Chap 13. Complex Numbers and Functions. Complex Differentiation

13.1. Complex Numbers and Their Geometric Representation. (Nothing New. Just Reminder)

Complex number $z = (x, y) = x + iy$.

x: real part, iy: imaginary part

$\mathrm{Re}\, z = x$

$\mathrm{Im}\, z = y$.

✓ addition, Multiplication.

$$z_1 = (x_1, y_1) \qquad z_2 = (x_2, y_2)$$

$$z_1 + z_2 = (x_1 + x_2,\ y_1 + y_2)$$

✓ $$z_1 z_2 = (x_1, y_1)(x_2, y_2) = (x_1 x_2 - y_1 y_2,\ x_1 y_2 + x_2 y_1)$$

when $x = 0$, pure imaginary $z = iy$

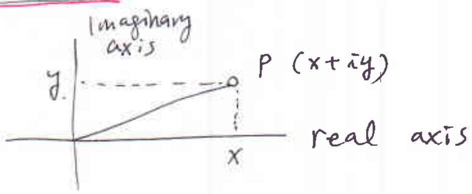
✓ subtraction,

$$z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2)$$

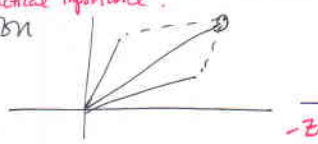
✓ Division $\dfrac{z_1}{z_2} = x + iy$ $\qquad x = \dfrac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} \qquad y = \dfrac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2}$

Consider the geometric representations of complex Numbers, which is of great practical importance.

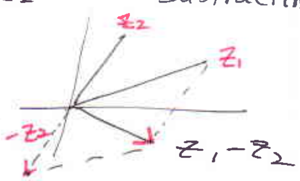
Complex plane



addition

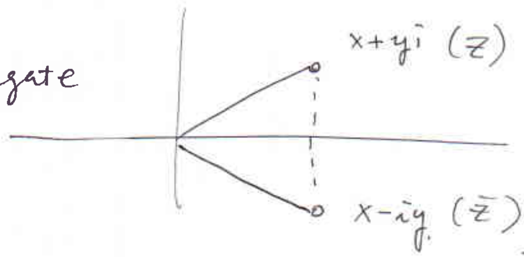


Subtraction



Conjugate Number

$\bar{z} = x - iy$. Complex Conjugate



$$\mathrm{Re}(z) = x = \frac{1}{2}(z + \bar{z})$$

$$\mathrm{Im}(z) = y = \frac{1}{2i}(z - \bar{z})$$

if z is real $z = \bar{z}$

$$\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2 \qquad \overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$$

$$\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2 \qquad \overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$$

13.2 Polar Form of Complex Numbers. Powers and Roots

(Another reminder)

$$x = r\cos\theta \qquad y = r\sin\theta$$

$$z = x + yi = r(\cos\theta + i\sin\theta)$$

$|z|$: distance of the point z from the origin

$|z_1 - z_2|$: distance between z_1 and z_2

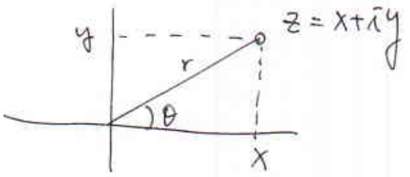
$$|z| = r = \sqrt{x^2+y^2} = \sqrt{z\bar{z}}$$

geometric interpretation $\quad\theta$ is called argument of z

arg z

$\therefore\ \theta = \arg z$, $\quad \tan\theta = \frac{y}{x}$

$\arg z = \operatorname{Arg} z \pm 2n\pi \quad (n = \pm 1, \pm 2 \ldots)$



$$r = |z| \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$-\pi < \operatorname{Arg} z \le \pi$

principal value.

