# 11.6 Orthogonal Series. Generalized Fourier Series.

→ includes (Fourier - Legendre Series, Fourier - Bessel Series)

Let $y_0, y_1 \dots$ be orthogonal. and let $f(x)$ be a function that can be represented by a convergent series

$$f(x) = \sum_{m=0}^{\infty} a_m y_m(x) = a_0 y_0 + a_1 y_1(x) + \cdots$$

orthogonal series or orthogonal expansion or generalized Fourier series

If $y_m$ : eigenfunctions of Sturm-Liouville problem.

→ eigenfunction expansion.

need to find $a_n$ (Fourier coefficient)

$$(f, y_n) = \int_a^b r f y_n \, dx = \int_a^b r\left(\sum_{m=0}^{\infty} a_m y_m\right) y_n \, dx = \sum_{m=0}^{\infty} a_m \int_a^b r y_m y_n \, dx = \sum_{m=0}^{\infty} a_m (y_m, y_n)$$

↳ multiply $r y_n$ and integrate.

$=0$, except when $m=n$.

$$a_n (y_n, y_n) = a_n \|y_n\|^2 \quad \text{thus} \quad (f, y_n) = a_n \|y_n\|^2$$

$$\therefore a_m = \frac{(f, y_m)}{\|y_m\|^2} = \frac{1}{\|y_m\|^2} \int_a^b r(x) f(x) y_m(x) \, dx$$

"Desired formula for the Fourier constants"

Ex. 1) Fourier - Legendre Series is an eigenfunction expansion. (page. 178) → p.179

$$f(x) = \sum_{m=0}^{\infty} a_m P_m(x) = a_0 P_0 + a_1 P_1(x) + a_2 P_2(x) \dots = a_0 + a_1 x + a_2\left(\tfrac{3}{2}x^2 - \tfrac{1}{2}\right) + \cdots$$

$$a_m = \frac{1}{\|P_m\|^2} \int_{-1}^{1} f(x) P_m(x) \, dx \qquad \|P_m\| = \sqrt{\int_{-1}^{1} P_m^2(x)\, dx} = \sqrt{\frac{2}{2m+1}} \quad (\text{tricky!})$$

$$P_n(x) = \sum_{m=0}^{M} (-1)^m \frac{(2n-2m)!}{2^n m! (n-m)! (n-2m)!} x^{n-2m}$$

$P_0(x) = 1$
$P_1(x) = x$
$P_2(x) = \frac{1}{2}(3x^2 - 1)$
$P_3(x) \dots$

$$a_m = \frac{2m+1}{2} \int_{-1}^{1} f(x) P_m(x) \, dx$$

for example if $f(x) = \sin \pi x$

$$a_m = \frac{2m+1}{2} \int_{-1}^{1} \sin \pi x \, P_m(x) \, dx$$

thus. $a_1 = \frac{3}{2} \int_{-1}^{1} x \sin \pi x \, dx = \frac{3}{\pi}$

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$$\therefore \sin \pi x = 0.95493 P_1(x) - 1.15824 P_3(x) + \cdots + 3 \times 10^{-7} P_{13}$$

why even-numbered ~~$P_n(x)$~~ coefficients are zero. → $\int_{-1}^{1} (\text{odd} \cdot \text{even}) = 0$

Ex 2) Fourier - Bessel series.

$$\tilde{x}^2 \ddot{J}_n(\tilde{x}) + \tilde{x} \dot{J}_n(\tilde{x}) + (\tilde{x}^2 - n^2) J_n(\tilde{x}) = 0$$

⇒ ($\dot{J}_n(\tilde{x}) = \frac{dJ_n}{d\tilde{x}}$, $\ddot{J}_n(\tilde{x}) = \frac{d^2 J_n}{d\tilde{x}^2}$)

set $\tilde{x} = kx$, $d\tilde{x} = k\,dx$, $\dot{J}_n = \frac{dJ_n}{d\tilde{x}} = \frac{dJ_n}{dx} \cdot \frac{1}{k} = J_n' \frac{1}{k}$, $\ddot{J}_n = \frac{1}{k^2} \frac{d^2 J_n}{dx^2}$

$J'' = \ddot{J}_n k^2$
$J' = k \dot{J}_n$

$x = \frac{\tilde{x}}{k}$

$$x^2 J''(kx) + x J_n'(kx) + (k^2 x^2 - n^2) J_n(kx) = 0$$

$(x J_n'(kx))'$

$x J''(kx) + J_n'(kx) + (k^2 x - \frac{n^2}{x}) J_n$

divide by $x$, → $(x J_n'(kx))' = x J_n''(kx) + J_n'(kx)$

Let $kx = \tilde{x}$: $\left(\frac{\tilde{x}}{k} J_n'(\tilde{x})\right)' = \frac{\tilde{x}}{k} J_n''(\tilde{x}) + J_n'(\tilde{x})$

$(k \dot{J}_n)' = k \dot{J}_n + k x \dot{J}_n = x k^2 \ddot{J}_n + k \dot{J}_n$

되이

$\lambda = k^2$

$$\therefore (x J_n'(kx))' + \left(-\frac{n^2}{x} + \lambda x\right) J_n(kx) = 0$$

$p(x) = x$
$q(x) = -\frac{n^2}{x}$
$r(x) = x$

→ S.L. problem $p(a) = 0$, $p(R) \neq 0$, $p(R) \neq 0$, $y_n'(R) y_m(b) - y_m'(R) y_n(b) = 0$

at $y_{m,n}(R) = 0$

$J_n(kR) = 0$ ← many zero at $\alpha_{n,m}$ according to figure (A85)

$kR = \alpha_{n,m}$

$k_{n,m} = \frac{\alpha_{n,m}}{R}$

orthogonal requirement

since $p(0) = 0$ → orthogonal on an interval $0 \le x \le R$.

