ENGINEERING MATHEMATICS II

010.141

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CHAP. 7
LINEAR ALGEBRA: MATRICS, VECTORS, DETEMINANTS. LINEAR SYSTEMS

ABSTRACT OF CHAP. 7

- ***Linear algebra in Chaps. 7 and 8*** discusses the theory and application of vectors and matrices, mainly related to linear systems of equations, eigenvalue problems, and linear transformation.

- ***Chapter 7*** concerns mainly systems of linear equations and linear transformations.
  - Systems of linear equations arise in structural analyses, electrical networks, mechanical frameworks, economic models, optimization problems, numerics (Chaps 21-23) for differential equations.

CHAP. 7.1 MATRICS, VECTORS: ADDITION AND SCALAR MULTIPLICATION

A matrix is a mathematical tool to represent linear system characteristics.

NOTATIONS AND CONCEPTS

- **Notations:** matrices are denoted by capital boldface letters, such as **A**; entries at the intersection of the i-row and j-column are denoted by a_{ij} .

$$\mathbf{A} = [a_{jk}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

- A m × n matrix means m-rows and n-columns; an n × n matrix is called ***square***.
- In a square matrix the elements a_{11} down to a_{nn} are said to be on the ***main diagonal***.
- Vector: a matrix with either one row or one column. Entries are called ***components***.

MATRIX EQUALITY

➤ Definition:

Equality of Matrices

Two matrices $\mathbf{A} = [a_{jk}]$ and $\mathbf{B} = [b_{jk}]$ are **equal**, written $\mathbf{A} = \mathbf{B}$, if and only if they have the same size and the corresponding entries are equal, that is, $a_{11} = b_{11}$, $a_{12} = b_{12}$, and so on. Matrices that are not equal are called **different**. Thus, matrices of different sizes are always different.

Addition of Matrices

The **sum** of two matrices $\mathbf{A} = [a_{jk}]$ and $\mathbf{B} = [b_{jk}]$ ***of the same size*** is written $\mathbf{A} + \mathbf{B}$ and has the entries $a_{jk} + b_{jk}$ obtained by adding the corresponding entries of **A** and **B**. Matrices of different sizes cannot be added.

Scalar Multiplication (Multiplication by a Number)

The **product** of any $m \times n$ matrix $\mathbf{A} = [a_{jk}]$ and any **scalar** c (number c) is written $c\mathbf{A}$ and is the $m \times n$ matrix $c\mathbf{A} = [ca_{jk}]$ obtained by multiplying each entry of **A** by c.

SCALAR MULTIPLICATION

Rules for Matrix Addition and Scalar Multiplication. From the familiar laws for the addition of numbers we obtain similar laws for the addition of matrices of the same size $m \times n$, namely,

$$
\begin{aligned}
&\text{(a)} \quad & \mathbf{A}+\mathbf{B} &= \mathbf{B}+\mathbf{A} \\
&\text{(b)} \quad & (\mathbf{A}+\mathbf{B})+\mathbf{C} &= \mathbf{A}+(\mathbf{B}+\mathbf{C}) \qquad (\text{written } \mathbf{A}+\mathbf{B}+\mathbf{C}) \\
&\text{(c)} \quad & \mathbf{A}+\mathbf{0} &= \mathbf{A} \\
&\text{(d)} \quad & \mathbf{A}+(-\mathbf{A}) &= \mathbf{0}.
\end{aligned}
$$

Here $\mathbf{0}$ denotes the **zero matrix** (of size $m \times n$), that is, the $m \times n$ matrix with all entries zero. (The last matrix in Example 5 is a zero matrix.)

Hence matrix addition is *commutative* and *associative* [by (3a) and (3b)].

Similarly, for scalar multiplication we obtain the rules

$$
(4) \qquad
\begin{aligned}
&\text{(a)} \quad & c(\mathbf{A}+\mathbf{B}) &= c\mathbf{A}+c\mathbf{B} \\
&\text{(b)} \quad & (c+k)\mathbf{A} &= c\mathbf{A}+k\mathbf{A} \\
&\text{(c)} \quad & c(k\mathbf{A}) &= (ck)\mathbf{A} \qquad (\text{written } ck\mathbf{A}) \\
&\text{(d)} \quad & 1\mathbf{A} &= \mathbf{A}.
\end{aligned}
$$

ADDITION AND SCALAR MULTIPLICATION OF VECTORS

- **Problem 1**

$$7A - 5B$$

where

$$A = \begin{bmatrix} 3 & 0 & 4 \end{bmatrix}$$
$$B = \begin{bmatrix} -1 & 8 & 2 \end{bmatrix}$$

$$7A = \begin{bmatrix} 21 & 0 & 28 \end{bmatrix}$$
$$5B = \begin{bmatrix} -5 & 40 & 10 \end{bmatrix}$$

$$7A - 5B = \begin{bmatrix} 26 & -40 & 18 \end{bmatrix}$$
$$7A^T - 5B^T = \begin{bmatrix} 26 \\ -40 \\ 18 \end{bmatrix}$$

ADDITION AND SCALAR MULTIPLICATION OF VECTORS

- **Problem 2**

$$5\left(C-2D\right)=?$$

$$C=\begin{bmatrix} 9 \\ 5 \\ 7 \end{bmatrix} \qquad D=\begin{bmatrix} 2 \\ -2 \\ 6 \end{bmatrix}$$

$$C-2D=\begin{bmatrix} 5 \\ 9 \\ -5 \end{bmatrix} \qquad 5\left(C-2D\right)=\begin{bmatrix} 25 \\ 45 \\ -25 \end{bmatrix}$$

$$5\left(2D-C\right)=?$$

$$=\begin{bmatrix} 20 \\ -20 \\ 60 \end{bmatrix}-\begin{bmatrix} 45 \\ 25 \\ 35 \end{bmatrix}=\begin{bmatrix} 25 \\ -25 \\ -25 \end{bmatrix}$$

HOMEWORK IN 7.1

- HW1. Recall Example 2 (Page 274). Compute the total revenue of the store when three products I, II, III are sold at \$100, \$120, and \$150, respectively.

- HW2. Build any matrix to represent your life characteristics. For instance, grade (% scale) vs. number of study hours per week for three courses (choose any) in the previous semester.

CHAP. 7.2
MATRIX MULTIPLICATION

The most popular use of matrix multiplication operation is a linear transformation of one system to the other system.

