Chap 12. PDEs

12.1 Basic Concepts of PDEs

PDEs

order

linear / nonlinear

homogeneous / nonhomogeneous

Ex1) Important 2nd-order PDEs

(1) ... (6)

$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$, $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ $\to (x, y, z)$ $\to (r, \theta, z)$ $\to (r, \theta, \phi)$

1D → 2D $\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$

$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ 2-D Laplace equation.

- Larger No. of Solution of a PDE : (7) $\left[u = x^2 y^2,\ e^x \cos y,\ \sin x \cosh y,\ \ln(x^2+y^2)\right]$

different each other, solution.

boundary condition :

initial condition : when t is variable. $(t=0)$

- $\boxed{\text{Superposition principle of a solution}}$

If u_1 and u_2 are solutions of a homogeneous linear PDE in some region R then $u = c_1 u_1 + c_2 u_2$ with any constants c_1 and c_2 is also a solution of that PDE in the region R.

Ex2) $u_{xx} - u = 0$ where $u(x, y)$

$u = Ae^x + Be^{-x}$ but A, B is not constant, but may be function of y

$\therefore u = A(y)e^x + B(y)e^{-y}$

Ex3) $u_{xy} = -u_x$ $u(x, y)$

$u_x = p$ $u_{xy} = p_y = -p$ $\therefore p_y/p = -1$ → $p = e^{-y} \cdot c$ (not c but $c(x)$)

$\therefore p = c(x)e^{-y}$

$u_x = p = c(x)e^{-y}$

$u = \int c(x) e^{-y} dx = e^{-y} \int c(x)dx + g(y) = f(x)e^{-y} + g(y)$

where $f(x) = \int c(x) dx$

12.2 Modelling : Vibrating String, Wave Equation

Deflected string at fixed time t.

~~Skip!~~

Consider the forces acting on a small portion of the string.

horizontal component. $T_1 \cos\alpha = T_2 \cos\beta = T = \text{const}$

Vertical : $T_2 \sin\beta - T_1 \sin\alpha = \rho\Delta x \frac{\partial^2 u}{\partial t^2}$ (Newton's 2nd law)

Δ tension = (mass·moving) mass acceleration

$\frac{T_2 \sin\beta}{T_2\cos\beta} - \frac{T_1 \sin\alpha}{T_1 \cos\alpha} = \frac{\rho\Delta x}{T}\frac{\partial^2 u}{\partial t^2}$

$\tan\beta - \tan\alpha = \frac{\rho\Delta x}{T}\frac{\partial^2 u}{\partial t^2}$

Slope of the string at x and $x+\Delta x$.

$\left.\frac{\partial u}{\partial x}\right|_{x+\Delta x} - \left.\frac{\partial u}{\partial x}\right|_{x} = \frac{\rho}{T}\Delta x \frac{\partial^2 u}{\partial t^2}$

$\frac{1}{\Delta x}\left(\left.\frac{\partial u}{\partial x}\right|_{x+\Delta x} - \left.\frac{\partial u}{\partial x}\right|_{x}\right) = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$

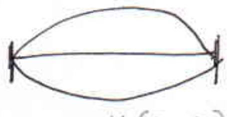
$\therefore \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$

where $c^2 = \frac{T}{\rho}$

One-Dimensional Wave equation.

$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$: one-D wave equation

12.3 Solution by separating variables. Use of Fourier Series.



1-D wave equation. $\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}$

2 BCs. $u(0,t)=0$, $u(L,t)=0$ for all $t \geq 0$

2 ICs. $u(x,0)=f(x)$ (initial deflection) $u_t(x,0)=g(x)$ $\frac{\partial u}{\partial t}$ (initial velocity) $0 \leq x \leq L$

Step 1. Two ODEs from the wave equation : $\boxed{\text{separating variables}}$

$u(x,t) = F(x)G(t)$

$\dfrac{\partial^2 u}{\partial t^2} = F(x)\ddot{G}(t)$ and $\dfrac{\partial^2 u}{\partial x^2} = G(t)F''(x)$

$F\ddot{G} = c^2 G F'' \longrightarrow \underbrace{\dfrac{\ddot{G}}{c^2 G}}_{\text{function of } t} = \dfrac{F''}{F}$ ↳ function of x.

Hence both sides must be constant because, if they were variable, then changing t or x would affect only one side, leaving the other unaltered,

$\therefore \dfrac{\ddot{G}}{c^2 G} = \dfrac{F''}{F} = k$ (arbitrary)

two ODEs
$$F'' - kF = 0$$
$$\ddot{G} - c^2 k G = 0$$

Step 2 Satisfying the BCs → Step 3. using Fourier series.

B.C. $u(0,t) = F(0)G(t) = 0$, $u(L,t) = F(L)G(t) = 0$. for all t.

If $G \equiv 0$, $u = FG = 0$, no interest. Hence $G \not\equiv 0$.