↳ need $y_n(b) = 0$, $J_n(kR) = 0$

5/2.

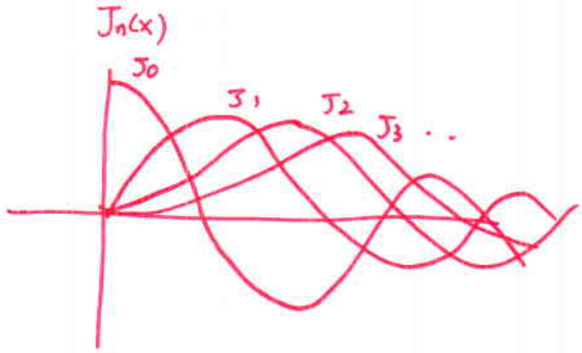
# Bessel Function

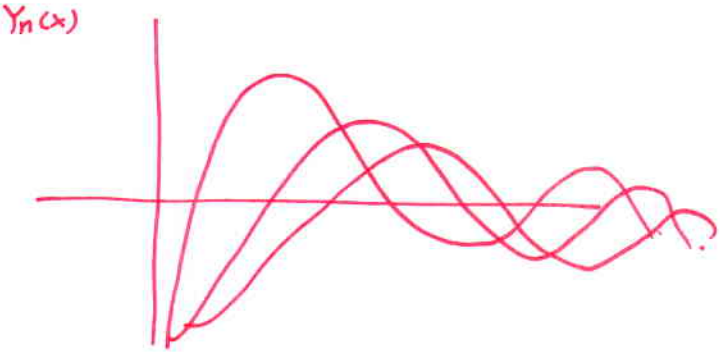
$$x^2y'' + xy' + (x^2 - \nu^2)y = 0$$

$$y = c_1 J_\nu(x) + c_2 Y_\nu(x)$$

$$J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(1+\nu+n)} \left(\frac{x}{2}\right)^{2n+\nu}$$ Bessel Function of the 1st kind of order $\nu$

$$Y_\nu(x) = \frac{(\cos \nu\pi) J_\nu(x) - J_{-\nu}(x)}{\sin \nu\pi}$$ Bessel Function of the 2nd kind of order $\nu$




Let $\nu = n$ $\quad n = 0, 1, 2 \ldots$

$$x^2y'' + xy' + (\alpha^2x^2 - n^2)y = 0.$$

$$\tilde{x}^2 \ddot{J}_n(\tilde{x}) + \tilde{x}\dot{J}_n(\tilde{x}) + (\tilde{x}^2 - n^2)J_n(\tilde{x}) = 0$$

$\dot{J}_n(\tilde{x}) = \dfrac{dJ_n}{d\tilde{x}}$

$\ddot{J}_n(\tilde{x}) = \dfrac{d^2J_n}{d\tilde{x}^2}$

Set $\tilde{x} = kx \qquad d\tilde{x} = k\,dx$

$$\dot{J}_n = \frac{dJ_n}{d\tilde{x}} = \frac{1}{k}\frac{dJ_n}{dx} = \frac{1}{k}J_n'$$

$$\ddot{J}_n = \frac{d^2J_n}{d\tilde{x}^2} = \frac{1}{k^2}J_n''$$

$$\therefore\ (kx)^2 \cdot \frac{1}{k^2}J_n'' + kx \cdot \frac{1}{k}J_n' + \left[(kx)^2 - n^2\right] J_n(kx) = 0$$

$$x^2J_n'' + xJ_n' + ((kx)^2 - n^2)\,J_n(kx) = 0$$

devide by $x$

$$\underline{xJ_n'' + J_n'} + \left(k^2x - \frac{n^2}{x}\right) J_n(kx)) = 0.$$

$k_1 f(a) + k_2 f'(b) = 0$

[illegible] $(xJ_n')' + \left(k^2x - \frac{n^2}{x}\right) J_n = 0$

$$P(x) = x \qquad \lambda = k^2 \qquad r(x) = x \qquad q(x) = -\frac{n^2}{x}$$

since $P(0) = 0$, on the interval $0 \le x \le R$

~~We need~~ [illegible] $\boxed{J_n(kR) = 0}$

So. [illegible] $= 0$

$$y_n'(kR)\,y_m(kR) - y_m'(kR)\,y_n(kR) = 0.$$

$kR = \alpha_{n,m}$

$Y_n(x)$ = unbounded at $x = 0$

So, $J_n$ can be only solution.

$$k_{n,m} = \frac{\alpha_{n,m}}{R}$$

$$(xJ_n'(kx))' = J_n'(kx) + xJ_n''(kx) \text{ —— ①}$$

$$\left.\begin{aligned}\dot{J}_n(\tilde{x}) &= \frac{dJ_n(\tilde{x})}{d\tilde{x}} \\ \ddot{J}_n(\tilde{x}) &= \frac{d^2J_n(\tilde{x})}{d\tilde{x}^2}\end{aligned}\right) \quad \tilde{x} = kx \Rightarrow \quad \begin{aligned}\dot{J}_n(\tilde{x}) &= \frac{1}{k}\frac{dJ_n(\tilde{x})}{dx} = \frac{1}{k}J_n' \\ \ddot{J}_n(\tilde{x}) &= \frac{1}{k^2}\frac{d^2J_n(\tilde{x})}{dx^2} = \frac{1}{k^2}J_n''\end{aligned}$$