MATRIX MULTIPLICATION

- Matrix algebraic operation: Addition, Subtraction, Scalar Multiplication, and **Matrix Multiplication**
- Two matrices may be multiplied, and produce a third matrix, provided they are conformable to multiplication.
- If there exists $\mathbf{A}_{mn}$ and $\mathbf{B}_{np}$ then these two matrices are conformable and the resultant matrix is of size 'mp', that is, $\mathbf{C}_{mp}$

MATRIX MULTIPLICATION DEFINITION

Multiplication of a Matrix by a Matrix

The **product** $\mathbf{C} = \mathbf{AB}$ (in this order) of an $m \times n$ matrix $\mathbf{A} = [a_{jk}]$ times an $r \times p$ matrix $\mathbf{B} = [b_{jk}]$ is defined if and only if $r = n$ and is then the $m \times p$ matrix $\mathbf{C} = [c_{jk}]$ with entries

$$c_{jk} = \sum_{l=1}^{n} a_{jl} b_{lk} = a_{j1}b_{1k} + a_{j2}b_{2k} + \cdots + a_{jn}b_{nk} \qquad \begin{array}{l} j = 1, \cdots, m \\ k = 1, \cdots, p\,. \end{array} \tag{1}$$

$$\begin{array}{ccccc} \mathbf{A} & \mathbf{B} & = & \mathbf{C} \\ [m \times n] & [n \times r] & = & [m \times r]\,. \end{array}$$

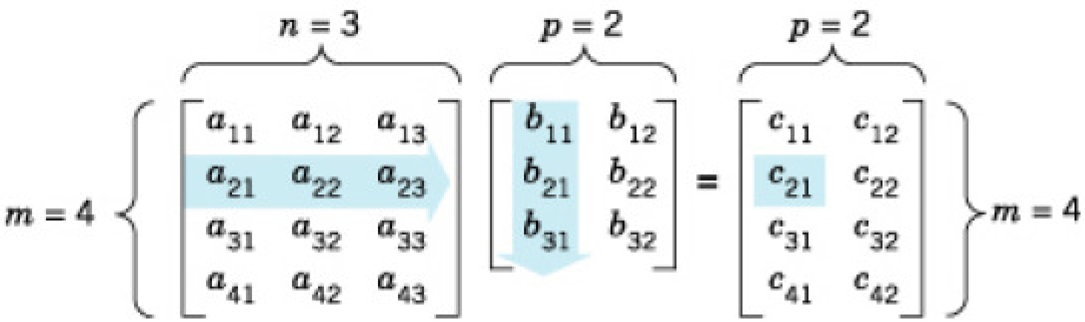


Fig. 155. Notations in a product **AB** = **C**

MULTIPLICATION EXAMPLES

- **Example**

$$\begin{bmatrix} 4 & 3 \\ 7 & 2 \\ 9 & 0 \end{bmatrix}_{3,2} \cdot \begin{bmatrix} 2 & 5 \\ 1 & 6 \end{bmatrix}_{2,2} = \begin{bmatrix} 8+3 & 20+18 \\ 14+2 & 35+12 \\ 18 & 45 \end{bmatrix}$$

- **Example**

$$\begin{bmatrix} 4 & 2 \\ 1 & 8 \end{bmatrix}_{2,2} \cdot \begin{bmatrix} 3 \\ 5 \end{bmatrix}_{2,1} = \begin{bmatrix} 12+10 \\ 3+40 \end{bmatrix}_{2,1} = \begin{bmatrix} 22 \\ 43 \end{bmatrix}$$

MULTIPLICATION RULES

- Multiplication is not commutative, in general.
- **AB** can be equal to 0, but this does not imply that either **A** or **B** is necessarily 0.
- Even if it is given that **AC = AD**, it is not necessarily true that **C = D**.
- Some rules include:

Our examples show that the ***order of factors*** in matrix products ***must always be observed very carefully***. Otherwise matrix multiplication satisfies rules similar to those for numbers, namely.

(2)

$$\begin{array}{lll}
\text{(a)} & (k\mathbf{A})\mathbf{B} = k(\mathbf{AB}) = \mathbf{A}(k\mathbf{B}) & \textit{Written } k\mathbf{AB} \textit{ or } \mathbf{A}k\mathbf{B} \\
\text{(b)} & \mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C} & \textit{Written } \mathbf{ABC} \\
\text{(c)} & (\mathbf{A}+\mathbf{B})\mathbf{C} = \mathbf{AC} + \mathbf{BC} & \\
\text{(d)} & \mathbf{C}(\mathbf{A}+\mathbf{B}) = \mathbf{CA} + \mathbf{CB} &
\end{array}$$

ROW AND COLUMN VECTOR MULTIPLICATION

- **Row × Column**

$$\begin{bmatrix} 3 & 6 & 1 \end{bmatrix}_{1,3} \cdot \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}_{3,1} = \begin{bmatrix} 3 + 12 + 4 \end{bmatrix}_{1,1}$$

- **Column × Row**

$$\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}_{3,1} \cdot \begin{bmatrix} 3 & 6 & 1 \end{bmatrix}_{1,3} = \begin{bmatrix} 3 & 6 & 1 \\ 6 & 12 & 2 \\ 12 & 24 & 4 \end{bmatrix}_{3,3}$$

MOTIVATION OF MULTIPLICATION

- Relation of x_1x_2-system to y_1y_2-system

$$y_1 = a_{11}x_1 + a_{12}x_2$$
$$y_2 = a_{21}x_1 + a_{22}x_2$$

$$\mathbf{Y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{Ax} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{bmatrix}.$$

- Further the relation of x_1x_2-system to w_1w_2-system

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{Bw} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} b_{11}w_1 + b_{12}w_2 \\ b_{21}w_1 + b_{22}w_2 \end{bmatrix}.$$

- Finally the relation of w_1w_2-system to y_1y_2-system

$$\mathbf{Y} = \mathbf{Cw} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} c_{11}w_1 + c_{12}w_2 \\ c_{21}w_1 + c_{22}w_2 \end{bmatrix}.$$

C = AB or

$$c_{11} = a_{11}b_{11} + a_{12}b_{21} \qquad c_{12} = a_{11}b_{12} + a_{12}b_{22}$$
$$c_{21} = a_{21}b_{11} + a_{22}b_{21} \qquad c_{22} = a_{21}b_{12} + a_{22}b_{22}.$$

TRANSPOSITION OF MATRICES

Transposition of Matrices and Vectors

The transpose of an $m \times n$ matrix $\mathbf{A} = [a_{jk}]$ is the $n \times m$ matrix $\mathbf{A}^\mathsf{T}$ (read *A transpose*) that has the first *row* of $\mathbf{A}$ as its first *column*, the second *row* of $\mathbf{A}$ as its second *column*, and so on. Thus the transpose of $\mathbf{A}$ in (2) is $\mathbf{A}^\mathsf{T} = [a_{kj}]$, written out

(9)
$$\mathbf{A}^\mathsf{T} = [a_{kj}] = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \cdot & \cdot & \cdots & \cdot \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}.$$

As a special case, transposition converts row vectors to column vectors and conversely.

