ENGINEERING MATHEMATICS II

010.141

Byeng Dong Youn (윤병동)

CHAP. 11
Fourier Series, Integrals, and Transforms

ABSTRACT OF CHAP. 11

- ***Fourier Analysis in Chap. 11*** concerns periodic phenomena—thinking of rotating parts of machines, alternating electric currents, or the motion of planets. However, the underlying ideas can also be extended to non-periodic phenomena.
  - **Fourier series**: Infinite series designed to represent general periodic functions in terms of simple ones (e.g., sines and cosines).
  - **Fourier series** is more general than Taylor series because many discontinuous periodic functions of practical interest can be developed in Fourier series.
  - **Fourier integrals** and **Fourier Transforms** extend the ideas and techniques of Fourier series to non-periodic functions and have basic applications to PDEs.

CHAP. 11.1
FOURIER SERIES

Infinite series designed to represent general periodic functions in terms of simple ones like cosines and sines.

PERIODIC FUNCTIONS

Fourier series are the basic tool for representing periodic functions, which play an important role in applications. A function $f(x)$ is called a **periodic function** if $f(x)$ is defined for all real x (perhaps except at some points, such as $x = \pm\pi/2, \pm 3\pi/2, \cdots$ for $\tan x$) and if there is some positive number p, called a **period** of $f(x)$, such that

(1) $$f(x+p) = f(x) \qquad \text{for all } x.$$

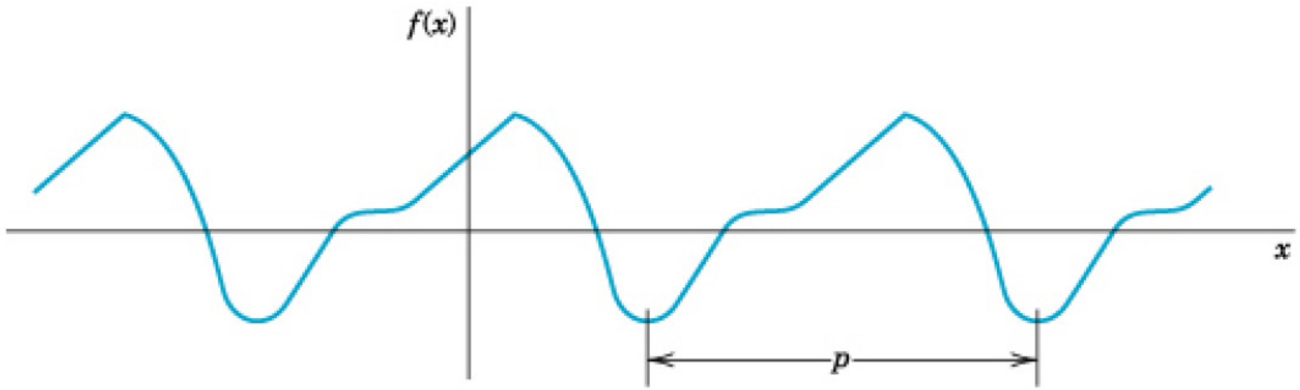


Fig. 255. Periodic function

- The smallest period is called a **fundamental period**.

PERIODIC FUNCTIONS

Our problem in the first few sections of this chapter will be the representation of various ***functions f(x) of period 2π*** in terms of the simple functions

(3) $1, \quad \cos x, \quad \sin x, \quad \cos 2x, \quad \sin 2x, \cdots, \quad \cos nx, \quad \sin nx, \cdots.$

All these functions have the period 2π. They form the so-called **trigonometric system**. Figure 256 shows the first few of them (except for the constant 1, which is periodic with any period).

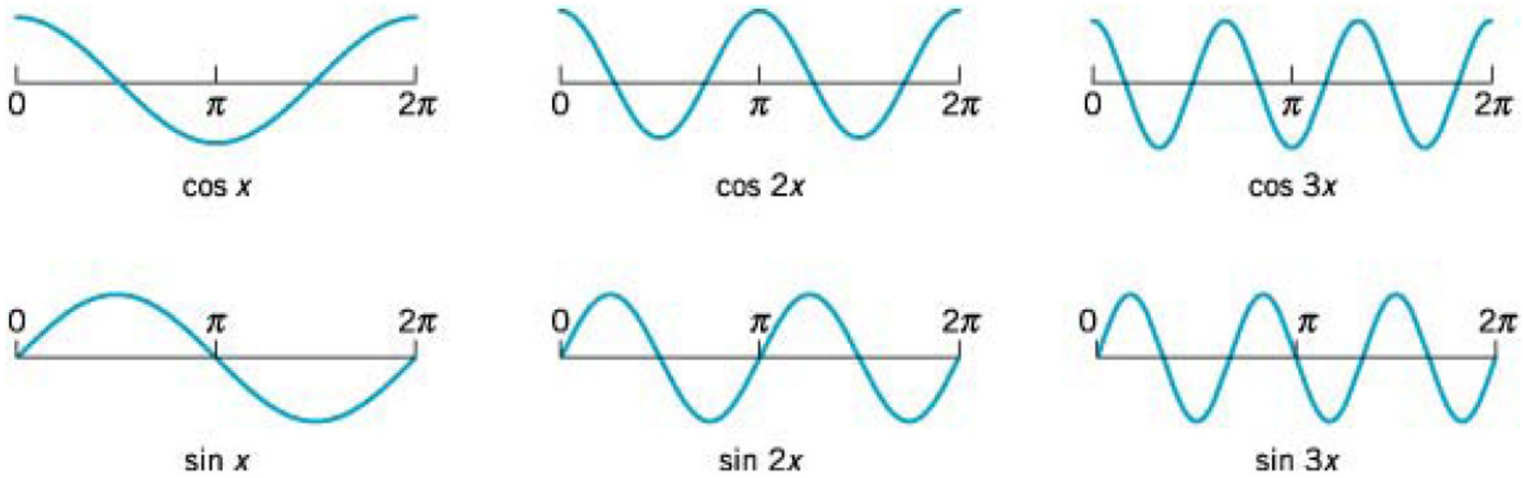


Fig. 256. Cosine and sine functions having the period 2π

FOURIER SERIES

Now suppose that $f(x)$ is a given function of period 2π and is such that it can be **represented** by a series (4) , that is, (4) converges and, moreover, has the sum $f(x)$. Then, using the equality sign, we write

(5) $$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

and call (5) the **Fourier series** of $f(x)$. We shall prove that in this case the coefficients of (5) are the so-called **Fourier coefficients** of $f(x)$, given by the **Euler formulas**

(6)

$$(a) \quad a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\,dx$$

$$(b) \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx \qquad n = 1, 2, \cdots$$

$$(c) \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx \qquad n = 1, 2, \cdots$$

EXAMPLE

EXAMPLE 1 Periodic Rectangular Wave (Fig. 257a)

Find the Fourier coefficients of the periodic function $f(x)$ in Fig. 257a. The formula is

$$(7)\qquad f(x) = \begin{cases} -k & \text{if} \quad -\pi < x < 0 \\ k & \text{if} \quad 0 < x < \pi \end{cases} \quad \text{and} \quad f(x + 2\pi) = f(x).$$

Functions of this kind occur as external forces acting on mechanical systems, electromotive forces in electric circuits, etc. (The value of $f(x)$ at a single point does not affect the integral; hence we can leave $f(x)$ undefined at $x = 0$ and $x = \pm\pi$.)