① $F'' - kF = 0$. $F(0)=0$ $F(L)=0$ —— (B.C)

$\therefore F(0)=0$ $F(L)=0$

k —— i) $k=0$ $F''=0$ $F=ax+b$ $a,b=0$ from B.C. $\therefore F\equiv 0$, no interest.

ii) $k=\mu^2>0$. $F = Ae^{\mu x} + Be^{-\mu x}$ $A,B=0$ from B.C → $F \equiv 0$ → $F(0)=0$ $A+B=0$ $A=-B$ $A(e^{\mu x}-e^{-\mu x})$ $A(e^{\mu L}-e^{-\mu L})$ $A=0 \to B=0$

iii) $k=-p^2$ $F''+p^2F=0$ $F(x) = A\cos px + B\sin px$.

$F(0)=A=0$ then $F(L) = B\sin pL = 0$.

since $B \neq 0$, $\therefore pL = n\pi$, so that $\left(p = \dfrac{n\pi}{L}\right)$ (n: integer)

Setting $B=1$, infinitely many solutions $F(x) = F_n(x)$

$F_n(x) = \sin \dfrac{n\pi}{L}x$ $(n = 1, 2, 3, \dots)$

② $\dfrac{\ddot{G}}{c^2 G} = -p^2$ $\ddot{G} + \lambda_n^2 G = 0$ where $\lambda_n = cp = \dfrac{cn\pi}{L}$.

$\therefore G_n(t) = B_n \cos\lambda_n t + B_n^* \sin\lambda_n t$

harmonic motion having the frequency $\lambda_n/2\pi = cn/2\pi$

λ changing by changing T (Tuning) $\left(c = \sqrt{\dfrac{T}{\rho}}\right)$

$u_n(x,t) = F_n(x)G_n(t) =$

$u_n(x,t) = (B_n\cos\lambda_n t + B_n^* \sin\lambda_n t)\sin\dfrac{n\pi}{L}x$

nth normal mode of the string

↳ $(n-1)$ nodes.

points of the string that do not move.

Fig 2.87 (p. 548)

→ eigenfunctions, or characteristic functions.

$\lambda_n = \dfrac{cn\pi}{L}$: eigenvalues or characteristic values.

$\{\lambda_1, \lambda_2 \cdots\}$ spectrum.

~~Bound~~ ~~BC~~. u_n satisfys BC, But not IC

$\sum u_n$ can do

Step 3. Solution of the Entire Problem. Fourier Series

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} (B_n \cos\lambda_n t + B_n^* \sin\lambda_n t) \sin\frac{n\pi}{L}x \qquad 0 \le x \le L$$

B_n, B_n^* : by using ICs (a) $\quad u(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{L}x = f(x)$ ← Given initial displacement

Fourier sine series. $\qquad$ p. 486. $\qquad$ odd function → with period of 2L

$$B_n = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi}{L}x \cdot dx$$

using ICs (b) Given initial velocity.

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = \sum_{n=1}^{\infty}\left(-B_n\lambda_n \cancel{\sin\lambda_n t} + B_n^*\lambda_n \cancel{\cos\lambda_n t}\right)\sin\frac{n\pi x}{L}\Big|_{t=0}$$

$$= \sum_{n=1}^{\infty} B_n^*\lambda_n \sin\frac{n\pi x}{L} = g(x)$$

↓ Fourier sine series of $g(x)$ $\qquad B_n^*\lambda_n = \frac{2}{L}\int_0^L g(x)\sin\frac{n\pi x}{L}dx$.

$$B_n^* = \frac{2}{cn\pi}\int_0^L g(x) \sin\frac{n\pi}{L}x \cdot dx \qquad (n = 1, 2, \dots).$$

For the sake of simplicity we consider only the case when $g(x) = 0$. Then $B_n^* = 0$.

$$u(x,t) = \sum_{n=1}^{\infty} B_n \underbrace{\cos\lambda_n t \cdot \sin\frac{n\pi x}{L}}_{\frac{1}{2}\left[\sin\left\{\frac{n\pi}{L}(x-ct)\right\} + \sin\left\{\frac{n\pi}{L}(x+ct)\right\}\right]} \qquad \lambda_n = \frac{cn\pi}{L}$$

$$\therefore\ u(x,t) = \frac{1}{2}\sum_{n=1}^{\infty} B_n \sin\left\{\frac{n\pi}{L}(x-ct)\right\} + \frac{1}{2}\sum_{n=1}^{\infty} B_n \sin\left\{\frac{n\pi}{L}(x+ct)\right\}$$

case when $g(x) = 0$.