$$\therefore J_n'(\tilde{x}) = k\dot{J}_n(\tilde{x})$$

$$\therefore J_n''(\tilde{x}) = k^2\ddot{J}_n(\tilde{x})$$

from ①

Right side $\quad xJ_n''(kx) = \frac{\tilde{x}}{k}J_n''(kx) = k\tilde{x}\ddot{J}_n(\tilde{x})$ —— ②

$$J_n'(kx) = k\dot{J}_n(\tilde{x}) \text{ —— ③}$$

$$\therefore xJ_n''(kx) + J_n'(kx) = \underline{k\tilde{x}\ddot{J}_n(\tilde{x}) + k\dot{J}_n(\tilde{x})}_{④}$$

left side

$$\frac{d}{dx}(xJ_n'(\tilde{x})) = \frac{d}{dx}\left(\frac{\tilde{x}}{k}k\dot{J}_n(\tilde{x})\right)$$

$$= \frac{k}{d\tilde{x}}(\tilde{x}\dot{J}_n(\tilde{x}))$$

$$= k(\dot{J}_n(\tilde{x}) + \tilde{x}\ddot{J}_n)$$

$$= k\dot{J}_n(\tilde{x}) + k\tilde{x}\ddot{J}_n$$

$$= ④.$$

$$\therefore (xJ_n'(kx))' = J_n'(kx) + xJ_n''(kx)$$

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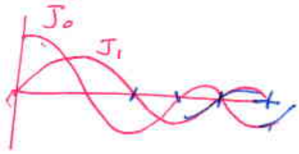
# Orthogonality of Bessel Functions.

$x^2y'' + xy' + (x^2-n^2)y = 0$

$J_n(x) = x^n \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+n} m!(n+m)!}$

page 190.




$J_n(k_{n,1}x)$, $J_n(k_{n,2}x)$ ... with $k_{n,m} = \frac{\alpha_{n,m}}{R}$

orthogonal set on the interval $0 \le x \le R$

w.r.t the weight function $r(x) = x$

(infinite many orthogonal set)

$$\int_0^R x J_n(k_{n,m}x) J_n(k_{n,j}x)\,dx = 0 \quad (j \ne m,\ n \text{ fixed})$$

i) $n$ is fixed. orthogonal set for $J_n$: $J_n(k_{n,1}x)$, $J_n(k_{n,2}x)$, $J_n(k_{n,3}x)$ ...

ii) fixed $k$, orthogonal set $J_0, J_1, J_2, J_3$ ...

Fourier-Bessel series.

$$f(x) = \sum_{m=1}^{\infty} a_m J_n(k_{n,m}x) = a_1 J_n(k_{n,1}x) + a_2 J_n(k_{n,2}x) + a_3 J_n(k_{n,3}x) + \dots$$

$$a_m = \frac{2}{R^2 J_{n+1}^2(\alpha_{n,m})} \int_0^R x f(x) J_n(k_{n,m}x)\,dx \qquad m = 1, 2 \dots$$

$$a_m = \frac{1}{\|y_m\|^2}\int_a^b r(x) f(x) y_m(x)\,dx$$

$y_m = J_n(k_{n,m}x)$

because

$$\| J_n(k_{n,m}x)\|^2 = \int_0^R x J_n^2(k_{n,m}x)\,dx = \frac{R^2}{2} J_{n+1}^2(k_{n,m}R)$$

(integration ??) w/o proof. $(xJ_n \cdot J_n)$

page 588, 589

$$\frac{d}{dx}(x^n J_n(x)) = x^n J_{n-1}(x)$$

$$\int x^n J_{n-1}(x) = x^n J_n(x)$$

$xJ_n \cdot J_n$

$$J_n(kx) = (kx)^n \sum_{n=0}^{\infty} \frac{(-1)^m (kx)^{2m}}{2^{2m+n} m!(m+n)!}$$

$$\frac{d}{dx} J_n(kx) = n(kx)^{n-1} k \sum_{n=0}^{\infty} \frac{(-1)^m (kx)^{2m}}{2^{2m+n} m!(m+n)}$$

$$+ (kx)^n \sum_{m=1} \frac{(-1)^m 2m (kx)^{2m-1} k}{2^{2m+n} m!(m+n)!}$$

$=$

Ex3) Special - Fourier - Bessel series,

$f(x) = 1-x^2$ and take $R=1$ and $n=0$ $\left[\text{in } f(x) = \sum_{m=1}^{\infty} a_m J_n(k_{n,m} x)\right]$ $\alpha_{n,m} = k_{n,m} R$

Set $[\lambda = \alpha_{0,m}]$ from Table.

Then $k_{n,m} = \alpha_{0,m} = \lambda = 2.404, 5.520, 8.654, 11.792, \ldots$ $a_m = \frac{2}{R^2 J_{n+1}^2(\alpha_{n,m})}\int_0^R x f(x) J_n(k_{n,m} x)$

$$a_m = \frac{2}{J_1^2(\lambda)} \int_0^1 x(1-x^2) J_0(\lambda x)\, dx$$

$[x J_1(\lambda x)]' = \lambda x J_0(\lambda x)$

$$= \frac{2}{J_1^2(\lambda)}\left[\frac{1}{\lambda}(1-x^2) x J_1(\lambda x)\Big|_0^1 - \frac{1}{\lambda}\int_0^1 x J_1(\lambda x)(-2x)\, dx\right]$$

$\frac{2}{\lambda^2} J_2(\lambda)$

$\left(\dfrac{4\left(\frac{0.52}{2.4} - 0\right)}{2.4^2 \cdot 0.52^2}\right) = 1.557$

$\left[a_m = \dfrac{4\left(\frac{2J_1}{\lambda} - J_0\right)}{\lambda^2 J_1^2(\lambda)}\right]$ $4\left(\frac{2}{\lambda} J_1(\lambda)\right) - J_0(\lambda)$

$$\therefore a_m = \frac{4 J_2(\lambda)}{\lambda^2 J_1^2(\lambda)} \quad (\lambda = \alpha_{0,m})$$

$\int [x^2 J_2(\lambda x)]' = \lambda x^2 J_1(\lambda x)$

recurrence $J_2 = 2x^{-1} J_1 - J_0$ $x$대신 $\lambda, \lambda_2$ 대입

$\int x^2 J_1(\lambda x) = \frac{1}{\lambda}\ldots$

$$\therefore 1-x^2 = 1.1081 J_0(2.405x) - 0.1398 J_0(5.520x) + \text{in terms of } J_0.$$

$J_{\nu-1} + J_{\nu+1} = \frac{2\nu}{x} J_\nu$

eigenfunction expansion.

