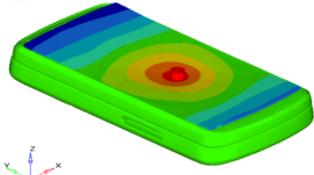
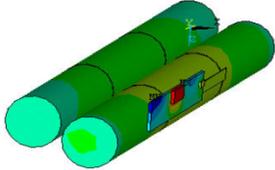
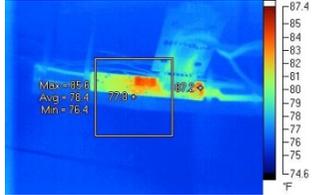


Introduction

- The *finite element method* (FEM) is a technique for analyzing the behavior of engineered structures subjected to a variety of loads. FEM is the most widely applied computer simulation method in engineering.
- The basic idea is to divide a complicated structure into small and manageable pieces (*discretization*) and solve the algebraic equation.

Applications of FEM in Engineering				
Simulation				
Experiment				
Engineered system	Vehicle	Tire	Cellular phone	Li-ion battery system
Predicted measure	Safety	Cornering force	Reliability	Thermal behavior

Available Commercial FEM Software

- ANSYS (General purpose)
- I-DEAS (Complete CAD/CAM/CAE package)
- HyperMesh (Pre/Post Processor)
- ABAQUS (Nonlinear and dynamic analyses)
- PATRAN (Pre/Post Processor)
- NASTRAN (General purpose)
- COSMOS (General purpose)
- Dyna-3D (Crash/impact analysis)

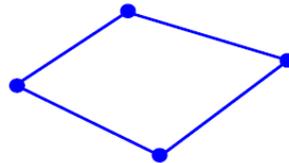
Types of Finite Elements

1-D (Line) Element



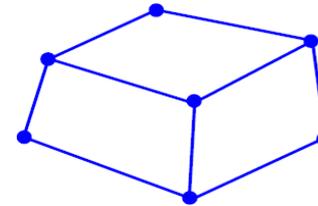
(Spring, truss, beam, pipe, etc.)

2-D (Plane) Element



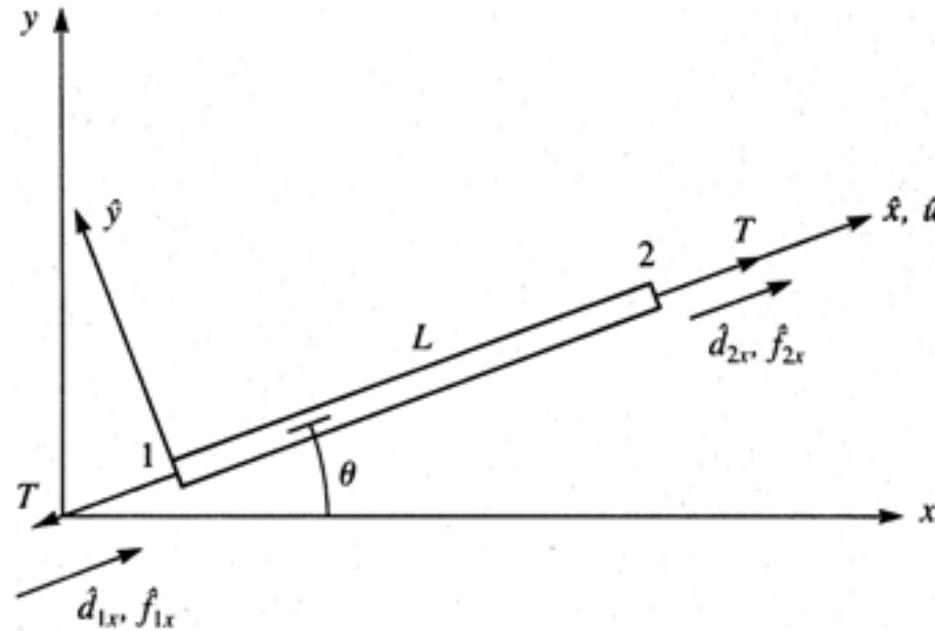
(Membrane, plate, shell, etc.)

3-D (Solid) Element



(3-D fields – temperature, displacement, stress, flow velocity, etc.)

Truss Element in 2D Space



x, y : global coordination system

\hat{u} : deformation

\hat{x}, \hat{y} : local coordination system

L : length

\hat{d} : displacement

\hat{f} : force

Linear Static Analysis

Most structural analysis problems can be **treated as linear static problems, based on the following assumptions.**

1. Small deformations (loading pattern is not changed due to the deformed shape)
2. Static loads (the load is applied to the structure in slow or steady fashion)
3. Elastic materials (no plasticity or failures)

Hooke's Law and Deformation Equation

$$\sigma = E\varepsilon \quad \varepsilon = \frac{d\hat{u}}{d\hat{x}}$$

Equilibrium

$$A\sigma_x = T = \text{Constant} \quad \frac{d}{d\hat{x}} \left(AE \frac{d\hat{u}}{d\hat{x}} \right) = 0$$

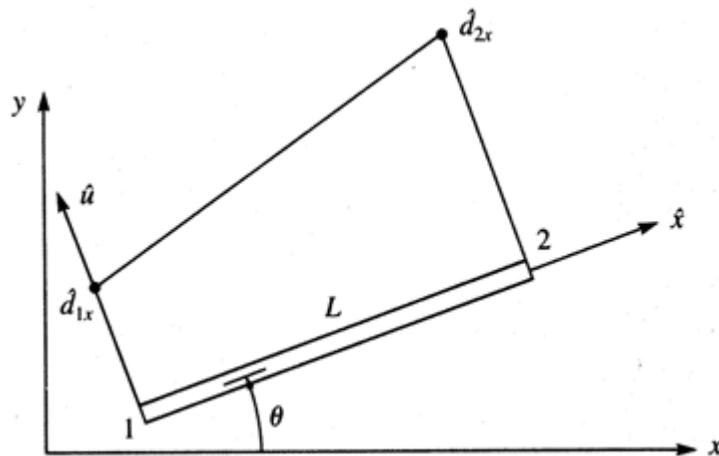
Assumptions

1. Truss cannot support shear force. $\hat{f}_{1y} = 0 \quad \hat{f}_{2y} = 0$
2. Ignore the effect of lateral deformation

Direct Stiffness Method

DSM is an approach to **calculate a stiffness matrix for a system by directly superposing the stiffness matrixes** of all elements. DSM is beneficial to get the stiffness matrix of relatively simple structures consist of several trusses or beams.