Rules for transposition are

(10)

(a) $(\mathbf{A}^\mathsf{T})^\mathsf{T} = \mathbf{A}$

(b) $(\mathbf{A} + \mathbf{B})^\mathsf{T} = \mathbf{A}^\mathsf{T} + \mathbf{B}^\mathsf{T}$

(c) $(c\mathbf{A})^\mathsf{T} = c\mathbf{A}^\mathsf{T}$

(d) $(\mathbf{AB})^\mathsf{T} = \mathbf{B}^\mathsf{T}\mathbf{A}^\mathsf{T}$.

TRANSPOSITION OF MATRICES

- Suppose the matrix **A** has m rows and n columns; then the transposed matrix $\mathbf{A}^{\mathrm{T}}$ has n rows and m columns, with the rows and columns of **A** interchanged.

$$\mathbf{A} = \begin{bmatrix} 5 & -8 & 1 \\ 4 & 0 & 0 \end{bmatrix}, \qquad \mathbf{A}^{\mathrm{T}} = \begin{bmatrix} 5 & 4 \\ -8 & 0 \\ 1 & 0 \end{bmatrix}$$

SPECIAL MATRICES

Symmetric and Skew-Symmetric Matrices. Transposition gives rise to two useful classes of matrices, as follows. *Symmetric matrices* and *skew-symmetric matrices* are ***square*** matrices whose transpose equals the matrix itself or minus the matrix, respectively:

(11) $$\mathbf{A}^{\mathsf{T}} = \mathbf{A} \quad (\text{thus } a_{kj} = a_{jk}), \qquad \mathbf{A}^{\mathsf{T}} = -\mathbf{A} \quad (\text{thus } a_{kj} = -a_{jk}, \text{ hence } a_{jj} = 0).$$

Symmetric Matrix

Skew Symmetric Matrix

$$\mathbf{A} = \begin{bmatrix} 20 & 120 & 200 \\ 120 & 10 & 150 \\ 200 & 150 & 30 \end{bmatrix} \quad \text{is symmetric, and} \qquad \mathbf{B} = \begin{bmatrix} 0 & 1 & -3 \\ -1 & 0 & -2 \\ 3 & 2 & 0 \end{bmatrix} \quad \text{is skew-symmetric.}$$

Triangular Matrices. Upper triangular matrices are square matrices that can have nonzero entries only on and ***above*** the main diagonal, whereas any entry below the diagonal must be zero. Similarly, **lower triangular matrices** can have nonzero entries only on and ***below*** the main diagonal. Any entry on the main diagonal of a triangular matrix may be zero or not.

$$\begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}, \quad \begin{bmatrix} 1 & 4 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & 6 \end{bmatrix}, \quad \begin{bmatrix} 2 & 0 & 0 \\ 8 & -1 & 0 \\ 7 & 6 & 8 \end{bmatrix}, \quad \begin{bmatrix} 3 & 0 & 0 & 0 \\ 9 & -3 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 9 & 3 & 6 \end{bmatrix}.$$

Upper triangular

Lower triangular

SPECIAL MATRICES

Diagonal Matrices. These are square matrices that can have nonzero entries only on the main diagonal. Any entry above or below the main diagonal must be zero.

If all the diagonal entries of a diagonal matrix $\mathbf{S}$ are equal, say, c, we call $\mathbf{S}$ a **scalar matrix** because multiplication of any square matrix $\mathbf{A}$ of the same size by $\mathbf{S}$ has the same effect as the multiplication by a scalar, that is,

$$\mathbf{AS} = \mathbf{SA} = c\mathbf{A}. \tag{12}$$

In particular, a scalar matrix whose entries on the main diagonal are all 1 is called a **unit matrix** (or **identity matrix**) and is denoted by $\mathbf{I}_n$ or simply by $\mathbf{I}$. For $\mathbf{I}$, formula (12) becomes

$$\mathbf{AI} = \mathbf{IA} = \mathbf{A}. \tag{13}$$

EXAMPLE 10 Diagonal Matrix D. Scalar Matrix S. Unit Matrix I

$$\mathbf{D} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \mathbf{S} = \begin{bmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{bmatrix}, \qquad \mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- For any square matrix **A** there exists an **I** such that: **AI = A**

SPECIAL MATRICES

- If **a** is a row vector of order n, and **b** is a column vector, also of order n, then the inner product or dot product of **a** and **b** is the scalar which is the sum of the products of the respective elements

- **Example**

$$a = \begin{bmatrix} 4 & -1 & 5 \end{bmatrix}$$

$$b = \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}$$

$$a \cdot b = 8 - 5 + 40 = 43 \quad \text{(a scalar)}$$

- Thus, matrix multiplication amounts to combinations of dot products of row and column vectors

MULTIPLICATION OF MATRICES BY MATRICES AND BT VECTORS

EXAMPLE 11 Computer Production. Matrix Times Matrix

Supercomp Ltd produces two computer models PC1086 and PC1186. The matrix **A** shows the cost per computer (in thousands of dollars) and **B** the production figures for the year 2005 (in multiples of 10000 units.) Find a matrix **C** that shows the shareholders the cost per quarter (in millions of dollars) for raw material, labor, and miscellaneous.

$$\mathbf{A} = \begin{array}{c} \text{PC1086} \quad \text{PC1186} \\ \begin{bmatrix} 1.2 & 1.6 \\ 0.3 & 0.4 \\ 0.5 & 0.6 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{Raw Components} \\ \text{Labor} \\ \text{Miscellaneous} \end{array} \qquad \mathbf{B} = \begin{array}{c} \text{Quarter} \\ 1 \quad 2 \quad 3 \quad 4 \\ \begin{bmatrix} 3 & 8 & 6 & 9 \\ 6 & 2 & 4 & 3 \end{bmatrix} \end{array} \begin{array}{l} \\ \\ \text{PC1086} \\ \text{PC1186} \end{array}$$

Solution:

$$\mathbf{C} = \mathbf{AB} = \begin{array}{c} \text{Quarter} \\ 1 \quad\quad 2 \quad\quad 3 \quad\quad 4 \\ \begin{bmatrix} 13.2 & 12.8 & 13.6 & 15.6 \\ 3.3 & 3.2 & 3.4 & 3.9 \\ 5.1 & 5.2 & 5.4 & 6.3 \end{bmatrix} \end{array} \begin{array}{l} \\ \\ \text{Raw Components} \\ \text{Labor} \\ \text{Miscellaneous} \end{array}$$

Since cost is given in multiples of \$1000 and production in multiples of 10 000 units, the entries of **C** are multiples of \$10 millions; thus $c_{11} = 13.2$ means \$132 million, etc.