$(x^\nu J_\nu)' = x^\nu J_{\nu-1}$

Mean Square Convergence. Completeness (?)

[Can be skipped!]

Sequence of function $f_k$ is convergent with limit $f$

if $\lim_{k\to\infty} \|f_k - f\| = 0$

or $\lim_{k\to\infty} \int_a^b r(x)[f_k(x) - f(x)]^2 dx = 0$

series $S_k(x)$ Converge and represent $f$ if

$$\lim_{k\to\infty} \int_a^b r(x)[S_k(x) - f(x)]^2 dx = 0$$

$y_n$ = orthonormal set $y_0, y_1, \ldots$

where $\boxed{S_k(x) = \sum_{m=0}^{k} a_m y_m(x)}$

$$\int_a^b r(x)[S_k(x) - f(x)]^2 dx = \int_a^b r S_k^2 dx - 2\int_a^b r f S_k dx + \int_a^b r f^2 dx$$

$$= \underbrace{\int_a^b r\left[\sum_{m=0}^{k} a_m y_m\right]^2 dx}_{\sum a_m^2} - 2\sum_{m=0}^{k} a_m \underbrace{\int_a^b r f y_m dx}_{a_m} + \int_a^b r f^2 dx$$

$$= -\sum_{m=0}^{k} a_m^2 + \int_a^b r f^2 dx \geq 0$$

Bessel's inequality $\sum_{m=0}^{k} a_m^2 \leq \|f\|^2 = \int_a^b r(x) f^2(x) dx$

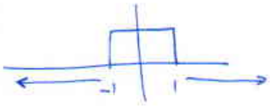
let $k \to \infty$ $\sum_{m=0}^{k} a_m^2 \leq \|f\|^2$

Parseval equality $\sum_{m=0}^{\infty} a_m^2 = \|f\|^2 = \int_a^b r(x) f^2(x) dx$

# 11.7. Fourier Integral.

extention to nonperiodic function

Ex 1. Rectangular wave



$$f_L(x) = \begin{cases} 0 & \text{if } -L < x < -1 \\ 1 & \text{if } -1 < x < 1 \\ 0 & \text{if } 1 < x < L \end{cases}$$

can extend to nonperiodic function : $f(x) = \lim_{L\to\infty} f_L(x) = \begin{cases} 1 & \text{if } -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

$f(x) = \frac{2}{L} \frac{\sin(n\pi/L)}{n\pi/L} \cdot \cos\frac{n\pi x}{L}$

since $f_L$ is even, $b_n = 0$ for all $n$

$a_0 = \frac{1}{2L}\int_{-1}^{1} dx = \frac{1}{L}$ $\qquad a_n = \frac{1}{L}\int_{-1}^{1} \cos\frac{n\pi x}{L}\, dx = \frac{2}{L}\int_0^1 \cos\frac{n\pi}{L}x\,dx = \frac{2}{L}\frac{\sin(n\pi/L)}{n\pi/L} = \frac{2}{L}\frac{\sin w_n}{w_n}$

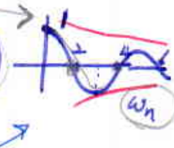
→ amplitude spectrum of $f_L$ because $|a_n|$ is the maximum amplitude of the wave $a_n \cos\left(\frac{n\pi x}{L}\right)$

Fig. 280.

$\left( \begin{array}{l} L = 2 \\ L = 4 \\ L = 16 \\ L = \infty \end{array} \right)$

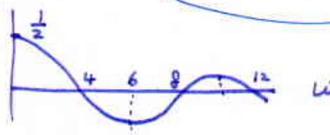
for $2L = 2^k$ amplitude $2^{k-1} - 1$

$\frac{2 \sin w_n}{L w_n}$ $\cos w_n x$ $2L = 4$ $L : 2$ $\left(\frac{\sin w_n}{w_n}\right)$




$2L = 8$

$L = 4$




$2L = 16$


→ gives an intuitive impression of what about to expect if

From Fourier Series to Fourier Integral

Consider any periodic function $f_L(x)$ of period $2L$

$f_L = a_0 + \sum_{n=1}^{\infty}(a_n \cos w_n t + b_n \sin w_n t)$ : $w_n = \frac{n\pi}{L}$

(fourier series)

$a_n, b_n$ from the Euler formula.

$$f_L(x) = \frac{1}{2L}\int_{-L}^{L} f_L(v)\,dv + \frac{1}{L}\sum_{n=1}^{\infty}\left[\cos w_n t \int_{-L}^{L} f_L(v)\cos w_n v \cdot dv + \sin w_n t \int_{-L}^{L} f_L(v)\sin w_n v\, dv\right]$$