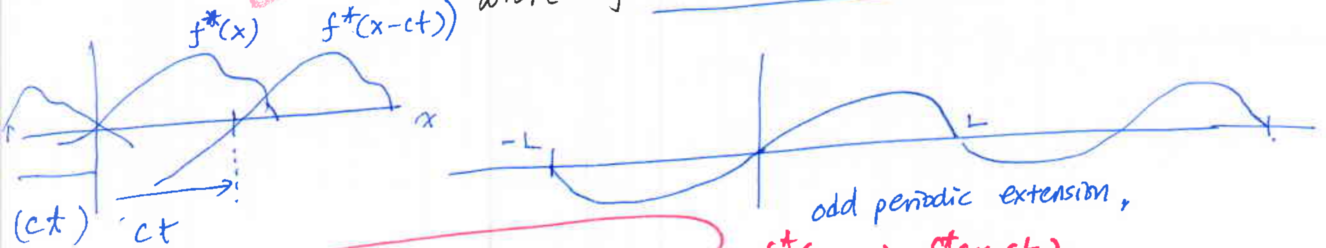
$$u(x,t) = \sum_{n=1}^{\infty} B_n \cos\lambda_n t \sin\frac{n\pi x}{L} \qquad \lambda_n = \frac{cn\pi}{L}.$$

→ Fourier sine series $\qquad$ x 대신 x-ct, x+ct 대입. $\quad \sin\frac{n\pi}{L}x$

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{L}x$$

$$\therefore\ u(x,t) = \frac{1}{2}\left[f^*(x-ct) + f^*(x+ct)\right]$$

where f^* : odd periodic extension of f with the period $2L$



odd periodic extension, $f^*(x+ct)$, $f^*(x-ct)$

∴ Superposition of these two waves

Ex 1. $f(x) = \begin{cases} \frac{2k}{L} x & \text{if } 0 < x < \frac{L}{2} \\ \frac{2k}{L}(L-x) & \text{if } \frac{L}{2} < x < L \end{cases}$ initial deflection, $g(x)=0$.

p. 489 Ex. 6.

since $g(x) \equiv 0$, $B_n^* = 0$.

- Fourier sine series coefficient
- odd extension

$B_n = \frac{8k}{n^2\pi^2} \sin\frac{n\pi}{2} \rightarrow$ odd. $n = 1\ 3\ 5\ 7\ 9$ $(1\ -1\ 1\ -1\ 1)$

$\lambda_n = \frac{cn\pi}{L}$

$$u(x,t) = \sum_{n=1}^{\infty} \frac{8k\,(-1)^{n+1}}{n^2\pi^2} \sin\frac{n\pi}{2L}x \cdot \cos\lambda_n t$$

$$= \frac{8k}{\pi^2}\left(\frac{1}{1^2}\sin\frac{\pi}{L}x \cos\frac{c\pi}{L}t - \frac{1}{3^2}\sin\frac{3\pi}{L}x\cos\frac{3\pi c}{L}t + - \cdots\right)$$

$\frac{1}{2}f^*(x)$ (at $t=0$ $\frac{1}{2}\left[f^*(x) + f^*(x)\right]$ ($u(x,0)$, $u(x, \frac{L}{5c})$)) Fig 291 설명 要

$\frac{1}{2}f^*(x-ct) + \frac{1}{2}f^*(x+ct)$

$\frac{1}{2}f^*(x-\frac{L}{5}) + \frac{1}{2}f(x+\frac{L}{5})$

12.4 D'Alembert's Solution of the Wave Eq'n Characteristics

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad u_{tt} = c^2 u_{xx}$$

$u(x,t) \rightarrow u(v,w)$

by introducing Two independent variables. $v = x+ct$ $w = x-ct$

$\hookrightarrow v_x = 1$ $w_x = 1$

$$u_x = u_v v_x + u_w w_x = u_v + u_w$$

chain rule

$$u_{xx} = (u_v + u_w)_x = (u_v+u_w)_v v_x + (u_v+u_w)_w w_x = \boxed{u_{vv} + 2u_{vw} + u_{ww}}$$

$u_t = u_v v_t + u_w w_t = cu_v - cu_w = c(u_v - u_w)$

$u_{tt} = c(u_v - u_w)_t = c(u_v-u_w)_v \cdot v_t + c(u_v-u_w)_w w_t$

$= c^2(u_{vv} - 2u_{wv} + u_{ww})$

$u_{tt} = c^2(u_{vv} - 2u_{vw} + u_{ww})$

$u_{tt} = c^2 u_{xx}$

$$u_{vw} = \frac{\partial^2 u}{\partial w \partial v} = 0.$$

integration w.r.t w: $\frac{\partial u}{\partial v} = h(v)$ $\xrightarrow{\text{integration w.r.t } v}$ $u = \int h(v)\,dv + \psi(w)$

$\downarrow \phi(v) + \psi(w)$

$= \phi(x+ct) + \psi(x-ct)$

h(v) and ψ(w) arbitrary functions of v and w.