Solution: From (6a) we obtain $a_0 = 0$.

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-k)\cos nx\,dx + \int_{0}^{\pi} k\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[-k\frac{\sin nx}{n}\Big|_{-\pi}^{0} + k\frac{\sin nx}{n}\Big|_{0}^{\pi}\right] = 0$$

because $\sin nx = 0$ at $-\pi$, 0, and π for all $n = 1, 2, \cdots$. Similarly, from (6c) we obtain

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-k)\sin nx\,dx + \int_{0}^{\pi} k\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[k\frac{\cos nx}{n}\Big|_{-\pi}^{0} - k\frac{\cos nx}{n}\Big|_{0}^{\pi}\right].$$

Since $\cos(-\alpha) = \cos\alpha$ and $\cos 0 = 1$, this yields

$$b_n = \frac{k}{n\pi}[\cos 0 - \cos(-n\pi) - \cos n\pi + \cos 0] = \frac{2k}{n\pi}(1 - \cos n\pi).$$

EXAMPLE

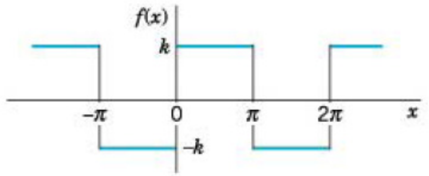
$$1 - \cos n\pi = \begin{cases} 2 & \text{for odd } n, \\ 0 & \text{for even } n. \end{cases} \qquad b_1 = \frac{4k}{\pi}, \qquad b_2 = 0, \qquad b_3 = \frac{4k}{3\pi}, \qquad b_4 = 0, \qquad b_5 = \frac{4k}{5\pi}, \cdots.$$

Since the a_n are zero, the Fourier series of $f(x)$ is

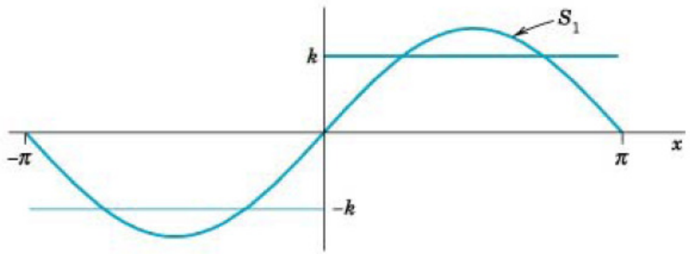
(8) $$\frac{4k}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right).$$

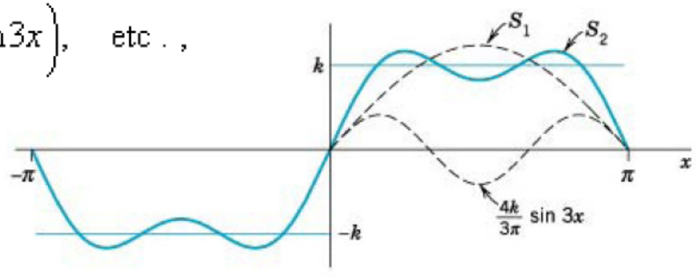
The partial sums are

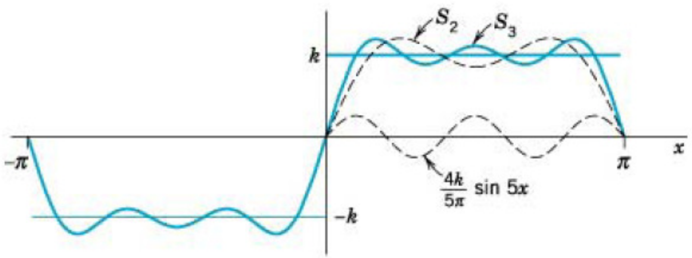
$$s_1 = \frac{4k}{\pi}\sin x, \qquad s_2 = \frac{4k}{\pi}\left(\sin x + \frac{1}{3}\sin 3x\right), \qquad \text{etc.},$$



(a) The given function $f(x)$ (Periodic rectangular wave)







(b) The first three partial sums of the corresponding Fourier series

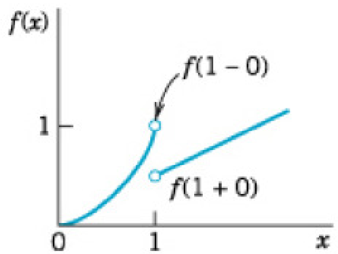
Fig. 257. Example 1

THEOREM 1

Orthogonality of the Trigonometric System (3)

The trigonometric system (3) *is orthogonal on the interval* $-\pi \leqq x \leqq \pi$ (hence also on $0 \leqq x \leqq 2\pi$ or any other interval of length 2π because of periodicity); *that is, the integral of the product of any two functions in* (3) *over that interval is* 0, *so that for any integers n and m,*

$$(9)\qquad \begin{aligned} &(a) \quad \int_{-\pi}^{\pi} \cos nx \cos mx \, dx = 0 \qquad (n \neq m) \\ &(b) \quad \int_{-\pi}^{\pi} \sin nx \sin mx \, dx = 0 \qquad (n \neq m) \\ &(c) \quad \int_{-\pi}^{\pi} \sin nx \cos mx \, dx = 0 \qquad (n \neq m \text{ or } n = m). \end{aligned}$$



THEOREM 2

Representation by a Fourier Series

Let f(x) be periodic with period 2π *and piecewise continuous* (see Sec. 6.1) *in the interval* $-\pi \leqq x \leqq \pi$. *Furthermore, let f(x) have a left-hand derivative and a right-hand derivative at each point of that interval. Then the Fourier series* (5) *of f(x)* [with coefficients (6)] *converges. Its sum is f(x), except at points* x_0 *where f(x) is discontinuous. There the sum of the series is the average of the left- and right-hand limits* of f(x) at* x_0.