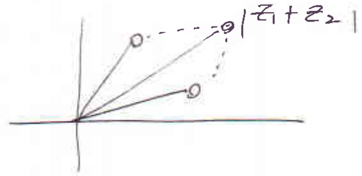
Ex 1)

$$z = 1 + i = \sqrt{2}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

$$|z| = \sqrt{2}$$
$$\arg z = \frac{\pi}{4} \pm 2n\pi$$
$$\operatorname{Arg} z = \frac{\pi}{4}$$

Triangle inequality.

$$|z_1 + z_2| \le |z_1| + |z_2|$$



generalized triangle inequality

$$|z_1 + z_2 + \cdots z_n| \le |z_1| + |z_2| + \cdots |z_n|.$$

Multiplication and Division in Polar Form

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1) \quad \text{and} \quad z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

$$z_1z_2 = r_1r_2\left[\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i\left(\sin\theta_1\cos\theta_2 + \sin\theta_2\cos\theta_1\right)\right]$$
$$= r_1r_2\left[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\right]$$

$$|z_1z_2| = |z_1||z_2|$$
$$\arg(z_1z_2) = \arg z_1 + \arg z_2$$

Division $\quad z_1 = \left(\frac{z_1}{z_2}\right)z_2 \qquad |z_1| = \left|\left(\frac{z_1}{z_2}\right)z_2\right| = \left|\frac{z_1}{z_2}\right||z_2| \qquad \therefore \frac{|z_1|}{|z_2|} = \left|\frac{z_1}{z_2}\right|$

By analogy,

$$\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$$

therefore

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\left[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right]$$

De Moivre's Formula. $(z_1 = z_2 = z)$, $z^2 = r^2(\cos 2\theta + i\sin 2\theta)$: by induction for $n = 1, 2, \cdots$)

$$z^n = r^n(\cos n\theta + i\sin n\theta) \quad (n = 0, 1, 2, \ldots)$$

For $|z| = r = 1$

$$(\cos\theta + i\sin\theta)^n = (\cos n\theta + i\sin n\theta)$$

$z^3 = 1$ $\quad (z^3 - 1) = 0$ $\quad z^2 + z + 1 = 0$

$z = 1$, ω, ω^2

Roots. if $z = w^n$ $\quad w = \sqrt[n]{z}$

we write z and w in the polar form,

$$z = r(\cos\theta + i\sin\theta) \quad \text{and} \quad w = R(\cos\phi + i\sin\phi)$$

By De Moivre's Formula

$$\omega^n = R^n(\cos n\phi + i\sin n\phi).$$

$R^n = r$ $\quad R = \sqrt[n]{r}$

$n\phi = \theta + 2k\pi$, thus $\phi = \dfrac{\theta}{n} + \dfrac{2k\pi}{n}$ where k is integer.

For $k = 0, 1, \ldots, n-1$. we get n distinct values of w.

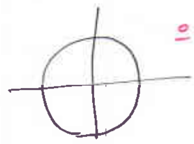
$\therefore$ $\sqrt[n]{z}$, for $z \neq 0$, has the n distinct values $(k = 0, 1, 2, \ldots n-1)$

$$w = \sqrt[n]{z} = \sqrt[n]{r}\left(\cos\frac{\theta + 2k\pi}{n} + i\sin\frac{\theta + 2k\pi}{n}\right)$$

for example) Taking $z = 1$. $|z| = r = 1$ $\quad \operatorname{Arg} z = 0$.

$$\sqrt[n]{1} = \cos\left(\frac{2k\pi}{n}\right) + i\sin\frac{2k\pi}{n} \qquad k = 0, 1, \ldots, n-1$$

n values : n^{th} roots of unity. $\rightarrow$ Fig.



으로 설명 (one by one.)

$\sqrt[3]{1}$ $\quad \sqrt[4]{1}$ $\quad \sqrt[5]{1}$ $\ldots$

$z^3 - 1 = 0$ $\quad z^4 - 1 = 0$ $\quad z^5 - 1 = 0$

If ω denotes the value corresponding to $k = 1$, then the n values of $\sqrt[n]{1}$

$=$ $1, \omega, \omega^2 \ldots \omega^{n-1}$ $\quad$ (n values of $\sqrt[n]{z}$)

of an arbitrary complex number z.

more generally, if w_1 is any nth root

w_1, $w_1\omega$, $w_1\omega^2$ $\ldots$ $w_1\omega^{n-1}$

because multiplying w by w^k correspond to increasing the argument of w_1 by $2k\pi/n$.