$\Delta w = w_{n+1} - w_n = \frac{(n+1)\pi}{L} - \frac{n\pi}{L} = \frac{\pi}{L}$ $\qquad \therefore \frac{1}{L} = \frac{\Delta w}{\pi}$

as $L \to \infty$

$$f_L(x) = \frac{1}{2L}\int_{-L}^{L} f_L(v)\,dv + \frac{1}{\pi}\sum_{n=1}^{\infty}\left[(\cos w_n x)\Delta w \int_{-L}^{L} f_L(v)\cos w_n v\, dv + (\sin w_n x)\Delta w \int_{-L}^{L} f_L(v)\sin w_n v\, dv\right]$$

as $L\to\infty$ becomes 0

$f(x) = \lim_{L\to\infty} f_L(x)$ : fourier integral

becomes integral from 0 to $\infty$

$L\to\infty$, $w_n \to w$

also $\Delta w = \pi/L$

$$\therefore f(x) = \frac{1}{\pi}\int_0^{\infty}\left[\cos wx \underbrace{\int_{-\infty}^{\infty} f(v)\cos wv \cdot dv}_{A(w)} + \sin wx \underbrace{\int_{-\infty}^{\infty} f(v)\sin wv\, dv}_{B(w)}\right] dw$$

$$\therefore f(x) = \int_0^{\infty}[A(w)\cos wx + B(w)\sin wx]\,dx$$

where $A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos wv\, dv$ $\qquad B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin wv \cdot dv$.

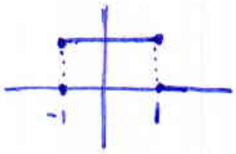
→ Fourier integral

Applications of Fourier Integral. 

Ex 2.

$$f(x) = \begin{cases} 1 & \text{if } |x| < 1 \\ 0 & \text{if } |x| > 1 \end{cases}$$

Find the Fourier integral representation.



$$A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos wv \cdot dv = \frac{1}{\pi}\int_{-1}^{1}\cos wv\,dv = \left.\frac{\sin wv}{\pi w}\right|_{-1}^{1} = \frac{\sin w}{\pi w} - \frac{\sin(-w)}{\pi w}$$

$$= \frac{2\sin w}{\pi w}$$

$$B(w) = \frac{1}{\pi}\int_{-1}^{1}\sin wv\,dv = 0. \quad (\because \text{even } f(x))$$

$$\therefore f(x) = \frac{2}{\pi}\int_{0}^{\infty}\frac{\cos wx \sin w}{w}\,dw = \begin{cases} 1 & \text{if } 0 \le x < 1 \\ \frac{1}{2}\left(=\frac{1}{2}(1+0)\right) & \text{if } x = 1 \quad \text{by Theorem.} \\ 0 & \text{if } x > 1 \end{cases}$$

Furthermore, from here,

$$\int_{0}^{\infty}\frac{\cos wx \sin w}{w}\,dw = \begin{cases} \pi/2 & : 0 \le x < 1 \\ \pi/4 & : x = 1 \\ 0 & : x > 1 \end{cases}$$

→ Dirichlet's discontinuous factor

Of particular interest is at $x = 0$ →

$$\int_{0}^{\infty}\frac{\sin w}{w}\,dw = \frac{\pi}{2} \quad \text{or} \quad \int_{0}^{u}\frac{\sin w}{w}\,dw = Si(u)$$

sine integral

as $u \to \infty$

$\left\{ Si(u) \xrightarrow{\text{요약은}} \text{Fig. 282} \right\}$

approximations are obtained by replacing $\infty$ by $a$.

$$f(x) = \frac{2}{\pi}\int_{0}^{a}\frac{\cos wx \sin w}{w}\,dw$$

$$= \frac{2}{\pi}\int_{0}^{a}\frac{\cos wx \sin w}{w}\,dw = \frac{1}{\pi}\int_{0}^{a}\frac{\sin(w+wx)}{w}\,dw + \frac{1}{\pi}\int_{0}^{a}\frac{\sin(w-wx)}{w}\,dw$$

$\Downarrow$ $w + wx = t$, $(1+x)dw = dt$ $\left(1+x = \frac{t}{w}\right)$, $\frac{dw}{w} = \frac{dt}{t}$

$0 \le w \le a \to 0 \le t \le (x+1)a.$

$\Downarrow$ $w - wx = -t$, $(1-x) = \frac{-t}{w}$, $\frac{dw}{w} = \frac{dt}{t}$

$(1-x)\,dw = -dt$

$0 \le w \le a \to 0 \le t \le (x-1)a.$

$\sin(-t) = -\sin t$

$$\frac{1}{\pi}\int_{0}^{(x+1)a}\frac{\sin t}{t}\,dt - \frac{1}{\pi}\int_{0}^{(x-1)a}\frac{\sin t}{t}\,dt$$

$$= \frac{1}{\pi}Si(a[x+1]) - \frac{1}{\pi}Si(a(x-1))$$

increase of $a$ $\Rightarrow$ oscillation toward the points of discontinuity ($-1$ and $1$)

Fig. 283

Gibbs phenomenon.

development of.