HOMEWORK IN 7.2

- HW1. Problem 19(d)
- HW2. Problem 23
- HW3. Problem 28(a) and 28(b)

CHAP. 7.3
LINEAR SYSTEMS OF EQUATIONS. GAUSS ELIMINATION

The most important use of matrices occurs in the solution of systems of linear equations, briefly called linear systems.

LINEAR SYSTEMS OF EQUATIONS

A **linear system of *m* equations in *n* unknowns** $x_1, \cdots, x_n$ is a set of equations of the form

(1)
$$\begin{aligned} a_1 x_1 &+ \cdots + a_{1n} x_n = b_1 \\ a_{21} x_1 &+ \cdots + a_{2n} x_n = b_2 \\ &\cdots\cdots\cdots\cdots\cdots \\ a_{m1} x_1 &+ \cdots + a_{mn} x_n = b_m \end{aligned}$$

The system is called *linear* because each variable x_j appears in the first power only, just as in the equation of a straight line. $a_{11}, \cdots, a_{mn}$ are given numbers, called the **coefficients** of the system. $b_1, \cdots, b_m$ on the right are also given numbers. If all the b_j are zero, then (1) is called a **homogeneous system**. If at least one b_j is not zero, then (1) is called a **nonhomogeneous system**.

A **solution** of (1) is a set of numbers $x_1, \cdots, x_n$ that satisfies all the m equations. A **solution vector** of (1) is a vector $\mathbf{x}$ whose components form a solution of (1) . If the system (1) is homogeneous, it has at least the **trivial solution** $x_1 = 0, \cdots, x_n = 0$.

Matrix Form of the Linear System (1) . From the definition of matrix multiplication we see that the m equations of (1) may be written as a single vector equation

(2)
$$\mathbf{Ax} = \mathbf{b}$$

where the **coefficient matrix** $\mathbf{A} = [a_{jk}]$ is the $m \times n$ matrix

LINEAR SYSTEMS OF EQUATIONS

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ \cdot \\ \cdot \\ \cdot \\ b_m \end{bmatrix}$$

are column vectors. We assume that the coefficients a_{jk} are not all zero, so that **A** is not a zero matrix. Note that **x** has n components, whereas **b** has m components. The matrix

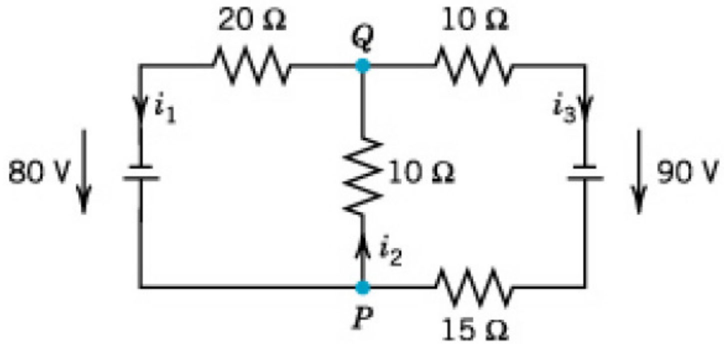
$$\tilde{\mathbf{A}} = \left[\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdots & \cdot & \cdot \\ a_{m1} & \cdots & a_{mn} & b_m \end{array}\right]$$

is called the **augmented matrix** of the system (1) . The dashed vertical line could be omitted (as we shall do later); it is merely a reminder that the last column of $\tilde{\mathbf{A}}$ does not belong to **A**.

- In a system of equations there may be a unique solution, infinitely many solutions, or no solution at all.

EXAMPLE OF LINEAR SYSTEMS

Node P gives the first equation, node Q the second, the right loop the third, and the left loop the fourth, as indicated in the figure.



$$
\begin{array}{lrcrcrcl}
\text{Node } P: & i_1 & - & i_2 & + & i_3 & = & 0 \\
\text{Node } Q: & -i_1 & + & i_2 & - & i_3 & = & 0 \\
\text{Right loop:} & & & 10i_2 & + & 25i_3 & = & 90 \\
\text{Left loop:} & 20i_1 & + & 10i_2 & & & = & 80
\end{array}
$$

Fig. 157. Network in Example 2 and equations relating the currents

GAUSS ELIMINATION

- The Gauss method of elimination is frequently used to solve the linear systems.
- Gauss elimination is a standard method for solving linear systems with the following rules:
  - Rule 1- rows may be interchanged;
  - Rule 2- one row may be multiplied by a non-zero constant
  - Rule 3- a multiple of one row may be added to another row
- We now call a linear system S_1 row-equivalent to a linear system S_2 if S_1 can be obtained from S_2 by (finitely many) row operations.

GAUSS ELIMINATION

THEOREM 1

Row-Equivalent Systems

Row-equivalent linear systems have the same set of solutions.

Because of this theorem, systems having the same solution sets are often called *equivalent systems*. But note well that we are dealing with ***row operations***. No column operations on the augmented matrix are permitted in this context because they would generally alter the solution set.

A linear system (1) is called **overdetermined** if it has more equations than unknowns, as in Example 2, **determined** if $m = n$, as in Example 1, and **underdetermined** if it has fewer equations than unknowns.

Furthermore, a system (1) is called **consistent** if it has at least one solution (thus, one solution or infinitely many solutions), but **inconsistent** if it has no solutions at all, as $x_1 + x_2 = 1, x_1 + x_2 = 0$ in Example 1.

GAUSS ELIMINATION

EXAMPLE 3 Gauss Elimination if Infinitely Many Solutions Exist

Solve the following linear systems of three equations in four unknowns whose augmented matrix is

$$(5)\quad \left[\begin{array}{cccc|c} 3.0 & 2.0 & 2.0 & -5.0 & 8.0 \\ 0.6 & 1.5 & 1.5 & -5.4 & 2.7 \\ 1.2 & -0.3 & -0.3 & 2.4 & 2.1 \end{array}\right]. \qquad \text{Thus,} \qquad \begin{aligned} 3.0x_1 + 2.0x_2 + 2.0x_3 - 5.0x_4 &= 8.0 \\ \boxed{0.6x_1} + 1.5x_2 + 1.5x_3 - 5.4x_4 &= 2.7 \\ \boxed{1.2x_1} - 0.3x_2 - 0.3x_3 + 2.4x_4 &= 2.1. \end{aligned}$$

Solution:

Step 1. ***Elimination of*** x_1 from the second and third equations by adding

$-0.6/3.0 = -0.2$ times the first equation to the second equation,

$-1.2/3.0 = -0.4$ times the first equation to the third equation.