Step 1: Determination of element type



Step 2: Determination of deformation

Assume a linear deformation function as

$$\hat{u} = a_1 + a_2 \hat{x} = \left(\frac{\hat{d}_{2x} - \hat{d}_{1x}}{L} \right) \hat{x} + \hat{d}_{1x}$$

In vector form

$$\hat{u} = \begin{bmatrix} N_1 & N_2 \end{bmatrix} \begin{Bmatrix} \hat{d}_{1x} \\ \hat{d}_{2x} \end{Bmatrix}$$

$$N_1 = 1 - \frac{\hat{x}}{L} \quad N_2 = \frac{\hat{x}}{L}$$

N_1, N_2 : shape function

Finite Element Method: Truss Element

Step 3: Strain and stress calculation

Deformation rate - strain

$$\varepsilon_x = \frac{d\hat{u}}{d\hat{x}} = \frac{\hat{d}_{2x} - \hat{d}_{1x}}{L}$$

Stress - strain

$$\sigma_x = E\varepsilon_x$$

Step 4: Derivation of element stiffness matrix

$$T = A\sigma_x = AE \left(\frac{\hat{d}_{2x} - \hat{d}_{1x}}{L} \right)$$

$$\hat{f}_{1x} = -T = \frac{AE}{L} (\hat{d}_{1x} - \hat{d}_{2x})$$

$$\hat{f}_{2x} = T = \frac{AE}{L} (\hat{d}_{2x} - \hat{d}_{1x})$$

$$\begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{2x} \end{Bmatrix} = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} \hat{d}_{1x} \\ \hat{d}_{2x} \end{Bmatrix}$$

$$\hat{k} = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

NOTE: In a truss element, stiffness (spring constant, k) is equivalent to AE/L

Finite Element Method: Truss Element

Step 5: Constitution of global stiffness matrix

Construct a global stiffness matrix in a global coordinate system

$$\underline{K} = [K] = \sum_{e=1}^N \underline{k}^{(e)}$$

$$\underline{F} = \{F\} = \sum_{e=1}^N \underline{f}^{(e)}$$

Step 6: Calculation of nodal displacement

- Use boundary conditions
- Solve the system of linear algebraic equations $\underline{F} = \underline{K} \underline{d}$ to calculate the nodal deformation in a truss structure

Step 7: Calculation of stress and strain in an element

Compute a strain and stress at any point within an element

Finite Element Method: Truss Element

Example (Truss element in 1D space)

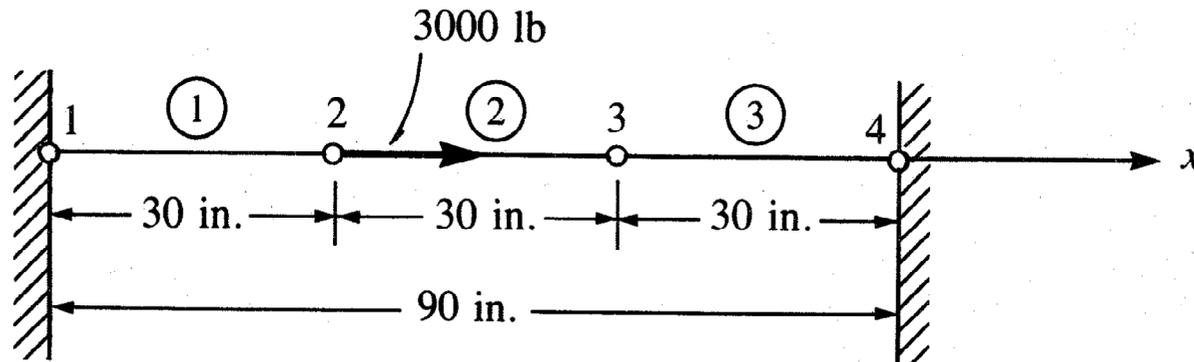
A structure consists of three beams (see below figure). Find (a) the global stiffness matrix in global coordinate system, (b) the displacements at the node 2 and 3, (c) the reaction forces at the node 2 and 3. 3000lb loading acts in the positive x-direction at node 2. The length of the elements is 30 in.

Young's modulus for the elements 1 and 2: $E = 30 \times 10^6$ psi

Cross-sectional area for the elements 1 and 2: $A = 1 \text{ in}^2$

Young's modulus for the elements 3: $E = 15 \times 10^6$ psi

Cross-sectional area for the elements 3: $A = 2 \text{ in}^2$



Finite Element Method: Truss Element

Step 4: Derivation of element stiffness matrix

$$\hat{k} = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

element #

$$\underline{k}^{(1)} = \underline{k}^{(2)} = \frac{(1)(30 \times 10^6)}{30} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 10^6 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\underline{k}^{(3)} = \frac{(2)(15 \times 10^6)}{30} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 10^6 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

node #

Step 5: Constitution of global stiffness matrix

$$\underline{K} = 10^6 \begin{bmatrix} d_{1x} & d_{2x} & d_{3x} & d_{4x} \\ 1 & -1 & 0 & 0 \\ -1 & 1+1 & -1 & 0 \\ 0 & -1 & 1+1 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

Answer of (a)

$$\begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \\ F_{4x} \end{Bmatrix} = 10^6 \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} d_{1x} \\ d_{2x} \\ d_{3x} \\ d_{4x} \end{Bmatrix}$$

Finite Element Method: Truss Element

Step 6: Calculation of nodal displacement

Using the boundary conditions $d_{1x} = d_{4x} = 0$

$$\begin{Bmatrix} 3000 \\ 0 \end{Bmatrix} = 10^6 \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{Bmatrix} d_{2x} \\ d_{3x} \end{Bmatrix}$$

So, the displacements for this system are

$$d_{2x} = 0.002 \text{ in.} \quad d_{3x} = 0.001 \text{ in.} \quad \text{Answer of (b)}$$

Substituting the displacements to the equation, the load can be obtained as follows.

$$F_{1x} = 10^6(d_{1x} - d_{2x}) = 10^6(0 - 0.002) = -2000lb \quad \text{Answer of (c)}$$

$$F_{2x} = 10^6(-d_{1x} + 2d_{2x} - d_{3x}) = 10^6[0 + 2(0.002) - 0.001] = 3000lb$$

$$F_{3x} = 10^6(-d_{2x} + 2d_{3x} - d_{4x}) = 10^6[-0.002 + 2(0.001) - 0] = 0$$

$$F_{4x} = 10^6(-d_{3x} + d_{4x}) = 10^6(-0.001 + 0) = -1000lb$$

Step 7: Calculation of stress and strain in an element

$$\sigma_x = \frac{F}{A}$$

Selection of Deformation Function

1. The deformation function needs to be continuous within an element.
2. The deformation function needs to provide continuity among elements.