$\rightarrow$ d'Alembert's solution

ICs $u(x,0) = f(x)$ $u_t(x,0) = g(x)$

$u_t(x,t) = c\phi'(x+ct) - c\psi'(x-ct)$

$u(x,0) = \phi(x) + \psi(x) = f(x)$ — ①

$u_t(x,0) = c\phi'(x) - c\psi'(x) = g(x)$ $\rightarrow$ $\phi(x) - \psi(x) = \frac{1}{c}\int_{x_0}^{x} g(s)\,ds + k(x_0)$ — ②

where $k(x_0) = \phi(x_0) - \psi(x_0)$

①+② $\phi(x) = \frac{1}{2}f(x) + \frac{1}{2c}\int_{x_0}^{x} g(s)\,ds + \frac{1}{2}\cancel{k(x_0)}$ — replace x by $x+ct$

①−② $\psi(x) = \frac{1}{2}f(x) - \frac{1}{2c}\int_{x_0}^{x} g(s)\,ds - \frac{1}{2}\cancel{k(x_0)}$ — replace x by $x-ct$

$\{\phi(x+ct) + \psi(x-ct)\}$ $\frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds$

$$u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds$$

if initial velocity $g(s) = 0$. $u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right]$ same! as bef

Characteristics. Types and Normal Forms of PDEs

(14) $Au_{xx} + 2Bu_{xy} + Cu_{yy} = F(x, y, u, u_x, u_y)$ called quasilinear because it is linear in the highest derivatives

Table.

Type	Defining condition	Example. in sec 12.1		
Hyperbolic	$AC - B^2 < 1$	wave eq'n	$u_{tt} - c^2 u_{xx} = 0$	$-c^2 < 0$
Parabolic	$AC - B^2 = 0$	heat eq'n	$u_t - c^2 u_{xx} = 0$	$0(c^2) = 0$
Elliptic	$AC - B^2 > 0$	Laplace eq'n	$u_{xx} + u_{yy} = 0$	$1 \cdot 1 > 0$

Transformation of (14) to Normal form.

(15) $A(y')^2 + 2By' + C = 0$ where $y' = \frac{dy}{dx}$

Solutions of (15) are called the characteristics of (14)

Type	New Variables		Normal Form
Hyperbolic	$v = \Phi$	$w = \Psi$	$u_{vw} = F_1$
parabolic	$v = x$	$w = \Phi = \Psi$	$u_{ww} = F_2$
Elliptic	$v = \frac{1}{2}(\Phi + \Psi)$	$w = \frac{1}{2i}(\Phi - \Psi)$	$u_{vv} + u_{ww} = F_3$

12.5 Modelling : Heat flow from a body in Space. Heat Eq'n

10-16 참조.

$\underline{v} = -K \nabla u$ velocity of the heat flow

u: Temperature.

T: region in the body bounded by a surface with outer normal vector $\underline{n}$

Total amount of heat that flows across S from T. $|\underline{v} \cdot \underline{n}| \Delta A$ heat leaving T

$$\iint_S \underline{v} \cdot \underline{n} \, dA$$

Gauss Divergence Theorem

$$\iint_S \underline{v} \cdot \underline{n} = -k \iint_S (\text{grad}\, u) \cdot \underline{n} \, dA = -k \iiint_T \text{div}(\text{grad}\, u) \, dx\,dy\,dz$$

$$= -k \iiint_T \nabla^2 u \, dx\,dy\,dz \quad \longleftarrow ①$$

Total amount of heat in T.

$$H = \iiint_T \sigma \rho u \, dx\,dy\,dz$$

the rate of decrease of H

$$\frac{\partial H}{\partial t} = -\iiint_T \sigma \rho \frac{\partial u}{\partial t} \, dx\,dy\,dz \quad ②$$

① = ②

$$\iiint_T \left(\frac{\partial u}{\partial t} - c^2 \nabla^2 u \right) dx\,dy\,dz = 0 \quad \text{where } c^2 = \frac{k}{\sigma \rho}$$

$\frac{\partial u}{\partial t} = c^2 \nabla^2 u$ (heat equation) $\therefore \frac{\partial u}{\partial t} = c^2 \nabla^2 u$.

12.6 Heat Equation: Solution by Fourier series — similar to 12.4
Steady 2-D Heat Problems
Dirichlet Problem.

governing eq'n for temperature in a body.

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

c^2: thermal diffusivity. $\frac{K}{\sigma\rho}$ (K: thermal conductivity; σ: specific heat; ρ: density)

0 $x=L$

consider bar.

BCs $u(0,t)=0$, $u(L,t)=0$ for all $t \geq 0$

keep the temperature zero.

IC $u(x,0) = f(x)$

Step 1. separating variables → 2 ODEs

$u(x,t) = F(x)G(t)$

$F\dot{G} = c^2 F''G \longrightarrow \frac{\dot{G}}{c^2 G} = \frac{F''}{F} = k = -p^2$ (already proves that only $k<0$ ~~applies~~ has meaning (↳ otherwise $u \equiv 0$))

$\longrightarrow F'' + p^2 F = 0, \quad \dot{G} + c^2 p^2 G = 0$

Step 2. Satisfying the B.C.