## Fourier Cosine Integral and Fourier Sine Integral

$f(x) = \int_0^\infty [A(w)\cos wx + B(w)\sin wx]\,dx$ 11-16

if $f$ is even, Fourier integral representation → (Fourier cosine integral)

$A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos wv\,dv$

$B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin wv\,dv$

$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} \underbrace{f(v)\sin \omega v}_{odd}\cdot dv = 0.$$

$$\therefore\ f(x) = \int_0^\infty A(\omega)\cos\omega x\,d\omega \quad \text{where} \quad A(\omega) = \frac{2}{\pi}\int_0^\infty f(v)\cos\omega v\cdot dv$$

if $f$ is odd, Fourier integral representation → **Fourier sine integral.**

$$f(x) = \int_0^\infty B(\omega)\sin\omega x\cdot d\omega \quad \text{where} \quad B(\omega) = \frac{2}{\pi}\int_0^\infty f(v)\sin\omega v\cdot dv$$

(Ex3) $f(x) = e^{-kx}$ (Laplace integral)

Derive

(a) Fourier cosine integral

$$A(w) = \frac{2}{\pi}\int_0^\infty f(v)\cos wv\,dv \Rightarrow \frac{2}{\pi}\int_0^\infty e^{-kv}\cos wv\,dv$$

integral by parts twice!

$$\int_0 e^{-kv}\cos wv\,dv = -\frac{1}{k}e^{-kv}\cos wv\Big|_0^\infty + \frac{w}{k}\int_0^\infty e^{-kv}\,(-\sin wu)\,dv$$

$$= -\frac{k}{k^2+w^2}e^{-kv}\left(-\frac{w}{k}\sin wv + \cos wv\right)$$

if $v=\infty$ $A(w)=0$

$v=0$ $-k/k^2+w^2$ $\therefore$ $A(w) = \dfrac{2k/\pi}{k^2+w^2}$

$$\therefore\ f(x) = \int_0^\infty A(w)\cos wx\,dx \Rightarrow e^{-kx} = \frac{2k}{\pi}\int_0^\infty \frac{\cos wx}{k^2+w^2}\,dw$$

$$\boxed{\int_0^\infty \frac{\cos wx}{k^2+w^2}\,dw = \frac{\pi}{2k}e^{-kx}}$$

(b) Fourier sine integral.

similarly. $B(\omega) = \frac{2}{\pi}\int_0^\infty e^{-kv}\sin wv\,dv.$

$$\Rightarrow \int_0 e^{-kv}\sin wv\,dv = -\frac{w}{k^2+w^2}e^{-kv}\left(\frac{k}{w}\sin wv + \cos wv\right)$$

(integral by parts twice!)

$$B(\omega) = \frac{2w/\pi}{k^2+w^2}$$

$$e^{-kx} = \frac{2}{\pi}\int_0^\infty \frac{w\sin wx}{k^2+w^2}\,dw$$

$$\therefore \int_0^\infty \frac{\omega\sin\omega x}{k^2+w^2}\,dw = \frac{\pi}{2}e^{-kx}$$

called. Laplace Integrals.

## 11.8 Fourier Cosine and Sine Transforms

ex) Laplace Transform.

( integral transform : a transformation in the form of integral that produces from given functions new functions depending on a different variable. )

### Fourier Cosine Transforms

Concerns even function $f(x)$

From Fourier Cosine integral

$$f(x) = \int_0^\infty A(\omega) \cos \omega x \, dx \quad \text{where } A(\omega) = \frac{2}{\pi} \int_0^\infty f(v) \cos \omega v \, dv$$

we set $A(\omega) = \sqrt{\frac{2}{\pi}} \hat{f}_c(\omega)$ → cosine.

$$\hat{f}_c(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos \omega x \, dx \longrightarrow \text{Fourier cosine transform}$$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_c(\omega) \cos \omega x \, dx \longrightarrow \text{inverse Fourier cosine Transform.}$$

symmetric distribution of $\frac{2}{\pi}$

$$\mathcal{F}_c(f) = \hat{f}_c$$

### Fourier sine Transform : odd function

$$\hat{f}_s(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin \omega x \, dx \qquad \mathcal{F}_s(f) = \hat{f}_s$$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_s(\omega) \sin \omega x \, dx$$

Ex 1.

$$f(x) = \begin{cases} k & \text{if } 0<x<a \\ 0 & \text{if } x>a. \end{cases} \qquad \mathcal{F}_c(f) \qquad \mathcal{F}_s(f) \;?$$

$$\hat{f}_c(\omega) = \sqrt{\frac{2}{\pi}} \int_0^a k \cos \omega x \, dx = \sqrt{\frac{2}{\pi}} \, k \, \frac{\sin a\omega}{\omega}$$

$$\hat{f}_s(\omega) = \sqrt{\frac{2}{\pi}} \int_0^a k \sin \omega x \, dx = \sqrt{\frac{2}{\pi}} \, k \left( \frac{1 - \cos a\omega}{\omega} \right)$$

Ex 2. $\mathcal{F}_c(e^{-x})$

integration by parts

$$\sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x} \cos \omega x \, dx = \sqrt{\frac{2}{\pi}} \frac{e^{-x}}{1+\omega^2} (-\cos \omega x + \omega \sin \omega x) \Big|_0^\infty$$

$$= \frac{\sqrt{2/\pi}}{1+\omega^2} \checkmark$$

linearity

$$\mathcal{F}_c(af+bg) = a\mathcal{F}_c(f) + b\mathcal{F}_c(f)$$
$$\mathcal{F}_s(af+bg) = a\mathcal{F}_s(f) + b\mathcal{F}_s(f)$$
linear operations"

Theorem 1. Cosine and sine Transforms of derivatives let $(f(x)\to 0 \text{ as } x\to\infty)$

$$\mathcal{F}_c(f'(x)) = w\,\mathcal{F}_s(f(x)) - \sqrt{\tfrac{2}{\pi}}\,f(0)$$
$$\mathcal{F}_s(f'(x)) = -w\,\mathcal{F}_c(f(x))$$

proof)
$$\mathcal{F}_c(f'(x)) = \sqrt{\tfrac{2}{\pi}}\int_0^\infty f'(x)\cos wx\,dx = \sqrt{\tfrac{2}{\pi}}\left[\left[f(x)\cos wx\right]_0^\infty + w\int_0^\infty f(x)\sin wx\,dx\right]$$
$$= -\sqrt{\tfrac{2}{\pi}}f(0) + w\,\mathcal{F}_s(f(x))$$

$$F_s(f'(x)) = \sqrt{\tfrac{2}{\pi}}\int_0^\infty f'(x)\sin wx\,dx = \sqrt{\tfrac{2}{\pi}}\left[f(x)\sin wx\right]_0^\infty - w\int_0^\infty f(x)\cos wx\cdot dx$$
$$= 0 - w\,\mathcal{F}_c(f(x))$$