$(z-1)(z^2 + z + 1) = 0$

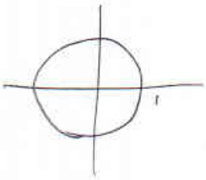
$\hookrightarrow$ if ω is solution, then ω^2 as well

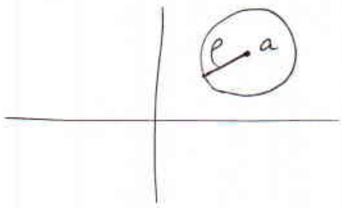
$$\omega^4 + \omega^2 + 1 = \omega + \omega^2 + 1 = 0.$$

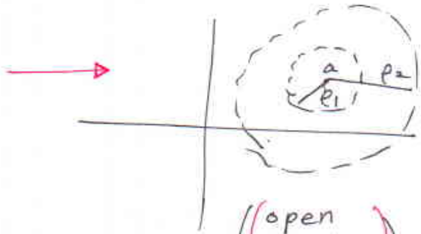
13.3 Derivative. Analytic Function.

Since the functions live in the complex plane, the concepts ('domain, derivative) are slightly different.

$|z| = 1$ unit circle $|z-a| = \rho$ $\rho_1 < |z-a| < \rho_2$







(open annulus)

neighborhood of a : $|z-a| < \rho$

if $\rho_1 \le |z-a| \le \rho_2$ Closed annulus.

$w = f(z)$

Complex function $w = f(z) = u(x,y) + i\,v(x,y)$ (Complex function $f(z)$ = equivalent to a pair of real functions $u(x,y)$ and $v(x,y)$)

Let $w = f(z) = z^2 + 3z$ Find u, v at $z = 1 + 3i$

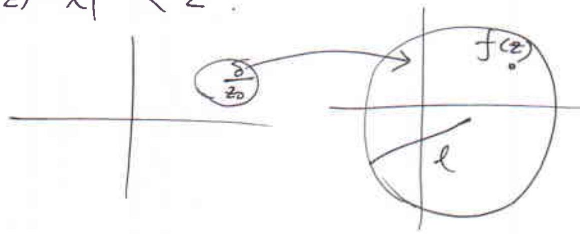
$z = (x+yi)$ $u^2 - v^2 + 2uvi + 3(u+vi)$

~~$(u^2 - v^2 + 3u)$~~

$(x^2 - y^2 + 3x) + (3y + 2xy)i$

$\therefore \left.\begin{matrix} x=1 \\ y=3 \end{matrix}\right) \Rightarrow (1-9+3) + (9+6)i$
$(-5+15i)$

limit $\lim_{z \to z_0} f(z) = l$.   for $|z - z_0| < \delta$ such that $|f(z) - l| < \varepsilon$.



continuous $\lim_{z \to z_0} f(z) = f(z_0)$

Derivative $(cf)' = cf'$ $(fg)' = f'g + fg'$ ...
$(f+g)' = f' + g'$

$f'(z_0) = \lim_{\Delta z \to 0} \dfrac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$

Same as in real calculus.

$f(z) = z^2$ $f'(z) = \lim_{\Delta z \to 0} \dfrac{(z+\Delta z)^2 - z^2}{\Delta z} = \lim_{\Delta z \to 0} (2z + \Delta z) = 2z$

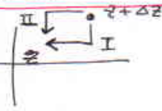
$f'(z_0) = \lim_{z \to z_0} \dfrac{f(z) - f(z_0)}{z - z_0}$

approach from any direction.

Ex. 4 $f(z) = \bar{z} = x - iy$.
$\Delta z = \Delta x + i\Delta y$.

$\dfrac{f(z+\Delta z) - f(z)}{\Delta z} = \dfrac{\overline{z + \Delta z} - \bar{z}}{\Delta z} = \dfrac{\overline{\Delta z}}{\Delta z} = \dfrac{\Delta x - i\Delta y}{\Delta x + i\Delta y}$

if $\Delta y = 0$, 1 (I)
$\Delta x = 0$ -1 (II)



$\boxed{\text{not differentiable}}$

Def) Analytic Function. (functions that are differentiable in some domain)

if $f(z)$ is defined and differentiable at any points in D.

said to be analytic at a point $z = z_0$ in D : if $f(z)$ is analytic in a neighborhood of z_0.