1-D linear deformation function satisfies (1) and (2). \rightarrow *compatibility*

3. The deformation function needs to express the displacement of a rigid body and the constant strain in element.

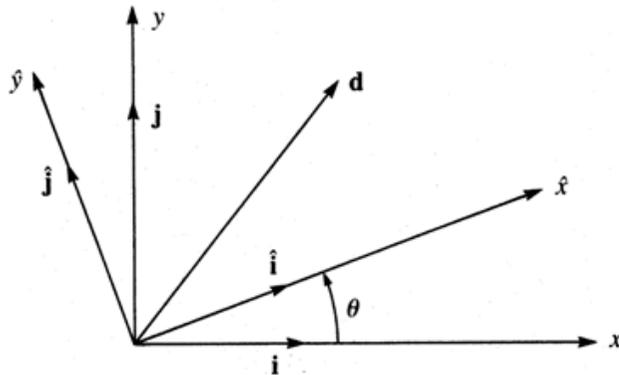
$\hat{u} = a_1 + a_2 \hat{x}$: 1-D linear deformation function \rightarrow *completeness*

a_1 considers the motion of a rigid body.

$a_2 \hat{x}$ considers the constant strain ($\varepsilon_x = d\hat{u}/d\hat{x} = a_2$)

Truss Element in 2D Space

Vector transformation in 2-D



Displacement vector \mathbf{d} in global coordinate and local coordinate systems.

$$\mathbf{d} = \underbrace{d_x \mathbf{i} + d_y \mathbf{j}}_{Global} = \underbrace{\hat{d}_x \hat{\mathbf{i}} + \hat{d}_y \hat{\mathbf{j}}}_{Local}$$

Relation between two coordination systems

$$\begin{Bmatrix} \hat{d}_x \\ \hat{d}_y \end{Bmatrix} = \begin{bmatrix} C & S \\ -S & C \end{bmatrix} \begin{Bmatrix} d_x \\ d_y \end{Bmatrix} \quad C = \cos \theta \quad S = \sin \theta \quad \text{Eq. (1)}$$

$$\begin{bmatrix} C & S \\ -S & C \end{bmatrix} : \text{Transformation Matrix}$$

Finite Element Method: Truss Element

Stiffness matrix in global coordinate system

The stiffness matrix of truss element in local coordinate system

$$\begin{Bmatrix} \hat{f}_1 \\ \hat{f}_2 \end{Bmatrix} = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} \hat{d}_{1x} \\ \hat{d}_{2x} \end{Bmatrix} \quad \underline{\hat{f}} = \underline{\hat{k}} \underline{\hat{d}} \quad : \text{Step 4 on page 6}$$

The stiffness matrix of truss element in global coordinate system

$$\begin{Bmatrix} f_{1x} \\ f_{1y} \\ f_{2x} \\ f_{2y} \end{Bmatrix} = \underline{k} \begin{Bmatrix} d_{1x} \\ d_{1y} \\ d_{2x} \\ d_{2y} \end{Bmatrix} \quad \underline{f} = \underline{k} \underline{d} \quad : \text{What we want to construct in 2-D space}$$

Let's calculate the relation between $\underline{\hat{k}}$ and \underline{k} .

Finite Element Method: Truss Element

From Eq.(1),

$$\begin{aligned} \hat{d}_{1x} &= d_{1x} \cos \theta + d_{1y} \sin \theta \\ \hat{d}_{2x} &= d_{2x} \cos \theta + d_{2y} \sin \theta \end{aligned} \quad \left\{ \begin{array}{c} \hat{d}_{1x} \\ \hat{d}_{2x} \end{array} \right\} = \begin{bmatrix} C & S & 0 & 0 \\ 0 & 0 & C & S \end{bmatrix} \left\{ \begin{array}{c} d_{1x} \\ d_{1y} \\ d_{2x} \\ d_{2y} \end{array} \right\}$$

Simply,

$$\boxed{\underline{\hat{d}} = \underline{T}^* \underline{d}} \quad \text{Eq. (2)} \quad \underline{T}^* = \begin{bmatrix} C & S & 0 & 0 \\ 0 & 0 & C & S \end{bmatrix}$$

Transform the loading in a same way,

$$\left\{ \begin{array}{c} \hat{f}_{1x} \\ \hat{f}_{2x} \end{array} \right\} = \begin{bmatrix} C & S & 0 & 0 \\ 0 & 0 & C & S \end{bmatrix} \left\{ \begin{array}{c} f_{1x} \\ f_{1y} \\ f_{2x} \\ f_{2y} \end{array} \right\}$$

Simply,

$$\boxed{\underline{\hat{f}} = \underline{T}^* \underline{f}} \quad \text{Eq. (3)}$$

Finite Element Method: Truss Element

From Eq.(2) and (3)

$$\underline{\hat{f}} = \underline{\hat{k}} \underline{\hat{d}} \quad \text{where,} \quad \underline{T}^* = \begin{bmatrix} C & S & 0 & 0 \\ 0 & 0 & C & S \end{bmatrix}$$
$$\underline{T}^* \underline{f} = \underline{\hat{k}} \underline{T}^* \underline{d}$$

We need **the inverse matrix of T^*** ; however, because T^* is not the **square matrix**, it cannot be immediately transformed.