This gives the following, in which the pivot of the next step is circled.

$$(6)\quad \left[\begin{array}{cccc|c} 3.0 & 2.0 & 2.0 & -5.0 & 8.0 \\ 0 & 1.1 & 1.1 & -4.4 & 1.1 \\ 0 & -1.1 & -1.1 & 4.4 & -1.1 \end{array}\right] \quad \begin{array}{l} \\ \text{Row 2} - 0.2\ \text{Row 1} \\ \text{Row 3} - 0.4\ \text{Row 1} \end{array} \qquad \begin{aligned} 3.0x_1 + 2.0x_2 + 2.0x_3 - 5.0x_4 &= 8.0 \\ 1.1x_2 + 1.1x_3 - 4.4x_4 &= 1.1 \\ \boxed{-1.1x_2} - 1.1x_3 + 4.4x_4 &= -1.1 \end{aligned}$$

GAUSS ELIMINATION

Step 2. ***Elimination of*** $\boldsymbol{x_2}$ from the third equation of (6) by adding

$1.1 / 1.1 = 1$ times the second equation to the third equation.

This gives

$$(7) \qquad \left[\begin{array}{cccc|c} 3.0 & 2.0 & 2.0 & -5.0 & 8.0 \\ 0 & 1.1 & 1.1 & -4.4 & 1.1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad \text{Row 3 + Row 2} \qquad \begin{aligned} 3.0x_1 + 2.0x_2 + 2.0x_3 - 5.0x_4 &= 8.0 \\ 1.1x_2 + 1.1x_3 - 4.4x_4 &= 1.1 \\ 0 &= 0\ . \end{aligned}$$

Back Substitution. From the second equation, $x_2 = 1 - x_3 + 4x_4$. From this and the first equation, $x_1 = 2 - x_4$. Since x_3 and x_4 remain arbitrary, we have infinitely many solutions. If we choose a value of x_3 and a value of x_4, then the corresponding values of x_1 and x_2 are uniquely determined.

On Notation. If unknowns remain arbitrary, it is also customary to denote them by other letters t_1, t_2, $\cdots$. In this example we may thus write $x_1 = 2 - x_4 = 2 - t_2$, $x_2 = 1 - x_3 + 4x_4 = 1 - t_1 + 4t_2$, $x_3 = t_1$ (first arbitrary unknown), $x_4 = t_2$ (second arbitrary unknown).

GAUSS ELIMINATION

EXAMPLE 4 Gauss Elimination if no Solution Exists

What will happen if we apply the Gauss elimination to a linear system that has no solution? The answer is that in this case the method will show this fact by producing a contradiction. For instance, consider

$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 2 & 1 & 1 & 0 \\ 6 & 2 & 4 & 6 \end{array}\right] \qquad \begin{aligned} 3x_1 + 2x_2 + x_3 &= 3 \\ \boxed{2x_1} + x_2 + x_3 &= 0 \\ \boxed{6x_1} + 2x_2 + 4x_3 &= 6. \end{aligned}$$

Step 1. ***Elimination of*** x_1 from the second and third equations by adding

$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 0 & -\frac{1}{3} & \frac{1}{3} & -2 \\ 0 & -2 & 2 & 0 \end{array}\right] \begin{array}{l} \\ \text{Row 2} - \frac{2}{3}\text{ Row 1} \\ \text{Row 3} - 2\text{ Row 1} \end{array} \qquad \begin{aligned} 3x_1 + 2x_2 + x_3 &= 3 \\ -\tfrac{1}{3}x_2 + \tfrac{1}{3}x_3 &= -2 \\ \boxed{-2x_2} + 2x_3 &= 0. \end{aligned}$$

Step 2. ***Elimination of*** x_2 from the third equation gives

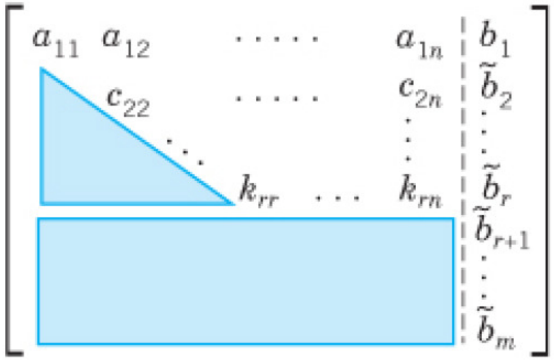
$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 0 & -\frac{1}{3} & \frac{1}{3} & -2 \\ 0 & 0 & 0 & 12 \end{array}\right] \begin{array}{l} \\ \\ \text{Row 3} - 6\text{ Row2} \end{array} \qquad \begin{aligned} 3x_1 + 2x_2 + x_3 &= 3 \\ -\tfrac{1}{3}x_2 + \tfrac{1}{3}x_3 &= -2 \\ 0 &= 12. \end{aligned}$$

The false statement $0 = 12$ shows that the system has no solution.

GAUSS ELIMINATION

Row Echelon Form and Information From It

At the end of the Gauss elimination (before the back substitution) the row echelon form of the augmented matrix will be



Here, $r \leqq m$ and $a_{11} \neq 0$, $c_{22} \neq 0$, $\cdots$, $k_{rr} \neq 0$,

a. **Exactly one solution.** if $r = n$ and $\tilde{b}_{r+1}, \cdots, \tilde{b}_m$, if present, are zero. To get the solution, solve the nth equation corresponding to (8) (which is $k_{nn}x_n = \tilde{b}_n$) for x_n, then the $(n-1)$st equation for x_{n-1}, and so on up the line. See Example 2, where $r = n = 3$ and $m = 4$.

b. **Infinitely many solutions.** if $r < n$ and $\tilde{b}_{r+1}, \cdots, \tilde{b}_m$, if present, are zero. To obtain any of these solutions, choose values of $x_{r+1}, \cdots, x_n$ arbitrarily. Then solve the rth equation for x_r, then the $(r-1)$st equation for x_{r-1}, and so on up the line. See Example 3.

c. **No solution.** if $r < m$ and one of the entries $\tilde{b}_{r+1}, \cdots, \tilde{b}_m$ is not zero. See Example 4, where $r = 2 < m = 3$ and $\tilde{b}_{r+1} = \tilde{b}_3 = 12$.