analyticity of $f(z)$ at z_0

$\longrightarrow$ $f(z)$ has a derivative at every point in some neighborhood of z_0

13.4 Cauchy-Rieman Equations. Laplace's Equation. → most important equation in this chapter.

provide a criterion for the analyticity of a complex function

$$w = f(z) = u(x,y) + i\,v(x,y)$$

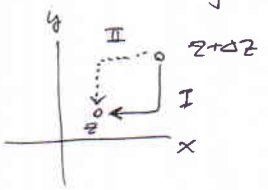
Theorem 1

f is analytic in a domain D if and only if

$$u_x = v_y, \quad u_y = -v_x \quad : \text{Cauchy-Rieman Eq'n.}$$

proof)

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$



two paths I and II

$$\Delta z = \Delta x + i\Delta y \qquad z + \Delta z = x + \Delta x + i(y + \Delta y)$$

$$\therefore f'(z) = \lim_{\Delta z \to 0} \frac{[u(x+\Delta x, y+\Delta y) + i\,v(x+\Delta x, y+\Delta y)] - [u(x,y) + i\,v(x,y)]}{\Delta x + i\Delta y}$$

① path I, $\Delta y \to 0$ first then $\Delta x \to 0$.

after $\Delta y = 0$, $\Delta z = \Delta x$

$$f'(z) = \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{v(x+\Delta x, y) - v(x,y)}{\Delta x}$$

$$= u_x + i v_x$$

Two must be the same.

② Similarly, path II, $\Delta x \to 0 \to \Delta y \to 0$.

$$f'(z) = \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) - u(x,y)}{i\Delta y} + i \lim_{\Delta y \to 0} \frac{v(x, y+\Delta y) - v(x,y)}{i\Delta y}$$

$$f'(z) = -i u_y + v_y$$

from previous example.

Ex 1)

$f(z) = z^2 = (x^2 - y^2) + 2xy\,i$ $\quad u_x = 2x,\ -u_y = 2y \quad v_y = 2x,\ v_x = 2y \quad \begin{pmatrix} u_x = v_y \\ -u_y = v_x \end{pmatrix}$ okay analytic.

$f(z) = \bar{z} = x - iy$ $\quad u_x = 1,\ u_y = 0 \quad v_y = -1,\ v_x = 0. \quad u_x \neq -v_y \Rightarrow$ not analytic

Theorem 2. Cauchy-Rieman Eq'n.

If $u(x,y)$, $v(x,y)$ have continuous first partial derivatives and satisfy the Cauchy-Rieman Eq'n in some domain D. Then complex function $f(z) = u(x,y) + i\,v(x,y)$ is analytic in D.

proof is optional

Ex 2) $f(z) = e^x(\cos y + i \sin y)$ analytic?

$u = e^x \cos y \qquad v = e^x \sin y$

$u_x = e^x \cos y \qquad v_y = e^x \cos y$

$u_y = -e^x \sin y \qquad v_x = e^x \sin y.$

Ex3) if $f(z)$ is analytic in a domain D

$f(z) = u + iv$ $|f(z)| = k = \text{const}$ in D.

then $f(z) = \text{const}$ in D

proof) $|f|^2 = |u+iv|^2 = u^2 + v^2 = k^2$

differentiation w.r.t. (x) $uu_x + vv_x = 0$

(y) $uu_y + vv_y = 0$

$v_x = -u_y$, $v_y = u_x$

$(uu_x - vu_y = 0) \times u$ To get rid of u_y

$(uu_y - vu_x = 0) \times v$

+

$(u^2+v^2)u_x = 0$

$(u^2+v^2)u_y = 0$ 마찬가지로 $\begin{pmatrix} x \leftrightarrow v \\ x \quad u \end{pmatrix}$ u_x 제거

(13.4) # 1

$u_x = u_y = 0$ $\Rightarrow$ $u_x = v_y = 0$

$u_y = -v_x = 0$

$u = \text{const}$ and $v = \text{const}$. hence $f = \text{const}$

① Cauchy - Rieman Eq'n in Polar form

$$u_r = \frac{1}{r} v_\theta$$

$$v_r = -\frac{1}{r} u_\theta$$

homework

Laplace Eq'n Harmonic functions.