Write $(\hat{f}_{1y}, \hat{d}_{1y})$ and $(\hat{f}_{2y}, \hat{d}_{2y})$

$$\begin{Bmatrix} \hat{d}_{1x} \\ \hat{d}_{1y} \\ \hat{d}_{2x} \\ \hat{d}_{2y} \end{Bmatrix} = \begin{bmatrix} C & S & 0 & 0 \\ -S & C & 0 & 0 \\ 0 & 0 & C & S \\ 0 & 0 & -S & C \end{bmatrix} \begin{Bmatrix} d_{1x} \\ d_{1y} \\ d_{2x} \\ d_{2y} \end{Bmatrix}$$

$$\underline{\hat{d}} = \underline{T} \underline{d}$$

Transformation Matrix

In a same way,

$$\underline{\hat{f}} = \underline{T} \underline{f}$$

Finite Element Method: Truss Element

Expanded 4×4 matrix \hat{k}

$$\begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{1y} \\ \hat{f}_{2x} \\ \hat{f}_{2y} \end{Bmatrix} = \frac{AE}{L} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \hat{d}_{1x} \\ \hat{d}_{1y} \\ \hat{d}_{2x} \\ \hat{d}_{2y} \end{Bmatrix}$$

The stiffness matrix of truss elements in global coordinate system

$$\underline{T} \underline{f} = \hat{k} \underline{T} \underline{d} \qquad \underline{f} = \underline{T}^{-1} \hat{k} \underline{T} \underline{d} = \underline{T}^T \hat{k} \underline{T} \underline{d}$$

T is an orthogonal matrix ($T^T = T^{-1}$).

$$\underline{k} = \underline{T}^T \hat{k} \underline{T}$$

Finite Element Method: Truss Element

So,

$$\underline{k} = \frac{AE}{L} \begin{bmatrix} C^2 & CS & -C^2 & -CS \\ & S^2 & -CS & -S^2 \\ & & C^2 & CS \\ \text{Symmetry} & & & S^2 \end{bmatrix} \quad \text{Eq. (4)}$$

$$\underline{k} = \underline{T}^T \hat{\underline{k}} \underline{T}$$

$$\underline{T} = \begin{bmatrix} C & S & 0 & 0 \\ -S & C & 0 & 0 \\ 0 & 0 & C & S \\ 0 & 0 & -S & C \end{bmatrix}$$

$$\hat{\underline{k}} = \frac{AE}{L} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Assemble the stiffness matrix for the whole system

$$\sum_{e=1}^N \underline{k}^{(e)} = \underline{K} \quad \sum_{e=1}^N \underline{f}^{(e)} = \underline{F}$$

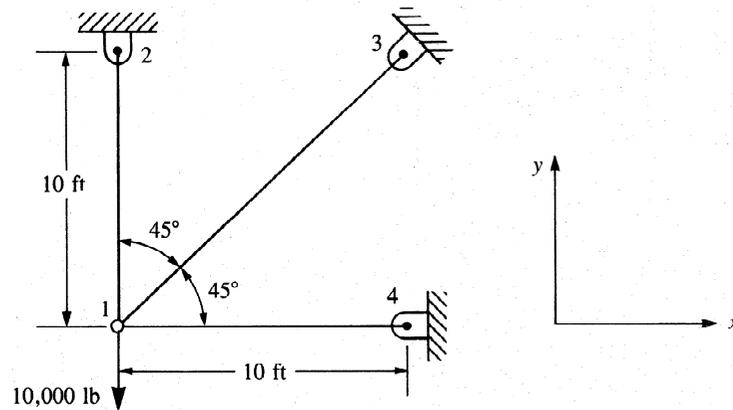
The relation between the nodal loading and displacement in global coordinate system

$$\underline{F} = \underline{K} \underline{d}$$

Finite Element Method: Truss Element

Example (Truss element in 2D space)

Three truss elements are assembled and 10,000lb loading is applied at node 1 as shown in figure. Find the displacements at node 1. For all elements, Young's modulus: $E = 30 \times 10^6 \text{ psi}$, cross-sectional area: $A = 2 \text{ in.}^2$.



Finite Element Method: Truss Element

Data for stiffness matrix calculation

Node	Element	θ	C	S	C^2	S^2	CS
1 and 2	1	90°	0	1	0	1	0
1 and 3	2	45°	$\sqrt{2}/2$	$\sqrt{2}/2$	1/2	1/2	1/2
1 and 4	3	0°	1	0	1	0	0

Step 4: Derivation of element stiffness matrix (from Eq. (4))

$$\underline{k}^{(1)} = \frac{(30 \times 10^6)(2)}{120} \begin{bmatrix} d_{1x} & d_{1y} & d_{2x} & d_{2y} \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

$$\underline{k}^{(2)} = \frac{(30 \times 10^6)(2)}{120 \times \sqrt{2}} \begin{bmatrix} d_{1x} & d_{1y} & d_{3x} & d_{3y} \\ 0.5 & 0.5 & -0.5 & -0.5 \\ 0.5 & 0.5 & -0.5 & -0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \end{bmatrix}$$

$$\underline{k}^{(3)} = \frac{(30 \times 10^6)(2)}{120} \begin{bmatrix} d_{1x} & d_{1y} & d_{4x} & d_{4y} \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Finite Element Method: Truss Element

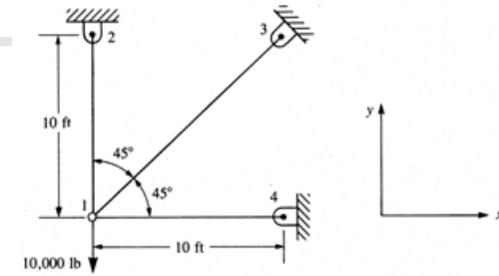
Step 5: Constitution of global stiffness matrix

$$\underline{K} = (500,000) \begin{bmatrix} d_{1x} & d_{1y} & d_{2x} & d_{2y} & d_{3x} & d_{3y} & d_{4x} & d_{4y} \\ 1.354 & 0.354 & 0 & 0 & -0.354 & -0.354 & -1 & 0 \\ 0.354 & 1.354 & 0 & -1 & -0.354 & -0.354 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ -0.354 & -0.354 & 0 & 0 & 0.354 & 0.354 & 0 & 0 \\ -0.354 & -0.354 & 0 & 0 & 0.354 & 0.354 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Finite Element Method: Truss Element

Step 6: Calculation of nodal displacement

Using boundary condition at node 2,3, and 4.



$$\begin{Bmatrix} 0 \\ -10,000 \\ F_{2x} \\ F_{2y} \\ F_{3x} \\ F_{3y} \\ F_{4x} \\ F_{4y} \end{Bmatrix} = (500,000) \begin{bmatrix} 1.354 & 0.354 & 0 & 0 & -0.354 & -0.354 & -1 & 0 \\ 0.354 & 1.354 & 0 & -1 & -0.354 & -0.354 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ -0.354 & -0.354 & 0 & 0 & 0.354 & 0.354 & 0 & 0 \\ -0.354 & -0.354 & 0 & 0 & 0.354 & 0.354 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \times \begin{Bmatrix} d_{1x} \\ d_{1y} \\ d_{2x} = 0 \\ d_{2y} = 0 \\ d_{3x} = 0 \\ d_{3y} = 0 \\ d_{4x} = 0 \\ d_{4y} = 0 \end{Bmatrix}$$