$F(x) = A\cos px + B\sin px$

from BCs $u(0,t) = F(0)G(t) = 0$ and $u(L,t) = F(L)G(t) = 0$

$F(0) = 0, \ F(L) = 0.$

$F(0) = A = 0. \quad F(L) = B\sin pL = 0 \quad p = \frac{n\pi}{L} \quad (n = 1, 2, \ldots)$

$F_n(x) = \sin \frac{n\pi}{L} x$

$\dot{G} + c^2 p^2 G = 0$ for p

$\dot{G} + c^2 \frac{n^2\pi^2}{L^2} G = 0$ or $\dot{G} + \lambda_n^2 G = 0$ where $\lambda_n = \frac{cn\pi}{L}$

general solution $G_n(t) = B_n e^{-\lambda_n^2 t}$

where B_n: constant.

$u_n(x,t) = F_n(x) G_n(t) = B_n \sin \frac{n\pi}{L} x \, e^{-\lambda_n^2 t}$ ~~with~~ Corresponding to the eigenvalues $\lambda_n = \frac{cn\pi}{L}$

$\longrightarrow$ eigenfunctions of the problem, which satisfies the BCs.

Step 3. Solution of the entire problem, Fourier series.

To obtain the solution that satisfies the ICs, we consider a series of eigenfunctions

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{L} x \, e^{-\lambda_n^2 t}$$

what is B_n?

From IC, $u(x,0) = f(x) = \boxed{\sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{L} x = f(x)}$

B_n: must be the coefficients of Fourier sine series

$$B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x \, dx$$

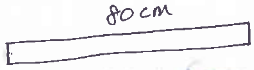
Due to the exponential term, $u(x,t) \to 0$ as $t \to \infty$

as $n \to \infty$, $u(x,t) \to 0$ faster.

Ex 1. Sinusoidal Initial Temperature.

$T_{max} \to 50$. when?

80 cm



insulated upper.

$f(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{L} x = f(x) = 100 \sin\left(\frac{\pi x}{80}\right)$

↳ $B_1 = 100$, $B_2, B_3 \ldots 0$

$u(0,t), \ u(L,t) = 0.$

we need $\lambda_1^2 = \frac{c^2\pi^2}{L^2}$

if $f(x) = 100 \sin\left(\frac{3\pi x}{80}\right)$

↳ $n=3$, λ_3^2

$c^2 = \frac{K}{\rho\sigma} = 1.158$

$\lambda_1^2 = \frac{1.158 \cdot \pi^2}{80^2} = 0.001785$

T_{max} 50

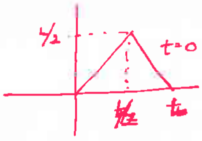
as $n \to \infty$, decay is faster

$\therefore u(x,t) = 100 \left(\sin \frac{\pi x}{80}\right) e^{-0.001785 t}$

$x =$

$100 \, e^{-0.001785 t} = 50$

Ex3. Triangular initial Temp. in a bar.



$$f(x) = \begin{cases} x & \text{if } 0 < x < L/2 \\ L-x & \text{if } L/2 < x < L \end{cases}$$

$$B_n = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi}{L}x\, dx = \frac{2}{L}\left[\int_0^{L/2} x \sin\frac{n\pi}{L}x\, dx + \int_{L/2}^{L} (L-x) \sin\frac{n\pi}{L}x\, dx\right)$$

$$\rightarrow \begin{cases} 0 & \text{if } n = \text{even} \\ \dfrac{4L}{n^2\pi^2} & \text{if } n = 1, 5, 9 \ldots) \\ -\dfrac{4L}{n^2\pi^2} & \text{if } n = 3, 7, 11 \ldots \end{cases}$$

Solution $u(x,t) = \frac{4L}{\pi^2}\left[\sin\frac{\pi x}{L} \exp\left[\left(-\left(\frac{c\pi}{L}\right)^2 t\right]\right. - \frac{1}{9}\sin\frac{3\pi x}{L} \exp\left[-\left(\frac{3c\pi}{L}\right)^2 t\right] + \cdots \right)$

Fig 295 (Temp. decreases to 0 with increasing t, due to the heat loss (cooling at both ends))

physical meaning

Ex4. Bar with insulated ends → no heat flow through the ends. or $\Delta T = 0$ heat flow = 0

$\therefore U_x(0,t) = U_x(L,t) = 0$

$F'(0)G(t) = 0$, $F'(L)G(t) = 0$

$F'(x) = -Ap\sin px + Bp\cos px$

→ $F'(0) = Bp = 0$, $B = 0$, $F'(L) = -Ap\sin pL = 0$

$p = p_n = n\pi/L$, $(n = 0, 1, 2 \ldots)$ $A = 1$.