Theorem 3) if $f(z) = u(x,y) + iv(x,y)$ is analytic.

Then both u, v satisfy Laplace's equation.

$$\nabla^2 u = u_{xx} + u_{yy} = 0.$$

$$\nabla^2 v = v_{xx} + v_{yy} = 0.$$

$\Rightarrow$ v is said to be a harmonic conjugate function of u in D.

proof) $u_x = v_y$ $\qquad u_y = -v_x$

x D'ff. $\downarrow$ $\qquad \downarrow$ y Diff

$u_{xx} = v_{yx}$ $\qquad u_{yy} = -v_{xy}$

since. $v_{yx} = v_{xy}$

$\therefore$ $u_{xx} + u_{yy} = 0$.

Solution = Harmonic function $\downarrow$ $\nabla^2 u = 0$

like wise $\therefore$ $v_{xx} + v_{yy} = 0$ $\nabla^2 v = 0$.

Ex4) How to find a harmonic Conjugate function by the Cauchy-Riemann Eq'n.

① harmonic $\rightarrow$ v

② harmonic conjugate function : v of u.

$u = x^2 - y^2 - y$.

$\nabla^2 u = 0$ $\quad \begin{pmatrix} u_x = 2x \\ u_y = -2y - 1 \end{pmatrix}$

$\therefore$ $v_y = u_x = 2x$ $\longrightarrow$ $v = 2xy + h(x)$

$v_x = -u_y = 2y + 1$ $\rightarrow$ ~~$v = 2$~~ $\downarrow$

$v_x = 2y + h'(x)$

$\frac{dh}{dx} = 1$ $\quad h = x + c$

$\therefore f(z) = (x^2 - y^2 - y) + i(2xy + x + c)$, $\qquad \therefore$ $v = 2xy + x + c$.

$= z^2 + iz + ic$

13.5 Exponential Function.

$$e^z = e^x(\cos y + i\sin y)$$

$e^x e^{iy}$ — Euler Formula.

e^z extends the real exponential function e^x of calculus in a natural fashion

i) $e^z = e^x$ for real $z = x$ $\cos y = 1$

ii) e^z is analytic. for all z

iii) $(e^z)' = e^z$, $e^{z_1+z_2} = e^{z_1}e^{z_2}$ ↳ can be proved

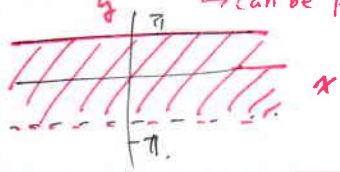
$re^{i\theta}$

$|e^z| = e^x$

$\arg e^z = y \pm 2n\pi$ $\quad (n = 0, 1, 2 \ldots)$

$-\pi < y \le \pi$: horizontal strip of width 2π, fundamental region of e^z

$e^{z+2\pi i} = e^z$ for all z



13.6 Trigonometric and Hyperbolic Function. Euler Formula

$$e^{ix} = \cos x + i\sin x$$
$$e^{-ix} = \cos x - i\sin x$$

~~Cosx~~ $\cos x = \frac{1}{2}(e^{ix} + e^{-ix})$

$\sin x = \frac{1}{2i}(e^{ix} - e^{-ix})$

for $z = x+yi$, by definition

$$\cos z = \frac{1}{2}(e^{iz} + e^{-iz})$$
$$\sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$$

Euler's formula is valid in complex.

$$e^{iz} = \cos z + i\sin z.$$

Ex1) show that

$$\cos z = \cos x \cdot \cosh y - i\sin x \sinh y$$
$$\sin z = \sin x \cosh y + i\cos x \sinh y$$

$$\cos z = \frac{1}{2}\left(e^{i(x+iy)} + e^{-i(x+iy)}\right) = \frac{1}{2}e^{-y}(\cos x + i\sin x) + \frac{1}{2}e^{y}(\cos x - i\sin x)$$

$$= \underbrace{\frac{1}{2}(e^y + e^{-y})}_{\cosh y}\cos x - \underbrace{\frac{1}{2}i(e^y - e^{-y})}_{\sinh y}\sin x$$

$$= \cosh y \cos x - i\sin x \sinh y$$

no real solution! but complex ~~Number~~ Solution is possible.