CONVERGENCE AND SUM OF FOURIER SERIES

Proof: For continuous f(x) with continuous first and second order derivatives.

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}\left(\frac{f(x)\sin nx}{n}\right)' dx - \frac{1}{\pi}\int_{-\pi}^{\pi} f' \frac{\sin nx}{n} dx$$

$$= \underbrace{\frac{1}{\pi}\frac{f(x)\sin nx}{n}\bigg|_{-\pi}^{\pi}}_{0} - \frac{1}{\pi n}\int_{-\pi}^{\pi} f' \sin nx\, dx$$

$$= -\frac{1}{\pi n}\int_{-\pi}^{\pi} f' \sin nx\, dx$$

CONVERGENCE AND SUM OF FOURIER SERIES (cont)

Repeating the process:

$$a_n = -\frac{1}{\pi n^2} \int_{-\pi}^{\pi} f''(x) \cos nx \, dx$$

Since f" is continuous on [– π, π]

$$\left| f''(x) \right| < M$$

$$\left| a_n \right| = \frac{1}{\pi n^2} \left| \int_{-\pi}^{\pi} f''(x) \cos nx \, dx \right| < \frac{1}{\pi n^2} \int_{-\pi}^{\pi} M \, dx$$

$$\left| a_n \right| < \frac{2\pi M}{\pi n^2} = \frac{2M}{n^2}$$

CONVERGENCE AND SUM OF FOURIER SERIES (cont)

Similarly for

$$|b_n| < \frac{2M}{n^2}$$

Hence

$$|f(x)| < |a_0| + 2M\left(1+1+\frac{1}{2^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{3^2}+\cdots\right)$$

which converges.

HOMEWORK IN 11.1

- HW1. Problem 9
- HW2. Problem 15

CHAP. 11.2
FUNCTIONS OF ANY PERIOD P=2L

general periodic functions with P=2L.

FUNCTIONS OF ANY PERIOD P = 2L

we thus obtain from (1) the **Fourier series** of the function $f(x)$ of period $2L$

(5)
$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{L} x + b_n \sin \frac{n\pi}{L} x \right)$$

with the **Fourier coefficients** of $f(x)$ given by the **Euler formulas**

(6)

(a) $$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\, dx$$

(b) $$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\, dx \qquad n = 1, 2, \cdots$$

(c) $$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\, dx \qquad n = 1, 2, \cdots .$$

EXAMPLE

EXAMPLE 1 Periodic Rectangular Wave

Find the Fourier series of the function (Fig. 259)

$$f(x) = \begin{cases} 0 & \text{if} \quad -2 < x < -1 \\ k & \text{if} \quad -1 < x < 1 \\ 0 & \text{if} \quad 1 < x < 2 \end{cases} \qquad p = 2L = 4, \quad L = 2.$$

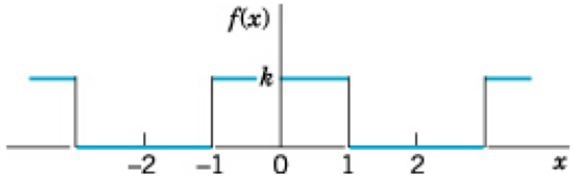


Fig. 259. Example 1

From (6a) we obtain $a_0 = k/2$ (verify!). From (6b) we obtain

$$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi x}{2}dx = \frac{1}{2}\int_{-1}^{1} k\cos\frac{n\pi x}{2}dx = \frac{2x}{n\pi}\sin\frac{n\pi}{2}.$$

Thus $a_n = 0$ if n is even and

$$a_n = 2k/n\pi \quad \text{if} \quad n = 1, 5, 9, \cdots, \qquad a_n = -2k/n\pi \quad \text{if} \quad n = 3, 7, 11, \cdots.$$

From (6c) we find that $b_n = 0$ for $n = 1, 2, \cdots$. Hence the Fourier series is

$$f(x) = \frac{k}{2} + \frac{2k}{\pi}\left(\cos\frac{\pi}{2}x - \frac{1}{3}\cos\frac{3\pi}{2}x + \frac{1}{5}\cos\frac{5\pi}{2}x - + \cdots\right).$$

FUNCTIONS OF ANY PERIOD (cont) P = 2L

Proof: Result obtained easily through change of scale.

$$v = \frac{\pi x}{L}$$

$$g(v) = a_0 + \sum_{n=1}^{\infty} a_n \cos nv + \sum_{n=1}^{\infty} b_n \sin nv$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(v) \cos nv \, dv = \frac{1}{\pi} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} \cdot \frac{\pi}{L} dx = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} dx$$

Same for a_0, b_n

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(v) dv = \frac{1}{2\pi} \int_{-L}^{L} f(x) \frac{\pi}{L} dx = \frac{1}{2L} \int_{-L}^{L} f(x) dx$$

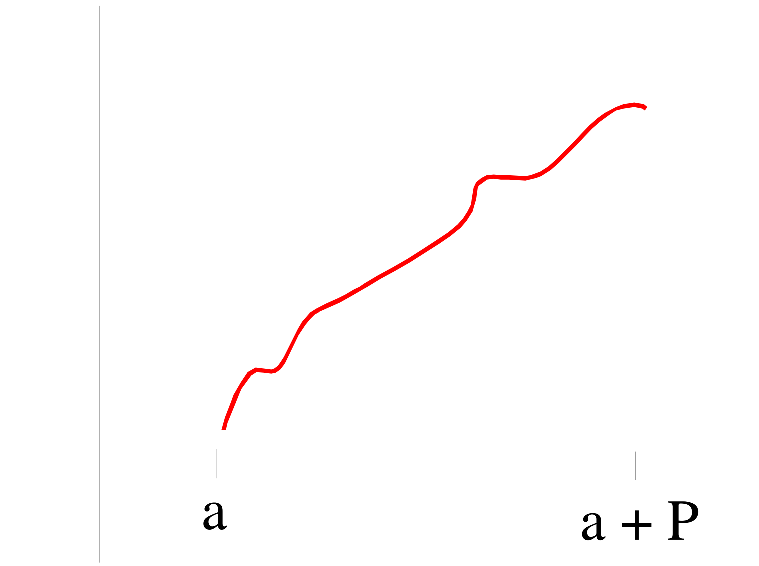
ANY INTERVAL (a, a + P)
P = PERIOD = 2L

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{2n\pi x}{P} + b_n \sin \frac{2n\pi x}{P} \right)$$

$$a_0 = \frac{1}{P} \int_a^{a+P} f(x)\,dx$$

$$a_n = \frac{2}{P} \int_a^{a+P} f(x) \cos \frac{2n\pi x}{P}\,dx$$

$$b_n = \frac{2}{P} \int_a^{a+P} f(x) \sin \frac{2n\pi x}{P}\,dx$$



HOMEWORK IN 11.2

- HW1. Problem 3
- HW2. Problem 5

CHAP. 11.3
EVEN AND ODD FUNCTIONS. HALF-RANGE EXPANSIONS

Take advantage of function properties.