$F_n(x) = \cos\left(\frac{n\pi x}{L}\right)$ $(n = 0, 1, 2, \ldots)$

G_n same as before.

$$u_n(x,t) = F_n(x)G_n(t) = A_n \cos\left(\frac{n\pi x}{L}\right) e^{-\lambda_n^2 t}$$

$\lambda_0 = 0$ (additional eigenvalue) → eigenfunction $u_0 = \text{const}$.

→ Solution of the problem if the initial temp $f(x)$ is const.

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = \sum_{n=0}^{\infty} A_n \cos\frac{n\pi x}{L} e^{-\lambda_n^2 t}$$

$$u(x,0) = \sum_{n=0}^{\infty} A_n \cos\frac{n\pi x}{L} = f(x) \longrightarrow$$ This time fourier cosine series.

$$A_0 = \frac{1}{L}\int_0^L f(x)\, dx$$

$$A_n = \frac{2}{L}\int_0^L f(x) \cos\frac{n\pi x}{L} dx.$$

Ex5. Triangular initial Temp. in a bar with insulated Ends.

$A_0 = \frac{L}{4}$ $A_n = \frac{2L}{n^2\pi^2}\left(2\cos\frac{n\pi}{2} - \cos n\pi - 1\right)$

$$u(x,t) = \frac{L}{4} - \frac{8L}{\pi^2}\left\{ \frac{1}{2^2}\cos\frac{2\pi x}{L} \exp\left(-\left(\frac{2c\pi}{L}\right)^2 t\right) \frac{1}{6^2} \cdots + \cdots \right)$$

as $t \to \infty$ $u \to \frac{L}{4}$ (not zero) different from Ex.3

12.7. Heat Eq'n : Modeling very long bars.

→ Solution by Fourier integrals and Transforms.

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

no BCs

$u(x,0) = f(x)$ $(-\infty < x < \infty)$ * main difference. of $0<x<L$

$u(x,t) = F(x)G(t)$

$F'' + pF = 0$ $\dot{G} + c^2p^2G = 0$

$F(x) = A\cos px + B\sin px$ $G(t) = e^{-c^2p^2t}$

for comparison

$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{L}x\, e^{-\lambda_n^2 t}$ $\lambda_n = \frac{cn\pi}{L}$

where $B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx$

$$u(x,t;p) = FG = (A\cos px + B\sin px)\, e^{-c^2p^2t}$$

Use of Fourier integral. (since $f(x)$ is non-periodic) → better to use Fourier intergral instead of Fourier series

→ A and B arbitrary → regard them as functions of p

So, $A(p)$ and $B(p)$

Solution → $u(x,t) = \int_0^\infty u(x,t;p)\,dp = \int_0^\infty [A(p)\cos px + B(p)\sin px]\, e^{-c^2p^2t}\,dp$

$u(x,0) = \int_0^\infty (A(p)\cos px + B(p)\sin px)\,dp = f(x)$

A(p) and B(p) from ICs

↓ Fourier integral! familiar form,

$A(p) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos pv\,dv$

$B(p) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin pv\,dv$.

using Fourier integral.

By entering A(p) & B(p),

$u(x,0) = \frac{1}{\pi}\int_0^\infty \left[\int_{-\infty}^{\infty} f(v)\cos(px - pv)\,dv\right]dp$ → by 11/6

Similarly

$u(x,t) = \frac{1}{\pi}\int_0^\infty \left[\int_{-\infty}^{\infty} f(v)\cos(px-pv)\, e^{-c^2p^2t}\,dv\right]dp$

by changing the order of integration. function of p.

$$u(x,t) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\left[\int_0^\infty e^{-c^2p^2t}\cos(px-pv)\,dp\right]dv \quad ①$$

by using the formula $\int_0^\infty e^{-s^2}\cos 2bs\,ds = \frac{\sqrt{\pi}}{2}e^{-b^2}$

① $s^2 = c^2p^2t$, ② $s = cp\sqrt{t}$ ③ $ds = c\sqrt{t}\,dp$ ④ $2bs = (x-v)p$ ⑤ $b = \frac{p(x-v)}{2s}$ ⑥ $b = \frac{x-v}{2c\sqrt{t}}$

$\therefore \int_0^\infty e^{-c^2p^2t}\cos(px-pv)\,dp = \frac{\sqrt{\pi}}{2c\sqrt{t}}\exp\left\{-\frac{(x-v)^2}{4c^2t}\right\}$

$$\boxed{\therefore u(x,t) = \frac{1}{2c\sqrt{\pi t}}\int_{-\infty}^{\infty} f(v)\exp\left\{-\frac{(x-v)^2}{4c^2t}\right\}dv} \quad ②$$

taking $z = \frac{(v-x)}{2c\sqrt{t}}$ → $v = x + 2cz\sqrt{t}$

$dz = \frac{1}{2c\sqrt{t}}\,dv$

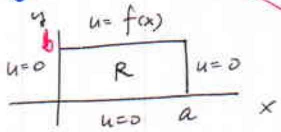
$$u(x,t) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} f(x + 2cz\sqrt{t})\, e^{-z^2}\,dz.$$

Steady 2-D Heat Problems. Laplace's Eq'n

Steady (or time-independent) $\frac{\partial u}{\partial t} = 0$

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

i) 1st BVP or Dirichlet problem



2nd BVP

ii) Neumann problem $u_n = \frac{\partial u}{\partial n}$ is prescribed.