HOMEWORK IN 7.3

- HW1. Problem 7
- HW2. Problem 10
- HW3. Problem 18
- HW4. Problem 22

CHAP. 7.4
LINEAR INDEPENDENCE. RANK OF A MATRIX. VECTOR SPACE

*To answer the questions of **existence and uniqueness of solutions**, the **rank of a matrix** and **vector space** shall be presented.*

LINEAR INDEPENDENCE

Linear Independence and Dependence of Vectors

Given any set of m vectors $\mathbf{a}_{(1)}, \cdots, \mathbf{a}_{(m)}$ (with the same number of components), a **linear combination** of these vectors is an expression of the form

$$c_1\mathbf{a}_{(1)} + c_2\mathbf{a}_{(2)} + \cdots + c_m\mathbf{a}_{(m)}$$

where $c_1, c_2, \cdots, c_m$ are any scalars. Now consider the equation

$$c_1\mathbf{a}_{(1)} + c_2\mathbf{a}_{(2)} + \cdots + c_m\mathbf{a}_{(m)} = \mathbf{0}. \tag{1}$$

Clearly, this vector equation (1) holds if we choose all c_j's zero, because then it becomes $\mathbf{0} = \mathbf{0}$. If this is the only m-tuple of scalars for which (1) holds, then our vectors $\mathbf{a}_{(1)}, \cdots, \mathbf{a}_{(m)}$ are said to form a *linearly independent set* or, more briefly, we call them **linearly independent**. Otherwise, if (1) also holds with scalars not all zero, we call these vectors **linearly dependent**, because then we can express (at least) one of them as a linear combination of the others. For instance, if (1) holds with, say, $c_1 \neq 0$, we can solve (1) for $\mathbf{a}_{(1)}$:

$$\mathbf{a}_{(1)} = k_2\mathbf{a}_{(2)} + \cdots + k_m\mathbf{a}_{(m)} \qquad \text{where } k_j = -c_j / c_1 .$$

(Some k_j's may be zero. Or even all of them, namely, if $\mathbf{a}_{(1)} = \mathbf{0}$.)

Why is it important? ***Existence and uniqueness of solutions.***

RANK OF A MATRIX

Rank of a Matrix

DEFINITION

The **rank** of a matrix **A** is the maximum number of linearly independent row vectors of **A**. It is denoted by rank **A**.

Our further discussion will show that the rank of a matrix is an important key concept for understanding general properties of matrices and linear systems of equations.

THEOREM 1

Row-Equivalent Matrices

Row-equivalent matrices have the same rank.

VECTOR SPACE

EXAMPLE 3 Determination of Rank

For the matrix in Example 2 we obtain successively

$$\mathbf{A} = \begin{bmatrix} 3 & 0 & 2 & 2 \\ -6 & 42 & 24 & 54 \\ 21 & -21 & 0 & -15 \end{bmatrix} \quad \text{(given)}$$

$$\begin{bmatrix} 3 & 0 & 2 & 2 \\ 0 & 42 & 28 & 58 \\ 0 & -21 & -14 & -29 \end{bmatrix} \begin{array}{l} \\ \text{Row } 2 + 2 \text{ Row1} \\ \text{Row } 3 - 7 \text{ Row1} \end{array}$$

$$\begin{bmatrix} 3 & 0 & 2 & 2 \\ 0 & 42 & 28 & 58 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} \\ \\ \text{Row } 3 + \frac{1}{2} \text{ Row } 2 \end{array}$$

THEOREM 2

Linear Independence and Dependence of Vectors

p vectors with n components each are linearly independent if the matrix with these vectors as row vectors has rank p, but they are linearly dependent if that rank is less than p.

RANK OF A MATRIX

THEOREM 3

Rank in Terms of Column Vectors

The rank r of a matrix $\mathbf{A}$ equals the maximum number of linearly independent **column** *vectors of* $\mathbf{A}$.

Hence $\mathbf{A}$ and its transpose $\mathbf{A}^T$ have the same rank.

THEOREM 4

Linear Dependence of Vectors

p vectors with $n < p$ components are always linearly dependent.

VECTOR SPACE

Vector Space

A **vector space** is a (nonempty) set V of vectors such that with any two vectors **a** and **b** in V all their linear combinations $\alpha\mathbf{a} + \beta\mathbf{b}$ (α, β any real numbers) are elements of V, and these vectors satisfy the laws

The maximum number of linearly independent vectors in V is called the **dimension** of V and is denoted by dim V. Here we assume the dimension to be finite; infinite dimension will be defined in Sec. 7.9.

A linearly independent set in V consisting of a maximum possible number of vectors in V is called a **basis** for V. Thus the number of vectors of a basis for V equals dim V.

The set of all linear combinations of given vectors $\mathbf{a}_{(1)}, \cdots, \mathbf{a}_{(p)}$ with the same number of components is called the **span** of these vectors. Obviously, a span is a vector space.

By a **subspace** of a vector space V we mean a nonempty subset of V (including V itself) that forms itself a vector space with respect to the two algebraic operations (addition and scalar multiplication) defined for the vectors of V.

VECTOR SPACE

THEOREM 5

Vector Space R^n

The vector space R^n consisting of all vectors with n components (n real numbers) has dimension n.

PROOF

A basis of n vectors is $\mathbf{a}_{(1)} = [1 \;\; 0 \;\; \cdots \;\; 0]$, $\mathbf{a}_{(2)} = [0 \;\; 1 \;\; 0 \;\; \cdots \;\; 0]$, $\cdots$, $\mathbf{a}_{(n)} = [0 \;\; \cdots \;\; 0 \;\; 1]$.

In the case of a matrix **A** we call the span of the row vectors the **row space** of **A** and the span of the column vectors the **column space** of **A**.

THEOREM 6

Row Space and Column Space

The row space and the column space of a matrix **A** *have the same dimension, equal to rank* **A**.

Finally, for a given matrix **A** the solution set of the homogeneous system $\mathbf{Ax} = \mathbf{0}$ is a vector space, called the **null space** of **A**, and its dimension is called the **nullity** of **A**. In the next section we motivate and prove the basic relation

$$\text{rank } A + \text{nullity } A = \text{Number of columns of } A. \tag{6}$$

BACKUPS

- p vectors $x_{(1)}$ … $x_{(p)}$ (with n components) are linearly independent if the matrix with rows $x_{(1)}$ … $x_{(p)}$ has rank p; they are linearly dependent if the rank is less than p.