Calculate unknown displacement

$$\begin{Bmatrix} 0 \\ -10,000 \end{Bmatrix} = (500,000) \begin{bmatrix} 1.354 & 0.354 \\ 0.354 & 1.354 \end{bmatrix} \begin{Bmatrix} d_{1x} \\ d_{1y} \end{Bmatrix}$$

$$d_{1x} = 0.414 \times 10^{-2} \text{ in.} \quad d_{1y} = -1.59 \times 10^{-2} \text{ in.}$$

Finite Element Method: Truss Element

Step 7: Calculation of stress and strain in an element ($\sigma = \frac{F}{A}$)

$$\sigma^{(1)} = \frac{30 \times 10^6}{120} [0 \quad -1 \quad 0 \quad 1] \left\{ \begin{array}{l} d_{1x} = 0.414 \times 10^{-2} \\ d_{1y} = -1.59 \times 10^{-2} \\ d_{2x} = 0 \\ d_{2y} = 0 \end{array} \right\} = 3965 \text{ psi}$$

$$\sigma^{(1)} = \frac{F_{2y}}{A}$$

$$\sigma^{(2)} = \frac{30 \times 10^6}{120\sqrt{2}} \left[\frac{-\sqrt{2}}{2} \quad \frac{-\sqrt{2}}{2} \quad \frac{\sqrt{2}}{2} \quad \frac{\sqrt{2}}{2} \right] \left\{ \begin{array}{l} d_{1x} = 0.414 \times 10^{-2} \\ d_{1y} = -1.59 \times 10^{-2} \\ d_{3x} = 0 \\ d_{3y} = 0 \end{array} \right\}$$

$$\sigma^{(2)} = \frac{\sqrt{2}F_{3x}}{A}$$

$$= 1471 \text{ psi}$$

and

$$\sigma^{(3)} = \frac{30 \times 10^6}{120} [-1 \quad 0 \quad 1 \quad 0] \left\{ \begin{array}{l} d_{1x} = 0.414 \times 10^{-2} \\ d_{1y} = -1.59 \times 10^{-2} \\ d_{4x} = 0 \\ d_{4y} = 0 \end{array} \right\} = -1035 \text{ psi}$$

$$\sigma^{(3)} = \frac{F_{4x}}{A}$$

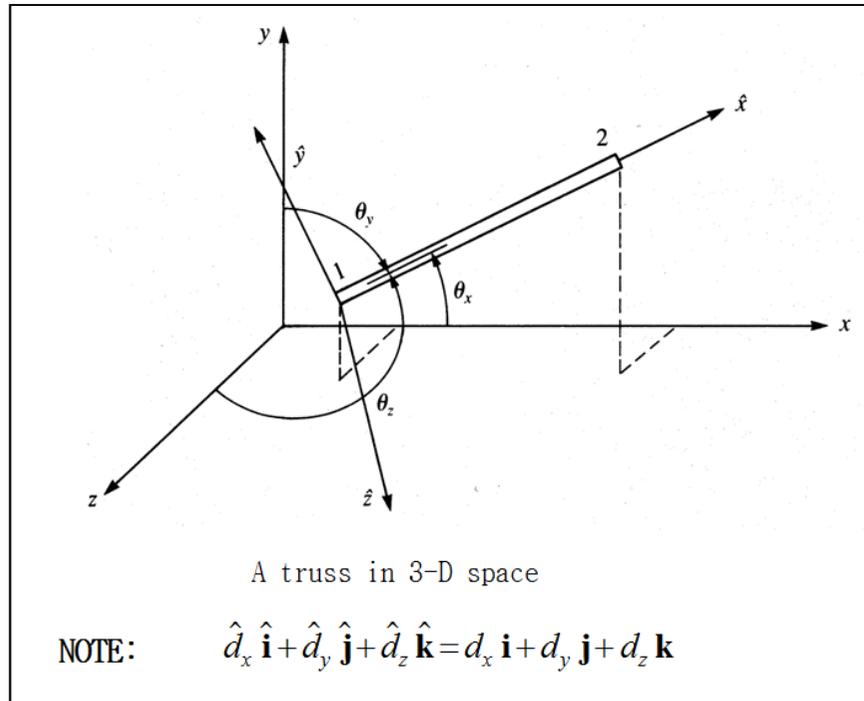
Finite Element Method: Truss Element

Step 8: Confirmation the loading equilibrium at node 1

$$\sum F_x = 0 \quad (1471 \text{ psi})(2 \text{ in.}^2) \frac{\sqrt{2}}{2} - (1035 \text{ psi})(2 \text{ in.}^2) = 0$$

$$\sum F_y = 0 \quad (3695 \text{ psi})(2 \text{ in.}^2) + (1471 \text{ psi})(2 \text{ in.}^2) \frac{\sqrt{2}}{2} - 10,000 = 0$$

Truss Element in 3D Space



$$\begin{Bmatrix} \hat{d}_{1x} \\ \hat{d}_{2x} \end{Bmatrix} = \begin{bmatrix} C_x & C_y & C_z & 0 & 0 & 0 \\ 0 & 0 & 0 & C_x & C_y & C_z \end{bmatrix} \begin{Bmatrix} d_{1x} \\ d_{1y} \\ d_{1z} \\ d_{2x} \\ d_{2y} \\ d_{2z} \end{Bmatrix}$$

$$\underline{T}^* = \begin{bmatrix} C_x & C_y & C_z & 0 & 0 & 0 \\ 0 & 0 & 0 & C_x & C_y & C_z \end{bmatrix}$$

We can get k using \underline{T}^* instead of \underline{T} ($\underline{k} = \underline{T}^T \hat{\mathbf{k}} \underline{T}$).