Ex2) $\cos z = 5$ (?)

$$\frac{1}{2}(e^{iz} + e^{-iz}) = 5 \Rightarrow e^{2iz} - 10e^{iz} + 1 = 0.$$

$(e^{iz}) = (e^{-y+xi}) = 5 \pm \sqrt{25-1} = 9.899$ & 0.101

$\therefore e^{-y} = 9.899$ or 0.101

↳ $y = \pm 2.292$

$e^{ix} = 1$ only real part

↳ $x = \pm 2n\pi$

$\therefore z = \pm 2n\pi \pm 2.292i$ $\quad (n = 0, 1, 2 \ldots 3)$

General formula. hold for complex values

$$\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2$$

$$\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \sin z_2 \cos z_1$$

$$\cos^2 z + \sin^2 z = 1$$

Hyperbolic functions

$$\cosh z = \frac{1}{2}(e^{z}+e^{-z})$$

$$\sinh z = \frac{1}{2}(e^{z}-e^{-z})$$

$\cosh iz = \cos z \qquad \sinh iz = i \sin z$

$\cos iz = \cosh z \qquad \sin iz = i \sinh z$

more complicated than real Logarithm

13.7 Logarithm. General Power. Principal Value.

$w = \ln z$ for $z = x+yi$ is defined for $z \neq 0$

$e^{w} = z \longrightarrow$ $w = u + iv$, $\quad z = re^{i\theta}$

$e^{w} = e^{u+iv} = re^{i\theta} = z \quad \left(e^{u} = r \Rightarrow u = \ln r, \quad v = \theta \right)$

$\left(\therefore w = u+iv, \quad \ln z = \ln r + i\theta \right)$

$\ln z = \ln r + i\theta \qquad \left(\begin{matrix} \theta = \arg z \\ r = |z| > 0 \end{matrix} \right)$ $\quad \hookrightarrow \theta = \arg z$ is infinitely many

Since the argument of z is determined only up to integer multiples of 2π, the complex natural logarithm $\ln z$ is infinitely many-valued.

principal value of $\ln z$ $\qquad$ $\ln z = \ln|z| + i \arg z$

$$\boxed{\mathrm{Ln}\, z = \ln|z| + i\,\mathrm{Arg}\, z \quad (z \neq 0)}$$ vs.

Since the other values of $\arg z$ differ by integer multiples of 2π,

$$\ln z = \mathrm{Ln}\, z \pm 2n\pi i \quad (n = 1, 2 \ldots)$$

For example, ① z is positive real, $\mathrm{Arg}\, z = 0$

② If z is negative real then $\mathrm{Arg}\, z = \pi$ (real part are same, imaginary part differ by integer multiples of 2π.)

$$\mathrm{Ln}\, z = \ln|z| + \pi i \quad (\text{from } \mathrm{Ln}\, z = \ln|z| + i\,\mathrm{Arg}\, z)$$

$$\ln e^{z} = z \pm 2n\pi i \qquad (n = 0, 1, \ldots)$$

$\ln(z_1 z_2) = \ln z_1 + \ln z_2, \qquad \ln(z_1/z_2) = \ln z_1 - \ln z_2$

Ex 2) Let $z_1 = z_2 = e^{\pi i} = -1$

$3-4i \qquad \tan\theta = -\frac{4}{3}$

$\ln(3-4i) = \ln 5 + i \arg(3-4i) = 1.609438 - 0.927295i \pm 2n\pi i$

$\ln 1 = 0, \pm 2\pi i, \pm 4\pi i \ldots$	$\mathrm{Ln}\, 1 = 0$
$\ln 4 = 1.386294 \pm 2n\pi i$	$\mathrm{Ln}\, 4 = 1.386294$
$\ln(-1) = \pm\pi i, \pm 3\pi i$	$\mathrm{Ln}(-1) = \pi i$
$\ln i = \frac{\pi}{2} i, -3\pi/2$	$\mathrm{Ln}(i) = \pi i/2$

Theorem 1. Analyticity of the Logarithm.