EVEN AND ODD FUNCTIONS

Examples: **x^4, cos x**

f(x) is an ***even*** *function* of x, if f(-x) = f(x). For example, f(x) = x sin(x), then

$$f(-x) = -x \sin(-x) = f(x)$$

and so we can conclude that x sin (x) is an even function.

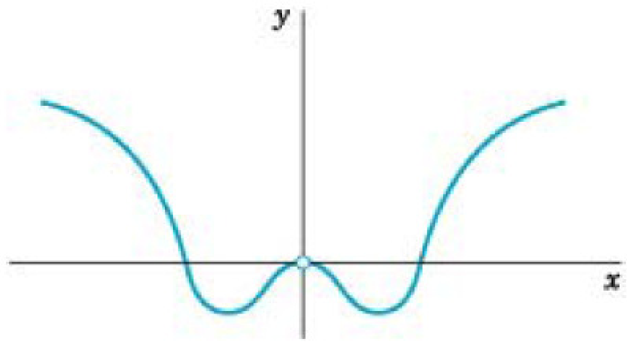


Fig. 262. Even function

EVEN AND ODD FUNCTIONS

Properties

1. If g(x) is an even function

$$\int_{-L}^{L} g(x)\,dx = 2\int_{0}^{L} g(x)\,dx$$

2. If h(x) is an odd function

$$\int_{-L}^{L} h(x)\,dx = 0$$

3. The product of an even and odd function is odd

THEOREMS

THEOREM 1

Fourier Cosine Series, Fourier Sine Series

The Fourier series of an ***even*** *function of period 2L is a* **"Fourier cosine series"**

(1) $$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{L} x \qquad (f \text{ even})$$

with coefficients (note: integration from 0 to L only!)

(2) $$a_0 = \frac{1}{L}\int_0^L f(x)dx, \quad a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx, \qquad n = 1, 2, \cdots.$$

The Fourier series of an ***odd*** *function of period 2L is a* **"Fourier sine series"**

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} x \qquad (f \text{ odd})$$

with coefficients

(4) $$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx.$$

THEOREM 2

Sum and Scalar Multiple

The Fourier coefficients of a sum $f_1 + f_2$ are the sums of the corresponding Fourier coefficients of f_1 and f_2.

The Fourier coefficients of cf are c times the corresponding Fourier coefficients of f.

EXAMPLE

EXAMPLE 3 Sawtooth Wave

Find the Fourier series of the function (Fig. 266)

$$f(x) = x + \pi \quad \text{if} \quad -\pi < x < \pi \quad \text{and} \quad f(x + 2\pi) = f(x).$$

Solution:

We have $f = f_1 + f_2$, where $f_1 = x$ and $f_2 = \pi$.

The Fourier coefficients of f_2 are zero, except for the first one (the constant term), which is π.

Since f_1 is odd, $a_n = 0$ for $n = 1, 2, \cdots$, and

$$b_n = \frac{2}{\pi}\int_0^{\pi} f_1(x) \sin nx \, dx = \frac{2}{\pi}\int_0^{\pi} x \sin nx \, dx.$$

Integrating by parts, we obtain

$$b_n = \frac{2}{\pi}\left[\frac{-x\cos nx}{n}\Bigg|_0^{\pi} + \frac{1}{n}\int_0^{\pi}\cos nx \, dx\right] = -\frac{2}{n}\cos n\pi.$$

Hence $b_1 = 2$, $b_2 = -2/2$, $b_3 = 2/3$, $b_4 = -2/4$, $\cdots$, and the Fourier series of $f(x)$ is

$$f(x) = \pi + 2\left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - + \cdots\right).$$

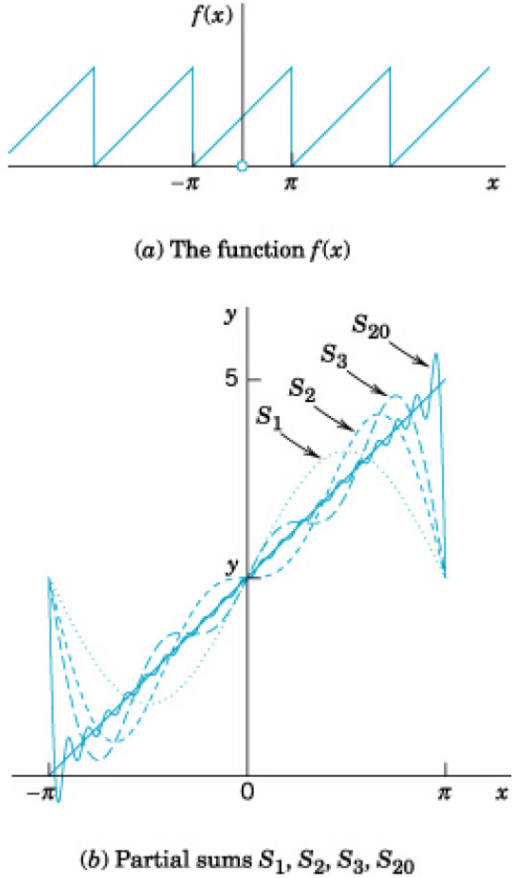


Fig. 266. Example 3

HOMEWORK IN 11.3

- HW1. Problem 11
- HW2. Problem 15

CHAP. 11.5
FORCED OSCILLATIONS

Connections with ODEs and PDEs.