Finite Element Method: Truss Element

\underline{k} is calculated by using $\underline{k} = (T^*)^T \hat{k} T^*$ as below.

$$\underline{k} = \begin{bmatrix} C_x & 0 \\ C_y & 0 \\ C_z & 0 \\ 0 & C_x \\ 0 & C_y \\ 0 & C_z \end{bmatrix} \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} C_x & C_y & C_z & 0 & 0 & 0 \\ 0 & 0 & 0 & C_x & C_y & C_z \end{bmatrix}$$

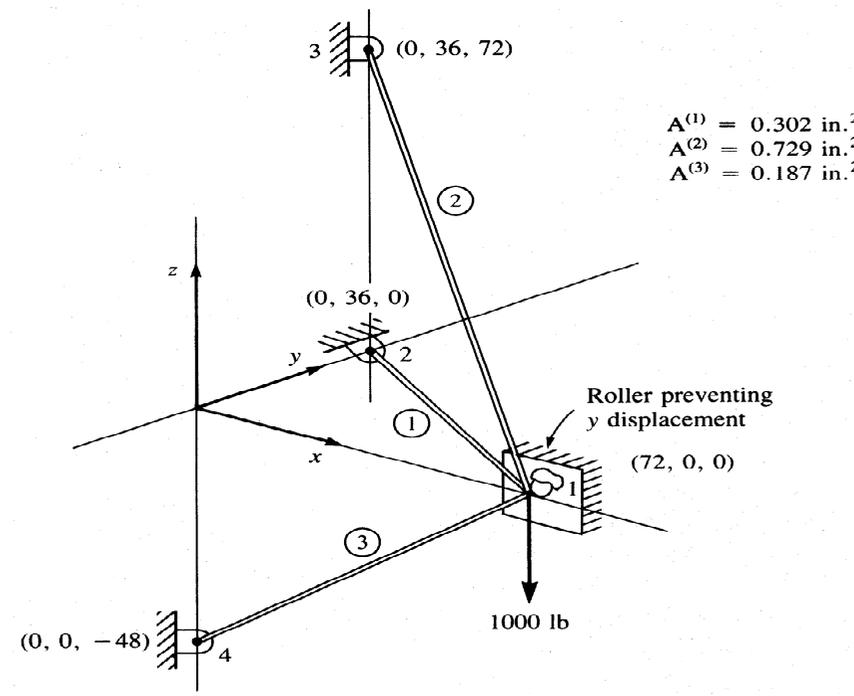
$$\underline{k} = \frac{AE}{L} \begin{bmatrix} C_x^2 & C_x C_y & C_x C_z & -C_x^2 & -C_x C_y & -C_x C_z \\ & C_y^2 & C_y C_z & -C_x C_y & -C_y^2 & -C_y C_z \\ & & C_z^2 & -C_x C_z & -C_y C_z & -C_z^2 \\ & & & C_x^2 & C_x C_y & C_x C_z \\ & & & & C_y^2 & C_y C_z \\ & & & & & C_z^2 \end{bmatrix}$$

Symmetry

Finite Element Method: Truss Element

Example (Truss element in 3D space)

Solve the 3-D truss problem as shown in below figure. The elastic moduli for all elements are $E=1.2 \times 10^6 \text{ psi}$. 1000lb loading is applied in the negative z-direction at node 1. Node 2, 3, and 4 are supported on sockets with balls (x,y, and z direction movements are constrained). The movement along the y direction is constrained by a roller at Node 1.



Finite Element Method: Truss Element

Step 4: Derivation of element stiffness matrix

Stiffness matrix of the elements \underline{k} :

$$\underline{k} = \frac{AE}{L} \begin{bmatrix} \underline{\lambda} & -\underline{\lambda} \\ -\underline{\lambda} & \underline{\lambda} \end{bmatrix}$$

where the submatrix $\underline{\lambda}$:

$$\underline{\lambda} = \begin{bmatrix} C_x^2 & C_x C_y & C_x C_z \\ C_y C_x & C_y^2 & C_y C_z \\ C_z C_x & C_z C_y & C_z^2 \end{bmatrix}$$

So, \underline{k} is defined as $\underline{\lambda}$ is defined.

For element 3

Directional cosine:

$$C_x = \frac{x_4 - x_1}{L^{(3)}} \quad C_y = \frac{y_4 - y_1}{L^{(3)}} \quad C_z = \frac{z_4 - z_1}{L^{(3)}}$$

Where length of the element $L^{(e)}$:

$$L^{(3)} = [(-72.0)^2 + (-48.0)^2]^{1/2} = 86.5 \text{ in.}$$

$$C_x = \frac{-72.0}{86.5} = -0.833$$

$$C_y = 0$$

$$C_z = \frac{-48.5}{86.5} = -0.550$$

Finite Element Method: Truss Element

Submatrix $\underline{\lambda}$

$$\underline{\lambda} = \begin{bmatrix} 0.69 & 0 & 0.46 \\ 0 & 0 & 0 \\ 0.46 & 0 & 0.30 \end{bmatrix}$$

Stiffness matrix of the element \underline{k}

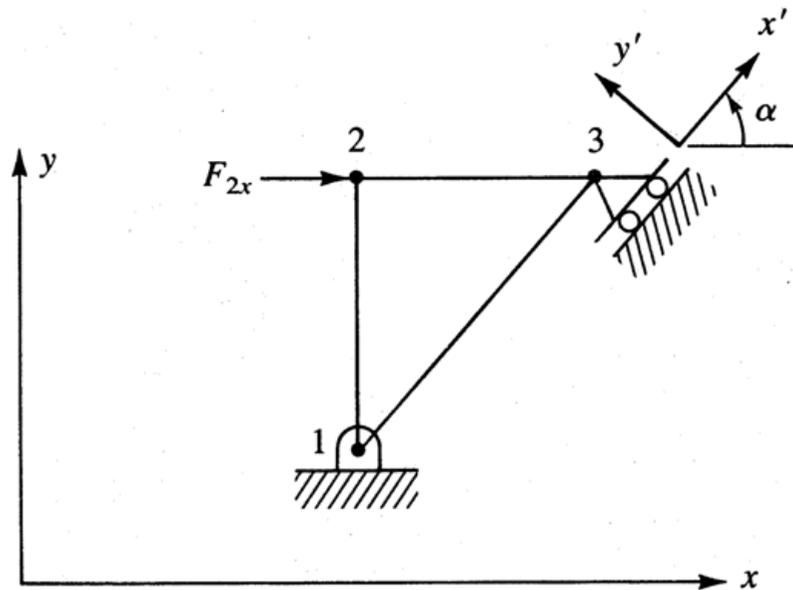
$$\underline{k}^{(3)} = \frac{(0.187)(1.2 \times 10^6)}{86.5} \begin{bmatrix} d_{1x} & d_{1y} & d_{1z} & d_{4x} & d_{4y} & d_{4z} \\ \underline{\lambda} & & & -\underline{\lambda} & & \\ -\underline{\lambda} & & & \underline{\lambda} & & \end{bmatrix}$$

