = \frac{\pi^2 (29 \times 10^6\,\text{psi})(8.00\,\text{in}^4)}{192^2} = 62.1\,\text{ksi}$$

(a) Allowable Load and Stress

For a factor of safety of 2: $P_{all} = \frac{P_{cr}}{F.S.} = \frac{62.1\,\text{ksi}}{2} = 31.1\,\text{kips}$

$$\sigma = \frac{P_{all}}{A} = \frac{31.1\,\text{ksi}}{3.54\,\text{in}^2} = 8.79\,\text{ksi}$$

Eccentric Loading: The Secant Formula (continued)



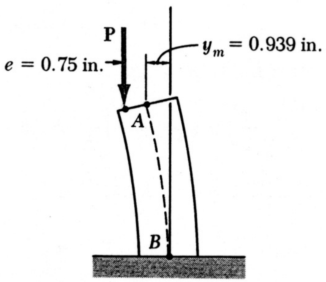


Fig. 1

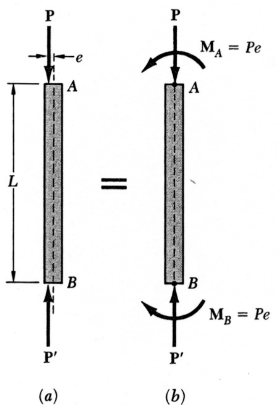


Fig. 2

(b) Eccentric Load.

Column AB (Fig. 1) and its loading are identical to the upper half of the upper half of the Fig. 2.

Horizontal deflection of point A:

$$y_{max} = e\left[\sec\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right) - 1\right] = (0.75\text{in})\left[\sec\left(\frac{\pi}{2\sqrt{2}}\right) - 1\right]$$

$$= 0.939\text{in}$$

Maximum normal stress:

$$\sigma_{max} = \frac{P}{A}\left[1 + \frac{ec}{r^2}\sec\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right)\right] = \frac{31.1\text{kips}}{3.54\text{in}^2}\left(1 + \frac{(0.75\text{in})(2\text{in})}{(1.50\text{in})^2}\sec\frac{\pi}{2\sqrt{2}}\right)$$

$$= 22.0\text{ksi}$$