Forced Oscillations

EXAMPLE 1

Forced Oscillations under a Nonsinusoidal Periodic Driving Force

In (1), let $m = 1$ (gm), $c = 0.05$ (gm/sec), and $k = 25$ (gm/sec^2), so that (1) becomes

$$y'' + 0.05y' + 25y = r(t) \tag{2}$$

where $r(t)$ is measured in gm · cm/sec^2. Let (Fig. 273)

$$r(t) = \begin{cases} t + \dfrac{\pi}{2} & \text{if} \quad -\pi < t < 0, \\ -t + \dfrac{\pi}{2} & \text{if} \quad 0 < t < \pi, \end{cases} \qquad r(t + 2\pi) = r(t).$$

Find the steady-state solution $y(t)$.

Forced Oscillations

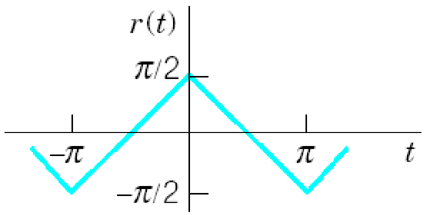


Fig. 273. Force in Example 1

Solution. We represent $r(t)$ by a Fourier series, finding

$$r(t) = \frac{4}{\pi}\left(\cos t + \frac{1}{3^2}\cos 3t + \frac{1}{5^2}\cos 5t + \cdots\right) \tag{3}$$

(take the answer to Prob. 11 in Problem Set 11.3 minus $\frac{1}{2}\pi$ and write t for x). Then we consider the ODE

$$y'' + 0.05y' + 25y = \frac{4}{n^2\pi}\cos nt \qquad (n = 1, 3, \cdots) \tag{4}$$

whose right side is a single term of the series (3). From Sec. 2.8 we know that the steady-state solution $y_n(t)$ of (4) is of the form

$$y_n = A_n \cos nt + B_n \sin nt. \tag{5}$$

Forced Oscillations

By substituting this into (4) we find that

$$(6) \qquad A_n = \frac{4(25 - n^2)}{n^2 \pi D_n}, \qquad B_n = \frac{0.2}{n \pi D_n}, \qquad \text{where} \qquad D_n = (25 - n^2)^2 + (0.05n)^2.$$

Since the ODE (2) is linear, we may expect the steady-state solution to be

$$(7) \qquad y = y_1 + y_3 + y_5 + \cdots$$

where y_n is given by (5) and (6). In fact, this follows readily by substituting (7) into (2) and using the Fourier series of $r(t)$, provided that termwise differentiation of (7) is permissible. (Readers already familiar with the notion of uniform convergence [Sec. 15.5] may prove that (7) may be differentiated term by term.)

From (6) we find that the amplitude of (5) is (a factor $\sqrt{D_n}$ cancels out)

$$C_n = \sqrt{A_n^2 + B_n^2} = \frac{4}{n^2 \pi \sqrt{D_n}}.$$

Forced Oscillations

Numeric values are

$$C_1 = 0.0531$$

$$C_3 = 0.0088$$

$$C_5 = 0.2037$$

$$C_7 = 0.0011$$

$$C_9 = 0.0003.$$

Figure 274 shows the input (multiplied by 0.1) and the output. For $n = 5$ the quantity D_n is very small, the denominator of C_5 is small, and C_5 is so large that y_5 is the dominating term in (7). Hence the output is almost a harmonic oscillation of five times the frequency of the driving force, a little distorted due to the term y_1, whose amplitude is about 25% of that of y_5. You could make the situation still more extreme by decreasing the damping constant c. Try it.

Forced Oscillations

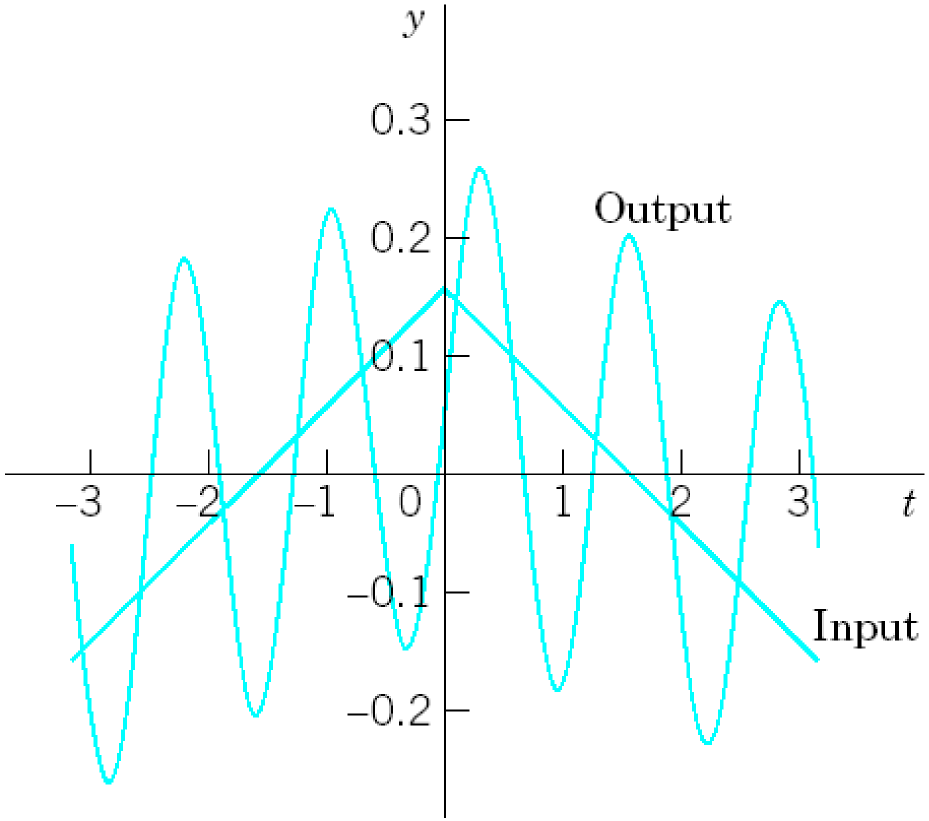


Fig. 274. Input and steady-state output in Example 1

HOMEWORK IN 11.5

- HW1. Problem 6

CHAP. 11.6 APPROXIMATION BY TRIGONOMETRIC POLYNOMIALS

Useful in approximation theory.

APPROXIMATION BY TRIGONOMETRIC POLYNOMIALS

Consider a function f(x), periodic of period 2π. Consider an approximation of f(x),

$$f(x) \approx F(x) = A_0 + \sum_{n=1}^{N} A_n \cos nx + B_n \sin nx$$

The **total square error of F**

$$E = \int_{-\pi}^{\pi} (f - F)^2 dx$$

is minimum when F's coefficients are the Fourier coefficients.