For element 1

$$L^{(1)} = 80.5 \text{ in.} \quad C_x = -0.89 \quad C_y = 0.45 \quad C_z = 0$$

$$\underline{\lambda} = \begin{bmatrix} 0.79 & -0.40 & 0 \\ -0.40 & 0.20 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \underline{k}^{(1)} = \frac{(0.320)(1.2 \times 10^6)}{80.5} \begin{bmatrix} d_{1x} & d_{1y} & d_{1z} & d_{2x} & d_{2y} & d_{2z} \\ \underline{\lambda} & & & -\underline{\lambda} & & \\ -\underline{\lambda} & & & \underline{\lambda} & & \end{bmatrix}$$

Inclined or Skewed and Support



To be convenient, apply the boundary condition $d'_{3,y}$ to the local coordinate $x' - y'$

Finite Element Method: Truss Element

Displacement vector transformation at node 3:

$$\begin{Bmatrix} d'_{3x} \\ d'_{3y} \end{Bmatrix} = \begin{bmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{bmatrix} \begin{Bmatrix} d_{3x} \\ d_{3y} \end{Bmatrix}$$

or

$$\{d'_3\} = [t_3] \{d_3\} \quad [t_3] = \begin{bmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{bmatrix}$$

Displacement vector transformation at all nodes:

$$\begin{aligned} \{d'\} &= [T_1] \{d\} \\ \{d\} &= [T_1]^T \{d'\} \end{aligned}$$

or

$$\begin{Bmatrix} d_{1x} \\ d_{1y} \\ d_{2x} \\ d_{2y} \\ d_{3x} \\ d_{3y} \end{Bmatrix} = [T_1]^T \{d'\} = \begin{bmatrix} [I] & [0] & [0] \\ [0] & [I] & [0] \\ [0] & [0] & [t_3]^T \end{bmatrix} \begin{Bmatrix} d'_{1x} \\ d'_{1y} \\ d'_{2x} \\ d'_{2y} \\ d'_{3x} \\ d'_{3y} \end{Bmatrix}$$

Finite Element Method: Truss Element

Loading vector transformation :

$$\text{In local coordinate system } \{f'_3\} = [t_3] \{f_3\}$$

In global coordinate system

$$\{f'\} = [T_1] \{f\} \quad \text{or} \quad \begin{Bmatrix} f_{1x} \\ f_{1y} \\ f_{2x} \\ f_{2y} \\ f'_{3x} \\ f'_{3y} \end{Bmatrix} = \begin{bmatrix} [I] & [0] & [0] \\ [0] & [I] & [0] \\ [0] & [0] & [t_3] \end{bmatrix} \begin{Bmatrix} f_{1x} \\ f_{1y} \\ f_{2x} \\ f_{2y} \\ f_{3x} \\ f_{3y} \end{Bmatrix}$$

Note: $f_{1x} = f'_{1x}, f_{1y} = f'_{1y}, f_{2x} = f'_{2x}, f_{2y} = f'_{2y}$

$$d_{1x} = d'_{1x}, d_{1y} = d'_{1y}, d_{2x} = d'_{2x}, d_{2y} = d'_{2y}$$

Finite Element Method: Truss Element

Transformation of the finite element formulation in global coordinates::

$$\{f\} = [K]\{d\}$$

$$\rightarrow [T_1]\{f\} = [T_1][K]\{d\}$$

$$\rightarrow [T_1]\{f\} = [T_1][K][T_1]^T \{d'\}$$

$$\rightarrow \begin{Bmatrix} f_{1x} \\ f_{1y} \\ f_{2x} \\ f_{2y} \\ f'_{3x} \\ f'_{3y} \end{Bmatrix} = [T_1][K][T_1]^T \begin{Bmatrix} d_{1x} \\ d_{1y} \\ d_{2x} \\ d_{2y} \\ d'_{3x} \\ d'_{3y} \end{Bmatrix}$$

Boundary condition: $d_{1x} = 0, d_{1y} = 0, d'_{3y} = 0$ $F_{2y} = 0, F'_{3x} = 0$ F_{2x}

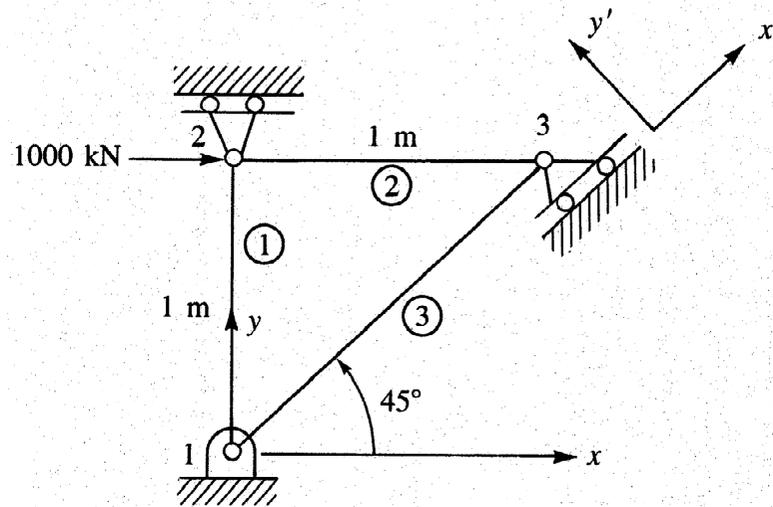
Calculation of displacement and loading: unknown displacements d_{2x}, d_{2y}, d'_{3x} reaction forces F_{1x}, F_{1y} and that of inclined roller F'_{3y}

Finite Element Method: Truss Element

Example (Truss element in inclined support)

Find displacements and reacting forces for the plane trusses in the figure. $E=210\text{ GPa}$ and

$A=6.00 \times 10^{-4}\text{ m}^2$ for nodes 1 and 2, and $A=6\sqrt{2} \times 10^{-4}\text{ m}^2$ for node 3.