For every $n = 0, \pm 1, \pm 2 \ldots$, $(\ln z = \mathrm{Ln}\, z \pm 2n\pi i)$, which is analytic, $\boxed{\text{except at } 0, \text{ and on the negative real axis,}}$ has the derivative

$$(\ln z)' = \frac{1}{z}$$

(Cauchy-Rieman equation satisfied?)

proof) $\ln z = \ln r + i(\theta + c) = \frac{1}{2}\ln(x^2+y^2) + i\left(\arctan\frac{y}{x} + c\right)$

$u_x = \frac{x}{x^2+y^2} = v_y = \frac{1}{1+(y/x)^2} \cdot \frac{1}{x}$

$u_y = \frac{y}{x^2+y^2} = -v_x = -\frac{1}{1+(y/x)^2}\left(-\frac{y}{x^2}\right)$

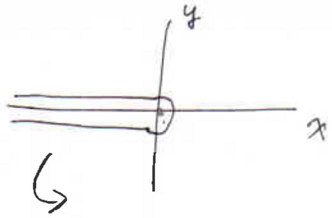
$\longrightarrow$ Cauchy-Riemann eq'n holds.

$$\therefore (\ln z)' = u_x + i v_x = \frac{x}{x^2+y^2} + i\frac{1}{1+(y/x)^2}\left(-\frac{y}{x^2}\right) = \frac{x-iy}{x^2+y^2} = \frac{1}{z}$$

branch

branch cut : negative real axis

principal branch : The branch for $n=0$.



z not 0 or negative real

General Powers.

$z = x+iy$.

$$\boxed{z^c = e^{c\ln z}}$$

$$z^c = e^{c \operatorname{Ln} z} \quad : \text{principal value of } z^c.$$

If $c = n = 1, 2, \ldots$

If $c = -1, -2, \ldots$

If $c = 1/n$. $\quad z^c = \sqrt[n]{z} = e^{(1/n)\ln z}$

Ex3) General Power

$\ln|z| + i\arg z = i\frac{\pi}{2} + \pm 2n\pi i = \ln z$ $\quad$ $i\frac{\pi}{2} = (\operatorname{Ln} z)$

$$i^i = e^{i\ln i} = \exp(i \ln i) = \exp\left(i\left(\frac{\pi}{2}i \pm 2n\pi i\right)\right)$$

$$= e^{-(\pi/2) \mp 2n\pi}$$

all real : principal value $(n=0)$

$$e^{-\pi/2}$$

$$(1+i)^{2-i} = \exp((2-i)\ln(1+i)) = \exp\left[(2-i)\left[\ln|z| + i\arg z \pm 2n\pi i\right]\right]$$

$\ln|z| = \sqrt{2}$, $\arg z = \frac{\pi}{4}$

$$\therefore \exp\left[2\ln\sqrt{2} + \frac{\pi}{4} \mp 2n\pi + i\left(-\ln\sqrt{2} + \frac{\pi}{2} \pm 4n\pi\right)\right]$$

$\ln 2^{-\frac{1}{2}}$ $\quad$ $\left((2)^{-\frac{1}{2}} + \frac{\pi}{2}\right)i$

$$2 \cdot e^{(\pi/4 \pm 2n\pi)}\left[\cos\left(\frac{\pi}{2} + \ln 2^{-\frac{1}{2}}\right)\right.$$

$\to -\sin\left(-\frac{1}{2}\ln 2\right)$

$+\cos$

$\sin\left(\frac{1}{2}\ln 2\right)$

$+ i\sin\left(\frac{\pi}{2} + \ln 2^{-\frac{1}{2}}\right)$

$\cos\left(\frac{1}{2}\ln 2\right)$

$+ i\cos\left(\frac{1}{2}\ln 2\right)$

h.w by 11/29

13.1	13. 14. 18
13.2	15. 22. 29
13.3	21
13.4	1. 15. 22
13.5	2. 13. 16
13.6	9. 11
13.7	8. 19. 25

$e^{i\theta} = \cos\theta + i\sin\theta$

$e^{i\theta}$

$\frac{1}{2}\ln\left(\frac{1}{2}\right)^{\frac{1}{2}}$

$\cos\left(\frac{\pi}{2} - \frac{1}{2}\ln 2\right)$

$-\sin\left(-\frac{1}{2}\ln 2\right) = \sin(=$