- p vectors with $n < p$ components are always linearly dependent.

- The vectors space $\mathbf{R}^n$ consisting of all vectors with n components has rank n.

BACKUPS

- **Problem 8**

$$\begin{matrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{matrix}$$

the first row is subtracted from row 2 and also from row 3, the result is:

$$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{matrix}$$

Then subtract row 2 from row 3 and get:

$$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}$$

This is sometimes called canonical form, and is obviously of rank 3, hence given vectors are linearly Independent.

BACKUPS

- **Problem 9**

$$\begin{matrix} 8 & -4 \\ -2 & 1 \\ 6 & -3 \end{matrix} \rightarrow \begin{matrix} 2 & -1 \\ 2 & -1 \\ 2 & -1 \end{matrix} \rightarrow \begin{matrix} 2 & -1 \\ 0 & 0 \\ 0 & 0 \end{matrix}$$

Therefore,

$$\text{Rank} = 1$$

BACKUPS

➢ Problem 10

$$
\begin{array}{ccccc} 3 & - & 1 & = & 5 \\ 2 & - & 4 & = & 6 \\ 10 & + & 0 & = & 14 \\ \hline \end{array}
\rightarrow
\begin{array}{ccccc} 1 & - & \frac{1}{3} & = & \frac{5}{3} \\ 1 & - & 2 & = & 3 \\ 1 & + & 0 & = & \frac{7}{5} \\ \hline \end{array}
\rightarrow
\begin{array}{ccccc} 1 & - & \frac{1}{3} & = & \frac{5}{3} \\ 0 & - & \frac{5}{3} & = & \frac{4}{3} \\ 0 & + & \frac{1}{3} & = & \left(\frac{7}{5} - \frac{5}{3}\right) \\ \hline \end{array}
$$

$$
\begin{array}{ccccc} 1 & - & \frac{1}{3} & = & \frac{5}{3} \\ 0 & + & 1 & = & -\frac{4}{5} \\ 0 & + & \frac{1}{3} & = & -\frac{4}{15} \\ \hline \end{array}
\rightarrow
\begin{array}{ccccc} 1 & - & \frac{1}{3} & = & \frac{5}{3} \\ 0 & + & 1 & = & -\frac{4}{5} \\ 0 & + & 1 & = & -\frac{4}{5} \\ \hline \end{array}
\rightarrow
\begin{array}{ccccc} 1 & + & \frac{1}{3} & = & \frac{5}{3} \\ 0 & + & 1 & = & -\frac{4}{5} \\ 0 & + & 0 & = & 0 \\ \hline \end{array}
$$

Therefore,

$$R = 2$$

HOMEWORK IN 7.4

- HW1. Problem 2
- HW2. Problem 6
- HW3. Problem 22
- HW4. Problem 27

CHAP. 7.5
SOLUTIONS OF LINEAR SYSTEMS: EXISTENCE, UNIQUENESS

Linear independence (or rank) shall answer the questions of ***existence, uniqueness, and general structure of the solution set of linear systems****.*

SOLUTION EXISTENCE AND UNIQUENESS

- A linear system of equations in n unknowns has a unique solution if the coefficient matrix and the augmented matrix have the same rank n, and infinitely many solution if that common rank is less than n. The system has no solution if those two matrices have different rank.

THEOREM 1

Fundamental Theorem for Linear Systems

a. Existence. *A linear system of m equations in n unknowns* $x_1, \cdots, x_n$

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned} \tag{1}$$

is **consistent**, *that is, has solutions, if and only if the coefficient matrix* **A** *and the augmented matrix* $\tilde{\mathbf{A}}$ *have the same rank. Here,*

$$\mathrm{A} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \quad \text{and} \quad \tilde{\mathrm{A}} = \left[\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdots & \cdot & \cdot \\ a_{m1} & \cdots & a_{mn} & b_m \end{array}\right]$$

SOLUTION EXISTENCE AND UNIQUENESS

b. Uniqueness. *The system* (1) *has precisely one solution if and only if this common rank r of* **A** *and* $\tilde{\mathbf{A}}$ *equals n.*

c. Infinitely many solutions. *If this common rank r is less than n, the system* (1) *has infinitely many solutions. All of these solutions are obtained by determining r suitable unknowns* (*whose submatrix of coefficients must have rank r*) *in terms of the remaining* $n - r$ *unknowns, to which arbitrary values can be assigned.* (See Example 3 in Sec. 7.3.)

d. Gauss elimination (Sec. 7.3). *If solutions exist, they can all be obtained by the Gauss elimination.* (This method will automatically reveal whether or not solutions exist; see Sec. 7.3.)

HOMOGENEOUS LINEAR SYSTEM

THEOREM 2

Homogeneous Linear System

A homogeneous linear system

$$
(4) \qquad \begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0
\end{aligned}
$$

always has the **trivial solution** $x_1 = 0, \cdots, x_n = 0$. *Nontrivial solutions exist if and only if rank* $\mathbf{A} < n$. *If* rank $\mathbf{A} = r < n$, *these solutions, together with* $\mathbf{x} = \mathbf{0}$, *form a vector space* (see Sec. 7.4) *of dimension* $n - r$, called the **solution space** of (4) .

In particular, if $\mathbf{x}_{(1)}$ *and* $\mathbf{x}_{(2)}$ *are solution vectors of* (4) , *then* $\mathbf{x} = \mathbf{c}_1\mathbf{x}_{(1)} + \mathbf{c}_2\mathbf{x}_{(2)}$ *with any scalars* c_1 *and* c_2 *is a solution vector of* (4) . (This ***does not hold*** for nonhomogeneous systems. Also, the term *solution space* is used for *homogeneous* systems only.)

The solution space of (4) is also called the **null space** of **A** because $\mathbf{Ax} = \mathbf{0}$ for every $\mathbf{x}$ in the solution space of (4) . Its dimension is called the **nullity** of **A**. Hence Theorem 2 states that

$$
(5) \qquad \text{rank } \mathbf{A} + \text{nullity } \mathbf{A} = n
$$

where n is the number of unknowns (number of columns of **A**).

NONHOMOGENEOUS LINEAR SYSTEM

THEOREM 4

Nonhomogeneous Linear System

If a nonhomogeneous linear system (1) *is consistent, then all of its solutions are obtained as*

(6) $$\mathbf{x} = \mathbf{x}_0 + \mathbf{x}_h$$

where $\mathbf{x}_0$ *is any* (fixed) *solution of* (1) *and* $\mathbf{x}_h$ *runs through all the solutions of the corresponding homogeneous system* (4) .