Finite Element Method: Truss Element

Element 1

$$\cos \theta = 0 \quad \sin \theta = 1$$

$$\underline{k}^{(1)} = \frac{(6.0 \times 10^{-4})(210 \times 10^9)}{1 \text{ m}} \begin{bmatrix} d_{1x} & d_{1y} & d_{2x} & d_{2y} \\ 0 & 0 & 0 & 0 \\ & 1 & 0 & -1 \\ & & 0 & 0 \\ \text{Symmetry} & & & 1 \end{bmatrix}$$

Element 2

$$\cos \theta = 1 \quad \sin \theta = 0$$

$$\underline{k}^{(2)} = \frac{(6.0 \times 10^{-4})(210 \times 10^9)}{1 \text{ m}} \begin{bmatrix} d_{2x} & d_{2y} & d_{3x} & d_{3y} \\ 1 & 0 & -1 & 0 \\ & 0 & 0 & 0 \\ & & 1 & 0 \\ \text{Symmetry} & & & 0 \end{bmatrix}$$

Finite Element Method: Truss Element

Element 3

$$\cos \theta = \frac{\sqrt{2}}{2} \quad \sin \theta = \frac{\sqrt{2}}{2}$$

$$\underline{k}^{(3)} = \frac{(6\sqrt{2} \times 10^{-4})(210 \times 10^9)}{\sqrt{2}} \begin{bmatrix} & d_{1x} & d_{1y} & d_{3x} & d_{3y} \\ & 0.5 & 0.5 & -0.5 & -0.5 \\ & & 0.5 & -0.5 & -0.5 \\ & & & 0.5 & 0.5 \\ & \text{Symmetry} & & & 0.5 \end{bmatrix}$$

Calculation of the matrix \underline{K} in global coordinates using direct stiffness method:

$$\underline{K} = 1260 \times 10^5 \text{ N/m} \begin{bmatrix} 0.5 & 0.5 & 0 & 0 & -0.5 & -0.5 \\ & 1.5 & 0 & -1 & -0.5 & -0.5 \\ & & 1 & 0 & -1 & 0 \\ & & & 1 & 0 & 0 \\ & & & & 1.5 & 0.5 \\ & \text{Symmetry} & & & & 0.5 \end{bmatrix}$$

Finite Element Method: Truss Element

Matrix \underline{T}_1 , which transforms the displacement at node 3 in global coordinates to the local coordinates $x'-y'$:

$$[\underline{T}_1] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & 0 & 0 & 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$$

Calculation of $\underline{K}^* = \underline{T}_1 \underline{K} \underline{T}_1^T$:

$$\underline{T}_1 \underline{K} = 1260 \times 10^5 \begin{bmatrix} 0.5 & 0.5 & 0 & 0 & -0.5 & -0.5 \\ 0.5 & 1.5 & 0 & -1 & -0.5 & -0.5 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ -0.707 & -0.707 & -0.707 & 0 & 1.414 & 0.707 \\ 0 & 0 & 0.707 & 0 & -0.707 & 0 \end{bmatrix}$$

And,

Finite Element Method: Truss Element

$$\underline{T}_1 \underline{K} \underline{T}_1^T = 1260 \times 10^5 \begin{bmatrix} d_{1x} & d_{1y} & d_{2x} & d_{2y} & d'_{3x} & d'_{3y} \\ 0.5 & 0.5 & 0 & 0 & -0.707 & 0 \\ 0.5 & 1.5 & 0 & -1 & -0.707 & 0 \\ 0 & 0 & 1 & 0 & -0.707 & 0.707 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ -0.707 & -0.707 & -0.707 & 0 & 1.500 & -0.500 \\ 0 & 0 & 0.707 & 0 & -0.500 & 0.500 \end{bmatrix}$$

Substituting the boundary conditions $d_{1x} = d_{1y} = d_{2y} = d'_{3y} = 0$ to the above equation,

$$\begin{Bmatrix} F_{2x} = 1000 \text{ kN} \\ F'_{3x} = 0 \end{Bmatrix} = (1260 \times 10^5) \begin{bmatrix} 1 & -0.707 \\ -0.707 & 1.50 \end{bmatrix} \begin{Bmatrix} d_{2x} \\ d'_{3x} \end{Bmatrix}.$$

Calculating displacements,

$$d_{2x} = 11.91 \text{ mm}$$

$$d'_{3x} = 5.613 \text{ mm}$$

Finite Element Method: Truss Element

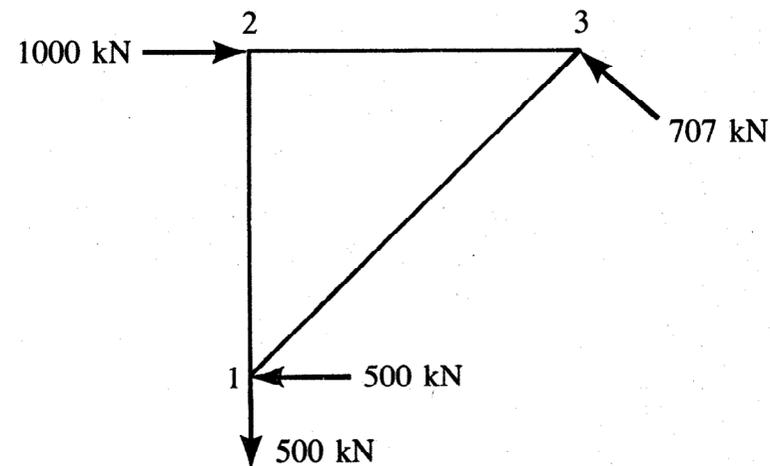
Calculating loadings,

$$F_{1x} = -500 \text{ kN}$$

$$F_{1y} = -500 \text{ kN}$$

$$F_{2y} = 0$$

$$F'_{3y} = 707 \text{ kN}$$



Free body diagram of truss structure

Finite Element Method: Truss Element

1. **Solution of finite element coincides with the exact solution at nodes.** The reason why it coincides with the exact solution at nodes is because it calculates the element nodal loading by the energy equivalent based on the linearly assumed displacement form at each element.
2. Although the nodal value for displacement coincides with the exact solution, the values between nodes are very inaccurate in the case of using few elements due to using linear displacement function at each element. However, **when the number of elements increases, the solution of finite element converges on the exact solution.**
3. Stress is derived from the slope of displacement curve as $\sigma = E \varepsilon = E(du/dx)$. Axial stress is constant at each element, for u of each element in the solution of finite element is a linear function. **The much more elements are needed for the modeling of the first derivative of the displacement function like the modeling of axial stress.**
4. **The most approximate value of the stress appears not at nodes, but at the central point.** It is because the derivative of displacement is calculated more accurately between nodes than at nodes.
5. **Stress is not continuous over the element boundaries.** Therefore the equilibrium is not satisfied over the element boundaries. Also, the equilibrium at each element is usually not satisfied.