Read Page 503 to see the derivation procedure of Eq. (6)

PARSEVAL'S THEOREM

The square error, call it E^*, is

(6) $$E^* = \int_{-\pi}^{\pi} f^2\,dx - \pi\left[2{a_0}^2 + \sum_{n=1}^{N}({a_n}^2 + {b_n}^2)\right].$$

where $a_n = A_n$ and $b_n = B_n$.

Parseval's theorem:

$$2a_0^2 + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right) \le \frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)\,dx$$

HOMEWORK IN 11.6

- HW1. Problem 2

CHAP. 11.7
FOURIER INTEGRAL

Extension of the Fourier series method to non-periodic functions.

FOURIER INTEGRALS

Since many problems involve functions that are nonperiodic and are of interest on the whole x-axis, we ask what can be done to extend the method of Fourier series to such functions. This idea will lead to "Fourier integrals."

we start from a special function f_L of period $2L$ and see what happens to its Fourier series if we let $L \to \infty$.

EXAMPLE 1 Rectangular Wave

Consider the periodic rectangular wave $f_L(x)$ of period $2L > 2$ given by

$$f_L(x) = \begin{cases} 0 & \text{if} \quad -L < x < -1 \\ 1 & \text{if} \quad -1 < x < 1 \\ 0 & \text{if} \quad 1 < x < L. \end{cases}$$

FOURIER INTEGRALS

The left part of Fig. 277 shows this function for $2L = 4, 8, 16$ as well as the nonperiodic function $f(x)$, we obtain from f_L if we let $L \to \infty$,

$$f(x) = \lim_{L \to \infty} f_L(x) = \begin{cases} 1 & \text{if } -1 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

We now explore what happens to the Fourier coefficients of f_L as L increases. Since f_L is even, $b_n = 0$ for n. For a_n the Euler formulas (6) , Sec. 11.2, give

$$a_0 = \frac{1}{2L}\int_{-1}^{1} dx = \frac{1}{L}, \quad a_n = \frac{1}{L}\int_{-1}^{1} \cos\frac{n\pi x}{L} dx = \frac{2}{L}\int_{0}^{1} \cos\frac{n\pi x}{L} dx = \frac{2}{L}\frac{\sin(n\pi / L)}{n\pi / L} .$$

FOURIER INTEGRALS

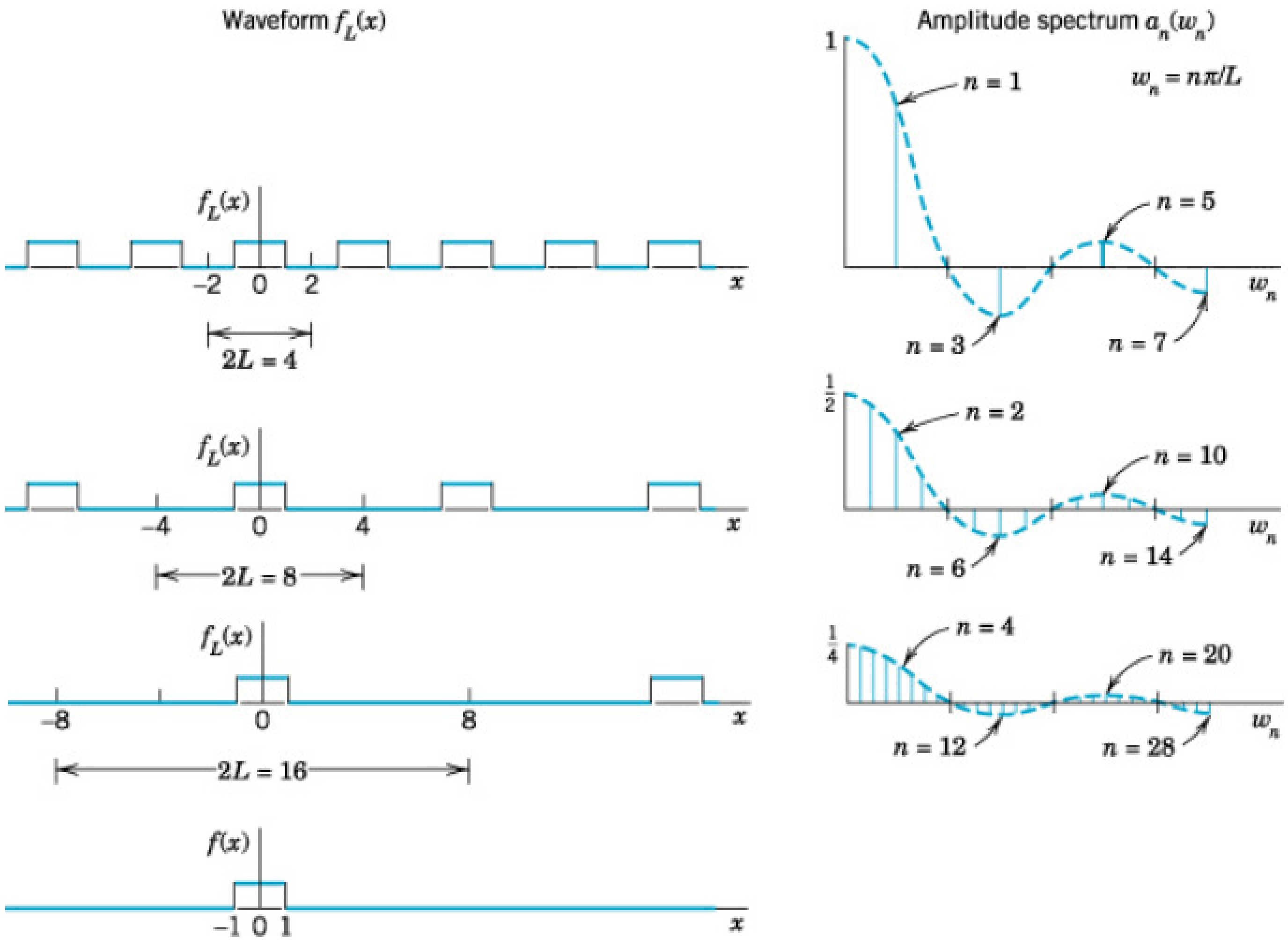


Fig. 277. Waveforms and amplitude spectra in Example 1

FOURIER COSINE AND SINE INTEGRALS

We now consider any periodic function $f_L(x)$ of period $2L$ that can be represented by a Fourier series

$$f_L(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos w_n x + b_n \sin w_n x), \quad w_n = \frac{n\pi}{L}$$

and find out what happens if we let $L \to \infty$. Together with Example 1 the present calculation will