3rd BVP

iii) Robin problem. u is prescribed on a portion of C

$u_n\left(\frac{\partial u}{\partial n}\right)$ is on rest of C.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

separating variable $u(x,y) = F(x)\,G(y)$

$$u_{xx} + u_{yy} = 0$$

$$F''G = -FG'' \Rightarrow \frac{1}{F}F'' = -\frac{G''}{G} = -k$$

$-p^2$

$F'' + kF = 0 \qquad F(0) = 0, \quad F(a) = 0$

$F'' + p^2F = 0$

$F = \cancel{A\cos px} + B\sin px$

$pa = n\pi$

$p = \frac{n\pi}{a}$

$k = \left(\frac{n\pi}{a}\right)^2 \qquad F(x) = F_n(x) = \sin\frac{n\pi}{a}x$

$$G'' - kG = 0 \quad G'' - \left(\frac{n\pi}{a}\right)^2 G = 0$$

$$G(y) = G_n(y) = A_n e^{\frac{n\pi y}{a}} + B_n e^{-\frac{n\pi y}{a}}$$

from B.C

$$\begin{cases} G_n(0) = 0. \quad A_n + B_n = 0 \quad B_n = -A_n \\ G_n(y) = A_n\left(e^{n\pi y/a} - e^{-n\pi y/a}\right) = \underbrace{2A_n}_{A_n^*} \sinh\frac{n\pi y}{a}. \end{cases}$$

A_n^* tentatively

Eigenfunction $u_n(x,y) = A_n^* \sin\frac{n\pi}{a}x \sinh\frac{n\pi y}{a}$.

$\hookrightarrow$ satisfies 3 BCs.

last BC : $u(x,b) = f(x)$ Consider the infinite series.

$$u(x,y) = \sum_{n=1}^{\infty} u_n(x,y)$$

constant.

$$u(x,b) = f(x) = \sum_{n=1}^{\infty} A_n^* \sin\frac{n\pi x}{a} \underbrace{\sinh\frac{n\pi b}{a}}$$

$$u(x,b) = \sum_{n=1}^{\infty} \underbrace{\left(A_n^* \sinh\frac{n\pi b}{a}\right)}_{b_n} \sin\frac{n\pi}{a}x = f(x)$$

Fourier Sine series.

$$u(x,y) = \sum_{n=1}^{\infty} u_n(x,y) = \sum_{n=1}^{\infty} A_n^* \sin\frac{n\pi x}{a} \cdot \sinh\frac{n\pi y}{a}$$

where $A_n^* = \frac{2}{a\sinh(n\pi b/a)} \int_0^a f(x)\sin\frac{n\pi x}{a}\,dx$

Ex 1. Temperature in an infinite bar.

$$f(x) = \begin{cases} u_0 = \text{const.} & \text{if } |x|<1 \\ 0 & \text{if } |x|>1. \end{cases}$$

~~$u(x,t) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} f(x+2cz\sqrt{t})\, e^{-z^2} dz$~~

from $u(x,t) = \frac{1}{2c\sqrt{\pi t}}\int_{-\infty}^{\infty} f(v)\exp\left\{-\frac{(x-v)^2}{4c^2t}\right\} dv$

$$u(x,t) = \frac{u_0}{2c\sqrt{\pi t}}\int_{-1}^{1}\exp\left(-\frac{(x-v)^2}{4c^2t}\right) dv$$

Fig 299

$$\frac{-1-x}{2c\sqrt{t}} < z < \frac{(1-x)}{2c\sqrt{t}}.$$

or $u(x,t) = \frac{u_0}{\sqrt{\pi}}\int_{-1-x/2c\sqrt{t}}^{(1-x)/2c\sqrt{t}} e^{-z^2} dz$ → error function

Fig

<Use of Fourier Transform>

Ex 2. Solve Ex 1. using Fourier transform. Take the Fourier transform with respect to x

$$\mathcal{F}(u_t) = c^2\mathcal{F}(u_{xx}) = c^2(-w^2)\mathcal{F}(u) = -c^2w^2\hat{u}$$

$$\mathcal{F}(u_t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u_t e^{-iwx} dx = \frac{1}{\sqrt{2\pi}}\frac{\partial}{\partial t}\int_{-\infty}^{\infty} u e^{-iwx} = \frac{\partial \hat{u}}{\partial t}$$