---

$$w\,\underbrace{F_s(f'(x))}_{-wF_c(f(x))} - \sqrt{\tfrac{2}{\pi}}\,f'(0) =$$

---

$$\mathcal{F}_c(f''(x)) = -w^2\,\mathcal{F}_c(f(x)) - \sqrt{\tfrac{2}{\pi}}\,f'(0)$$
$$\mathcal{F}_s(f''(x)) = -w^2\,\mathcal{F}_s(f(x)) + \sqrt{\tfrac{2}{\pi}}\,w f(0)$$
$$\searrow -w F_c(f'(x)) = -w^2\mathcal{F}_s(f(x)) + \sqrt{\tfrac{2}{\pi}}\,w f(0)$$

Ex 3. $\mathcal{F}_c(e^{-ax})$ of $f(x) = e^{-ax}$, where $a>0$

$$(e^{-ax})'' = a^2 e^{-ax}$$
$$a f(x) = f''(x)$$

Cosine Transform $\to$ $$a^2\mathcal{F}_c(f) = \mathcal{F}_c(f'')$$
$$= -w^2\mathcal{F}_c(f) - \sqrt{\tfrac{2}{\pi}}\,f'(0)$$
$$= -w^2 F_c(f) + a\sqrt{\tfrac{2}{\pi}}$$
$$\therefore F_c(a^2+w^2) = a\sqrt{2/\pi}$$
$$\therefore \mathcal{F}_c(e^{-ax}) = \sqrt{\tfrac{2}{\pi}}\left(\frac{a}{a^2+w^2}\right)$$

## 11.9 Fourier Transform. Discrete and Fast Fourier Transform.

Fourier sine / Cosine transform → real transform

11-19

obtained from the Complex form of the Fourier integral.

$e^{iwx}$, $e^{-iwv}$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(v)\, e^{iw(x-v)}\, dv\, dw$$

**Fourier integral**

$$f(x) = \int_0^{\infty} [A(w)\cos wx + B(w)\sin wx]\, dw$$

where $A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos wv\, dv$

$B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin wv\, dv$

$$\therefore f(x) = \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} \underbrace{f(v)}_{\text{not depend on } w}\,[\underbrace{\cos wv \cos wx + \sin wv \sin wx}_{\cos(wx-wv)}]\, dv\, dw$$

even function of $w$ $\left(\frac{1}{2}\ \begin{matrix} 0 \leftrightarrow \infty \\ -\infty \leftrightarrow \infty \end{matrix}\right)$

$$\therefore \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(v)\cos(wx-wv)\, dv\right] dw \quad ①$$

② : $\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(v)\underbrace{\sin(wx-wv)}_{\text{odd of } w}\, dv\, dw = 0$

since $\int \text{odd}(w)\, dw = 0$.

Euler formula $e^{ix} = \cos x + i\sin x$

Taking $wx - wv \Rightarrow x$, multiply by $f(v)$

$f(v)\, e^{i(wx-wv)} = f(v)\cos(wx-wv) + i f(v)\sin(wx-wv)$

① + i②

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(v)\, e^{-iwv}\, dv\right] e^{iwx}\, dw$$

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{-iwx}\, dx \qquad \text{Fourier Transform of } f$$

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(w)\, e^{iwx}\, dw \qquad \text{inverse Fourier Transform.}$$

**Theorem 1** : Existence of the Fourier Transform.

if $f(x)$ is absolutely integrable, piecewise continuous.

→ exists.

**Ex 1)** Fourier Transform

$f(x) = 1$ if $|x| < 1$
$0$ otherwise

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{-iwx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-1}^{1} e^{-iwx}\, dx = \frac{1}{-iw\sqrt{2\pi}}\left(e^{-iw} - e^{iw}\right) \quad \text{not end !}$$

$e^{iw} = \cos w + i\sin w$
$e^{-iw} = \cos w - i\sin w$

$$= \frac{1}{-iw\sqrt{2\pi}}\, 2i\sin w = \sqrt{\frac{2}{\pi}}\,\frac{\sin w}{w}$$

**Ex 2)** $\mathcal{F}(e^{-ax})$ if $x > 0$
$0$ if $x < 0$

$$\frac{1}{\sqrt{2\pi}}\int_0^{\infty} e^{-ax} e^{-iwx}\, dx = \frac{1}{\sqrt{2\pi}}\left.\frac{e^{-(a+iw)x}}{-(a+iw)}\right|_0^{\infty} = \frac{1}{\sqrt{2\pi}\,(a+iw)}$$

Physical interpretation : Spectrum. ( skip! )



$\hat{f}(w)$ spectral density.