CHAP. 7.7
DETERMINANTS. CRAMER'S RULE

Determinant were originally introduced for solving linear systems but delivers important implications in eigenvalue problems.

DETERMINANTS

A **determinant of order** n is a scalar associated with an $n \times n$ (hence ***square!***) matrix $\mathbf{A} = [a_{jk}]$, which is written

(1)
$$D = \det \mathbf{A} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

(3a)
$$D = a_{j1}C_{j1} + a_{j2}C_{j2} + \cdots + a_{jn}C_{jn} \qquad (j = 1, 2, \cdots, \text{ or } n)$$

or

(3b)
$$D = a_{1k}C_{1k} + a_{2k}C_{2k} + \cdots + a_{nk}C_{nk} \qquad (k = 1, 2, \cdots, \text{ or } n)$$

Here,

$$C_{jk} = (-1)^{j+k} M_{jk}$$

and M_{jk} is a determinant of order $n - 1$, namely, the determinant of the submatrix of **A** obtained from **A**

M_{jk} is called the **minor** *of* a_{jk} *in* D, and C_{jk} the **cofactor** *of* a_{jk} *in* D.

For later use we note that (3) may also be written in terms of minors

(4a)
$$D = \sum_{k=1}^{n} (-1)^{j+k} a_{jk} M_{jk} \qquad (j = 1, 2, \cdots, \text{ or } n)$$

DETERMINANTS

EXAMPLE 2 Expansions of a Third-Order Determinant

$$D = \begin{vmatrix} 1 & 3 & 0 \\ 2 & 6 & 4 \\ -1 & 0 & 2 \end{vmatrix} = 1\begin{vmatrix} 6 & 4 \\ 0 & 2 \end{vmatrix} - 3\begin{vmatrix} 2 & 4 \\ -1 & 2 \end{vmatrix} + 0\begin{vmatrix} 2 & 6 \\ -1 & 0 \end{vmatrix} = 1(12-0) - 3(4+4) + 0(0+6) = -12.$$

This is the expansion by the first row. The expansion by the third column is

$$D = 0\begin{vmatrix} 2 & 6 \\ -1 & 0 \end{vmatrix} - 4\begin{vmatrix} 1 & 3 \\ -1 & 0 \end{vmatrix} + 2\begin{vmatrix} 1 & 3 \\ 2 & 6 \end{vmatrix} = 0 - 12 + 0 = -12,$$

Verify that the other four expansions also give the value -12.

EXAMPLE 3 Determinant of a Triangular Matrix

$$\begin{vmatrix} -3 & 0 & 0 \\ 6 & 4 & 0 \\ -1 & 2 & 5 \end{vmatrix} = -3\begin{vmatrix} 4 & 0 \\ 2 & 5 \end{vmatrix} = -3 \cdot 4 \cdot 5 = -60.$$

Inspired by this, can you formulate a little theorem on determinants of triangular matrices? Of diagonal matrices?

DETERMINANTS

THEOREM 1

Behavior of an *n*th-Order Determinant under Elementary Row Operations

a. *Interchange of two rows multiplies the value of the determinant by* -1.

b. *Addition of a multiple of a row to another row does not alter the value of the determinant.*

c. *Multiplication of a row by a nonzero constant* c *multiplies the value of the determinant by* c. (This holds also when $c = 0$, but gives no longer an elementary row operation.)

THEOREM 2

Further Properties of *n*th-Order Determinants

(a)–(c) *in Theorem 1 hold also for columns.*

(d) ***Transposition*** *leaves the value of a determinant unaltered.*

(e) ***A zero row or column*** *renders the value of a determinant zero.*

(f) ***Proportional rows or columns*** *render the value of a determinant zero. In particular, a determinant with two identical rows or columns has the value zero.*

DETERMINANTS

THEOREM 3

Rank in Terms of Determinants

An $m \times n$ *matrix* $\mathbf{A} = [a_{jk}]$ *has rank* $r \geqq 1$ *if and only if* $\mathbf{A}$ *has an* $r \times r$ *submatrix with nonzero determinant, whereas every square submatrix with more than r rows that* $\boldsymbol{A}$ *has (or does not have!) has determinant equal to zero.*

In particular, if $\mathbf{A}$ *is square,* $n \times n$*, it has rank* n *if and only if*

$$\det \mathbf{A} \neq 0.$$

CRAMER'S RULE

THEOREM 4

Cramer's Theorem (Solution of Linear Systems by Determinants)

a. *If a linear system of n equations in the same number of unknowns* $x_1, \cdots, x_n$

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots\cdots\cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots\cdots\cdots + a_{2n}x_n &= b_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots&\cdots\cdots\cdots\cdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots\cdots\cdots + a_{nn}x_n &= b_n \end{aligned}$$

has a nonzero coefficient determinant $D = \det \mathbf{A}$, *the system has precisely one solution. This solution is given by the formulas*

$$(7) \qquad x_1 = \frac{D_1}{D}, \qquad x_2 = \frac{D_2}{D}, \cdots, \qquad x_n = \frac{D_n}{D} \qquad \text{(Cramer's rule)}$$

where D_k *is the determinant obtained from D by replacing in D the kth column by the column with the entries* $b_1, \cdots, b_n$.

b. *Hence if the system* (6) *is **homogeneous** and* $D \neq 0$, *it has only the trivial solution* $x_1 = 0$, $x_2 = 0, \cdots, x_n = 0$. *If* $D = 0$, *the homogeneous system also has nontrivial solutions.*

CRAMER'S RULE

$$x = \frac{\begin{matrix} 20 & 2 & 3 \\ 13 & 3 & 1 \\ 0 & 6 & 2 \end{matrix}}{91} = \frac{120 + 234 - (52 + 120)}{91} = \frac{354 - 172}{91} = \frac{182}{91} = 2$$

$$y = \frac{\begin{matrix} 1 & 20 & 3 \\ 7 & 13 & 1 \\ 1 & 0 & 2 \end{matrix}}{91} = \frac{26 + 20 - 39 - 280}{91} = \frac{-319 + 46}{91} = \frac{-273}{91} = -3$$

$$z = \frac{\begin{matrix} 1 & 2 & 20 \\ 7 & 3 & 13 \\ 1 & 6 & 0 \end{matrix}}{91} = \frac{26 + 840 - (60 + 78)}{91} = \frac{866 - 138}{91} = \frac{728}{91} = 8$$

Therefore,

$$x = 2, \quad y = -3, \quad z = 8$$

HOMEWORK IN 7.7

- HW1. Problem 11
- HW2. Problem 19
- HW3. Problem 24 (b)