Ex2

$$\therefore \frac{\partial \hat{u}}{\partial t} = -c^2w^2\hat{u} \quad \text{(1st ODE)}$$

$$\hat{u}(w,t) = C(w)\, e^{-c^2w^2t}$$

IC. $\hat{u}(w,0) = C(w) = \hat{f}(w) = \mathcal{F}(f)$

$$\hat{u}(w,t) = \hat{f}(w)\, e^{-c^2w^2t}$$

inversion formula. $\mathcal{F}^{-1}(\hat{u}) =$ $u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(w)\, e^{-c^2w^2t}\, e^{iwx} dw$

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(v)\, e^{-iwv} dv$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} f(v)\left[\int_{-\infty}^{\infty} e^{-c^2w^2t}\, e^{i(wx-wv)} dw\right] dv.$$

Euler formuler $\cancel{i\sin(wx-wv)}$ (due to odd function

$$① = \therefore u(x,t) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\left[\int_0^{\infty} e^{-c^2w^2t}\cos(wx-wv)\, dw\right] dv$$

Same as that obtained from Fourier integral

Ex3. Solve ~~it~~ by method of convolution!

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(w)\, e^{-c^2w^2t}\, e^{iwx} dw$$

$f*g$ page 528. 11.9 $\hat{f}\cdot\hat{g}$ (13)

$\mathcal{F}(f*g) = \sqrt{2\pi}\,\mathcal{F}(f)\,\mathcal{F}(g)$ <inverse transform>

$$u(x,t) = (f*g)(x) = \int_{-\infty}^{\infty}\hat{f}(w)\,\hat{g}(w)\, e^{iwx} dw$$

where $\hat{g}(w) = \frac{1}{\sqrt{2\pi}} e^{-c^2w^2t}$ ⇒ we need to obtain $g = \mathcal{F}^{-1}(\hat{g})$

by the definition of convolution

$$(f*g)(x) = \int_{-\infty}^{\infty} f(p)\, g(x-p)\, dp \rightarrow \mathcal{F}(f*g) = \sqrt{2\pi}\,\mathcal{F}(f)\,\mathcal{F}(g),\quad f*g = \mathcal{F}^{-1}(\hat{f}\cdot\hat{g}) \rightarrow \int_{-\infty}^{\infty}\hat{f}(w)\hat{g}(w)e^{iwx} dw$$

using Fourier transform

$$\mathcal{F}(e^{-ax^2}) = \frac{1}{\sqrt{2a}} e^{-w^2/4a} \qquad c^2t = \frac{1}{4a} \text{ or } a = \frac{1}{4c^2t}$$

$$\mathcal{F}(e^{-x^2/4c^2t}) = \sqrt{2c^2t}\, e^{-c^2w^2t} = \sqrt{2c^2t}\,\sqrt{2\pi}\,\hat{g}(w)$$

$$g(x) = \therefore \mathcal{F}^{-1}(\hat{g}(w)) = \frac{1}{\sqrt{2c^2t}\sqrt{2\pi}} e^{-x^2/4c^2t}$$

②

replacing x with $x-p$. $u(x,t) = (f*g)(x) = \frac{1}{2c\sqrt{\pi t}}\int_{-\infty}^{\infty} f(p)\exp\left\{-\frac{(x-p)^2}{4c^2t}\right\} dp$

same as above!

Ex4

Fourier sine transform applied to the heat equation : Use of Fourier Sine Transform (skip!)

$u(x,0) = f(x)$

$u(0,t) = 0.$ $\qquad f(0) = u(0,0) = 0$

0 ———— ∞

$$\mathcal{F}_s(u_t) = \frac{\partial \hat{u}_s}{\partial t} = c^2 \mathcal{F}_s(u_{xx}) = -c^2 w^2 \mathcal{F}_s(u) = -c^2 w^2 \hat{u}_s(w,t)$$

$$\mathcal{F}_s(f''(x)) = -w^2 \mathcal{F}_s(f(x)) + \sqrt{\tfrac{2}{\pi}}\, w f(0)$$

$$\frac{\partial \hat{u}_s}{\partial t} + c^2 w^2 \hat{u}_s = 0 \qquad \hat{u}_s(w,t) = C(w)\, e^{-c^2 w^2 t}$$

From the initial condition $u(x,0) = f(x)$, $\hat{u}_s(w,0) = \hat{f}_s(w) = C(w)$

$$\therefore \hat{u}_s(w,t) = \hat{f}_s(w)\, e^{-c^2 w^2 t}.$$

$$\hat{f}_s(w) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin wx\, dx$$

Taking inverse Fourier sine Transform and substitution

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_s(w) \sin wx\, dw$$

$$\hat{f}_s(w) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(p) \sin wp\, dp.$$

$$u(x,t) = \frac{2}{\pi} \int_0^\infty \int_0^\infty f(p) \sin wp\; e^{-c^2 wt} \sin wx\, dp\, dw$$

$$\mathcal{F}_s^{-1}(\hat{u}_s(w,t))$$