$\int_{-\infty}^{\infty} |\hat{f}(w)|^2 dw$ total energy

$my'' + ky = 0$

---

Linearity of the Fourier Transform

$$\mathcal{F}(af+bg) = a\mathcal{F}(f) + b\mathcal{F}(g)$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(af+bg)\,e^{-iwx}dx = a\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f\,e^{-iwx}dx + b\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g\,e^{-iwx}dx$$

$$= a\mathcal{F}(f) + b\mathcal{F}(g)$$

Derivative $f(x) \longrightarrow$ Fourier Transform

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f'(x)\,e^{-iwx}dx = \frac{1}{\sqrt{2\pi}}\left[f(x)e^{-iwx}\Big|_{-\infty}^{\infty} - (-iw)\int_{-\infty}^{\infty} f(x)\,e^{-iwx}dx\right]$$

$\mathcal{F}(f'(x)) = iw\mathcal{F}\{f(x)\}$

$f(\infty) \to 0$. $f(-\infty) \to 0$

$iw\,\mathcal{F}\{f(t)\}$

$\mathcal{F}(f''(x)) = -w^2\mathcal{F}(f(x))$ $\mathcal{F}(f'') = iw\mathcal{F}(f'(t)) = -w^2\mathcal{F}(f)$

Ex3). $\mathcal{F}(xe^{-x^2}) = \mathcal{F}\left(-\frac{1}{2}(e^{-x^2})'\right)$ Due to linearity.

$= -\frac{1}{2}\mathcal{F}\{(e^{-x^2})'\} = -\frac{1}{2}iw\mathcal{F}(e^{-x^2})$ Table I

$= -\frac{1}{2}iw\left(\frac{1}{\sqrt{2}}e^{-w^2/4}\right)$

$= \frac{-iw}{2\sqrt{2}}e^{-w^2/4}$

$$\underbrace{\int_{-\infty}^{\infty} e^{-x^2}e^{-iwx}dx}\qquad \int_{-\infty}^{\infty} e^{-(x^2+iwx)}dx$$

$\mathcal{F}(e^{-ax^2}(a>0))$

$= \frac{1}{\sqrt{2a}}e^{-w^2/4a}$

Not $\mathcal{F}(f\cdot g) = \mathcal{F}(f)\mathcal{F}(g)$ No.

Convolution.

$$h(x) = (f*g)(x) = \int_{-\infty}^{\infty} f(p)g(x-p)\,dp = \int_{-\infty}^{\infty} f(x-p)g(p)\,dp$$

"regarded as product of Fourier Transform."

Theorem 4. Convolution Theorem.

$$\mathcal{F}(f*g) = \sqrt{2\pi}\,\mathcal{F}(f)\,\mathcal{F}(g) = \sqrt{2\pi}\,\hat{f}(w)\,\hat{g}(w)$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \sqrt{2\pi}\,\hat{f}(w)\,\hat{g}(w)\,e^{iwx}\,dw = \int_{-\infty}^{\infty}\hat{f}(w)\,\hat{g}(w)\,e^{iwx}\,dw$$

proof)
$$\mathcal{F}(f*g) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(p)g(x-p)\,dp\,e^{-iwx}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(p)\,g(x-p)\,e^{-iwx}\,dx\,dp$$

$x-p=q$. $x=p+q$. $dx = dq$

$$\boxed{f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(w)\,e^{iwx}\,dw}$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(p)g(q)\,e^{-iw(p+q)}\,dq\,dp$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(p)\,e^{-iwp}\,dp\int_{-\infty}^{\infty} g(q)\,e^{-iwq}\,dq$$

$$= \frac{1}{\sqrt{2\pi}}\left[\sqrt{2\pi}\,\mathcal{F}(f)\right]\left[\sqrt{2\pi}\,\mathcal{F}(g)\right] = \sqrt{2\pi}\,\mathcal{F}(f)\,\mathcal{F}(g)$$

$\mathcal{F}^{-1}\mathcal{F}(f*g) = f*g =$

taking inverse Fourier Transform

$$(f*g) = \int_{-\infty}^{\infty}\hat{f}(w)\,\hat{g}(w)\,e^{iwx}\,dx$$

useful!

---

DFT (Discrete Fourier Transform)

dealing with sampled values rather than with functions

$$x_k = \frac{2\pi k}{N} \qquad k = 0, 1, \dots N-1.$$

$f(x)$ is being sampled at these points

$$g(x) = \sum_{n=0}^{N-1} c_n e^{inx_k} \quad : \text{complex trigonometric polynomial}$$

$$g(x_k) = f(x_k)$$

$$f_k = f(x_k) = g(x_k) = \sum_{n=0}^{N-1} c_n e^{inx_k}$$

we must determine $c_0, c_1, \dots c_{N-1}$

# Summary of Chap 11.

Fourier series : periodic function f(x) of period $p = 2L$

"set Ground work for solving PDE in Chap 12."

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{L}x + b_n \sin\frac{n\pi}{L}x\right)$$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi}{L}x\,dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi}{L}x\,dx$$

If f(x) is
- even $\longrightarrow$ fourier cosine series
- odd $\longrightarrow$ fourier sine series

- Orthogonality of trigonometric system.
- Replacement the trigonometric system by orthogonal.

  $\longrightarrow$ Sturm-Liouville problems.

  $\longrightarrow$ Solution of S-L problems : eigenfunctions.

Generalized fourier series : Fourier-Legendre, Fourier-Bessel series

Fourier integral : extension to nonperiodic functions f(x)

$$f(x) = \int_0^{\infty}[A(w)\cos wx + B(w)\sin wx]\,dw.$$

where $A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos wv\,dv$, $B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin wv\,dv$.

in complex form

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(w)\,e^{iwx}\,dw$$

where $\hat{f}(w) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\,e^{-iwx}\,dx$.

Fourier transform.

Fourier cosine transform

$$\hat{f}_c(w) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos wx\,dx$$

Fourier sine transform

$$\hat{f}_s(w) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\sin wx\,dx$$

h.w. (1차)

11.1 : 9. 13. 20
11.2 : 10. 29
11.3 : 6
11.4 : 4. 11
11.5 : 10. 12 — by 10/23 (화)

11.6 : 2. 8 (11-2)

11.7 : 1. 7. 17
11.8 : 1, 12 — by 11/22 (화)
11.9 : 7. 13 — by 11/6 (화)