If we insert a_n and b_n from the Euler formulas (6) , Sec. 11.2, and denote the variable of integration by v, the Fourier series of $f_L(x)$ becomes

$$f_L(x) = \frac{1}{2L}\int_{-L}^{L} f_L(v)dv + \frac{1}{L}\sum_{n=1}^{\infty}\left[\cos w_n x \int_{-L}^{L} f_L(v)\cos w_n v dv + \sin w_n x \int_{-L}^{L} f_L(v)\sin w_n v dv\right].$$

We now set

$$\Delta w = w_{n+1} - w_n = \frac{(n+1)\pi}{L} - \frac{n\pi}{L} = \frac{\pi}{L}.$$

Then $1/L = \Delta w/\pi$, and we may write the Fourier series in the form

$$(1)\quad f_L(x) = \frac{1}{2L}\int_{-L}^{L} f_L(v)dv + \frac{1}{\pi}\sum_{n=1}^{\infty}\left[(\cos w_n x)\Delta w \int_{-L}^{L} f_L(v)\cos w_n v dv + (\sin w_n x)\Delta w \int_{-L}^{L} f_L(v)\sin w_n v dv\right]$$

This representation is valid for any fixed L, arbitrarily large, but finite.

FOURIER COSINE AND SINE INTEGRALS

We now let $L \to \infty$ and assume that the resulting nonperiodic function

$$f(x) = \lim_{L \to \infty} f_L(x)$$

is **absolutely integrable** on the x-axis; that is, the following (finite!) limits exist:

(2) $$\lim_{a \to -\infty} \int_a^0 |f(x)|\,dx + \lim_{b \to \infty} \int_0^b |f(x)|\,dx \left(\text{written} \int_{-\infty}^{\infty} |f(x)|\,dx\right).$$

Then $1/L \to 0$,

(3) $$f(x) = \frac{1}{\pi}\int_0^{\infty}\left[\cos wx \int_{-\infty}^{\infty} f(v)\cos wv\,dv + \sin wx \int_{-\infty}^{\infty} f(v)\sin wv\,dv\right]dw.$$

If we introduce the notations

(4) $$A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos wv\,dv, \quad B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin wv\,dv$$

we can write this in the form

(5) $$f(x) = \int_0^{\infty} [A(w)\cos wx + B(w)\sin wx]\,dw.$$

This is called a representation of $f(x)$ by a **Fourier integral**.

FOURIER COSINE AND SINE INTEGRALS

$$f(x) = \int_0^{\infty} \left[A(\omega)\cos\omega x + B(\omega)\sin\omega x \right] d\omega$$

$$A(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v)\cos(\omega v)\,dv$$

$$B(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v)\sin(\omega v)\,dv$$

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx \right)$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cdot \cos nx\,dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cdot \sin nx\,dx$$

THEOREM1: EXISTENCE

If f(x) is piecewise continuous in every finite interval and has a right hand and left hand derivative at every point and if

$$\lim_{a \to -\infty} \int_a^0 |f(x)|\,dx + \lim_{b \to \infty} \int_0^b |f(x)|\,dx$$

exists, then f(x) can be represented by the Fourier integral.

The F.I. equals the average of the left-hand and right-hand limit of f(x) where f(x) is discontinuous.

FOURIER COSINE AND SINE INTEGRALS

For an **even** or **odd** function the F.I. becomes much simpler.

If f(x) is **even**

$$A(\omega) = \frac{2}{\pi}\int_0^\infty f(v)\cos\omega v\, dv$$

$$f(x) = \int_0^\infty A(\omega)\cos(\omega x)\, d\omega \quad \text{Fourier cosine integral}$$

If f(x) is **odd**

$$B(\omega) = \frac{2}{\pi}\int_0^\infty f(v)\sin\omega v\, dv$$

$$f(x) = \int_0^\infty B(\omega)\sin(\omega x)\, d\omega \quad \text{Fourier sine integral}$$

EXAMPLE

Consider $f(x) = e^{-kx}$ $x > 0,\ k > 0$

Evaluate the Fourier cosine integral A(ω) and sine integral B(ω).

For Fourier cosine integral, $A(w) = \frac{2}{\pi}\int_0^\infty e^{-kv}\cos wv\, dv$

Integration by parts gives $\int e^{-kv}\cos wv\, dv = -\frac{k}{k^2+w^2}e^{-kv}\left(-\frac{w}{k}\sin wv + \cos wv\right)$

$$A(w) = \frac{2k/\pi}{k^2+w^2}$$

Fourier cosine integral is

$$f(x) = e^{-kx} = \frac{2k}{\pi}\int_0^\infty \frac{\cos wx}{k^2+w^2}dw \quad (x > 0, k > 0)$$

EXAMPLE

For Fourier sine integral, $B(w) = \frac{2}{\pi}\int_0^\infty e^{-kv} \sin wv \, dv$

Integration by parts gives $\int e^{-kv} \sin wv \, dv = -\frac{w}{k^2 + w^2} e^{-kv}\left(\frac{k}{w}\sin wv + \cos wv\right)$

$$B(w) = \frac{2w/\pi}{k^2 + w^2}$$

Fourier sine integral is

$$f(x) = e^{-kx} = \frac{2}{\pi}\int_0^\infty \frac{w \sin wx}{k^2 + w^2} dw$$

Laplace integrals

$$\int_0^\infty \frac{\cos wx}{k^2 + w^2} dw = \frac{\pi}{2k} e^{-kx} \quad (x > 0, k > 0)$$

$$\int_0^\infty \frac{w \sin wx}{k^2 + w^2} dw = \frac{\pi}{2} e^{-kx} \quad (x > 0, k > 0)$$

HOMEWORK IN 11.7

- HW1. Problem 1
- HW2. Problem 7

CHAP. 11.8
FOURIER COSINE AND SINE TRANSFORMS

An integral transform is a transformation in the form of an integral that produces from given functions new functions depending on a different variable.

FOURIER SINE AND COSINE TRANSFORMS

For an **even** function, the Fourier integral is the Fourier cosine integral

$$f(x) = \int_0^\infty A(\omega)\cos(\omega x)\, d\omega \qquad A(\omega) = \frac{2}{\pi}\int_{-\infty}^{\infty} f(v)\cos\omega v\, dv = \sqrt{\frac{2}{\pi}}\,\hat{f}_c(\omega)$$

Then

$$\hat{f}_c(\omega) = \sqrt{\frac{\pi}{2}}\,A(\omega) = \sqrt{\frac{\pi}{2}}\,\frac{2}{\pi}\int_0^\infty f(v)\cos\omega v\, dv = \sqrt{\frac{2}{\pi}}\int_0^\infty f(v)\cos\omega v\, dv$$

$$f(x) = \int_0^\infty \sqrt{\frac{2}{\pi}}\,\hat{f}_c(\omega)\cos\omega x\, d\omega = \sqrt{\frac{2}{\pi}}\int_0^\infty \hat{f}_c(\omega)\cos\omega x\, d\omega$$

$\hat{f}_c(\omega)$ is defined as the Fourier cosine transform of f.

f is the inverse Fourier cosine transform of $\hat{f}_c$.

$$\mathcal{F}_c\{f\} = \hat{f}_c, \quad \mathcal{F}_c^{-1}\{\hat{f}_c\} = f$$

FOURIER SINE AND COSINE TRANSFORMS (cont)

Similarly for **odd** function

$$F\ \text{sine}\ T \rightarrow \hat{f}_s(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\ \sin \omega x\ dx$$

$$Inverse\ F\ \text{sine}\ T \rightarrow f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_s(\omega)\ \sin \omega x\ d\omega$$

$$\mathcal{F}_s\{f\} = \hat{f}_s, \quad \mathcal{F}_s^{-1}\{\hat{f}_s\} = f$$

NOTATION AND PROPERTIES

$$\mathcal{F}_c\{f\} = \hat{f}_c, \quad \mathcal{F}_s\{f\} = \hat{f}_s$$

$$\mathcal{F}_c^{-1}\{\hat{f}_c\} = f, \quad \mathcal{F}_s^{-1}\{\hat{f}_s\} = f$$

(1) $F_c\{af + bg\} = aF_c\{f\} + bF_c\{g\}$

(2) $F_s\{af + bg\} = aF_s\{f\} + bF_s\{g\}$

(3) $\mathcal{F}_c\{f'(x)\} = \omega \mathcal{F}_s\{f\} - \sqrt{\frac{2}{\pi}}\, f(0)$

(4) $\mathcal{F}_s\{f'(x)\} = -\omega \mathcal{F}_c\{f\}$

(5) $\mathcal{F}_c\{f''\} = -\omega^2 \mathcal{F}_c\{f\} - \sqrt{\frac{2}{\pi}}\, f'(0)$

(6) $\mathcal{F}_s\{f''\} = -\omega^2 \mathcal{F}_s\{f\} + \sqrt{\frac{2}{\pi}}\, \omega f'(0)$

TRANSFORMS OF DERIVATIVES

Let $f(x)$ be continuous and absolutely integrable on the x-axis, let $f'(x)$ be piecewise continuous on every finite interval, and let let $f(x) \to 0$ *as* $x \to \infty$. *Then*

(8)

$$\text{(a)}\quad \mathcal{F}_c\left\{f'(x)\right\} = w\,\mathcal{F}_s\left\{f(x)\right\} - \sqrt{\frac{2}{\pi}}\,f(0),$$

$$\text{(b)}\quad \mathcal{F}_s\left\{f'(x)\right\} = -w\,\mathcal{F}_c\left\{f(x)\right\}.$$

PROOF

$$\begin{aligned}\mathcal{F}_c\left\{f'(x)\right\} &= \sqrt{\frac{2}{\pi}}\int_0^\infty f'(x)\cos wx\,dx \\ &= \sqrt{\frac{2}{\pi}}\left[f(x)\cos wx\Big|_0^\infty + w\int_0^\infty f(x)\sin wx\,dx\right] \\ &= -\sqrt{\frac{2}{\pi}}\,f(0) + w\,\mathcal{F}_s\left\{f(x)\right\};\end{aligned}$$

and similarly,

$$\begin{aligned}\mathcal{F}_s\left\{f'(x)\right\} &= \sqrt{\frac{2}{\pi}}\int_0^\infty f'(x)\sin wx\,dx \\ &= \sqrt{\frac{2}{\pi}}\left[f(x)\sin wx\Big|_0^\infty - w\int_0^\infty f(x)\cos wx\,dx\right] \\ &= 0 - w\,\mathcal{F}_c\left\{f(x)\right\}.\end{aligned}$$

FOURIER TRANSFORM

The F.I. is:

$$f(x) = \int_0^\infty \left[A(\omega)\cos\ \omega x + B(\omega)\sin\ \omega x \right] dx$$

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\ \cos\ \omega v\ dv$$

$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\ \sin\ \omega v\ dv$$

Replacing

$$f(x) = \frac{1}{\pi}\int_0^\infty \left[\int_{-\infty}^{\infty} f(v)\left[\cos\omega v\cos\omega x + \sin\omega v\sin\omega x\right] dv \right] d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{\left[\int_{-\infty}^{\infty} f(v)\cos(\omega x - \omega v)\, dv \right]}_{G(\omega)=\text{Even function in } \omega} d\omega$$

FOURIER TRANSFORM

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(v)\cos\omega(x-v)\,dv\right]d\omega$$

$$g(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\underbrace{\left[\int_{-\infty}^{\infty} f(v)\sin\omega(x-v)\,dv\right]}_{F(\omega)=\text{odd function in }\omega}d\omega = 0$$

Now, $f(v)\cos(wx-wv) + if(v)\sin(wx-wv) = f(v)e^{i(wx-wv)}$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[F(\omega)+iG(\omega)\right]d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(v)\; e^{i\omega(x-v)}dv\right]d\omega \qquad \text{Complex Fourier Integral}$$

FOURIER TRANSFORM (cont)

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \underbrace{\left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(v)\, e^{-i\omega v} dv \right]}_{\hat{f}(\omega) \equiv \text{Fourier Transform of } f} e^{i\omega x} d\omega$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega)\, e^{i\omega x} d\omega$$

NOTATION AND PROPERTIES

$$\mathcal{F}\{af + bg\} = a\mathcal{F}\{f\} + b\mathcal{F}\{g\}$$

$$\mathcal{F}\{f'\} = i\omega \mathcal{F}\{f\}$$

$$\mathcal{F}\{f''\} = -\omega^2 \mathcal{F}\{f\}$$

$$\mathcal{F}\{f * g\} = \sqrt{2\pi}\mathcal{F}\{f\} + \mathcal{F}\{g\}$$