Chap 15. Power series. Taylor series

$\int f(z)\,dz$ ① Cauchy's integral formula $\longleftarrow$ Cauchy's integral Theorem

② residue integration. (requires a thorough understanding of power series, Taylor series.)

$\hookrightarrow$ series solution

Any analytic function can be represented by Taylor series.

15.1. Sequences. Series. Convergence Tests.

Sequences $\downarrow$ 수열   급수   수렴 / 발산 (?)

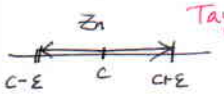
sequence

Convergent sequence $\displaystyle\lim_{n\to\infty} z_n = c$

$\updownarrow$

divergent sequence.

real sequence



ex $\left(\dfrac{i^n}{n}\right)$ as $n\to\infty$ $z_n \to 0$. Convergent sequence.

i^n as $n\to\infty$ $\pm1, \pm i$ divergent.

oscillation

Theorem 1. $z_n = x_n + iy_n$. as $n\to\infty$ $z_n = a+bi \iff \displaystyle\lim_{n\to\infty} x_n = a$ & $\displaystyle\lim_{n\to\infty} y_n = b$

Series

series. $\displaystyle\sum_{m=1}^{\infty} z_m = z_1 + z_2 + \cdots$

$S_n = z_1 + z_2 + \cdots + z_n$

nth partial sum.

Convergent series. $\displaystyle\lim_{n\to\infty} S_n = S$. $S = \displaystyle\sum_{m=1}^{\infty} z_m = z_1 + z_2 + \cdots$

Remainder or error $R_n = z_{n+1} + z_{n+2} + z_{n+3} + \cdots$

as $n\to\infty$ $R_n \to 0$.

Theorem 2 $z_m := x_m + iy_m$. $\sum z_m \Rightarrow u+iv \iff \sum x_m = u$, $\sum y_m = v$.

Theorem 3. [If series $z_1 + z_2 + \cdots$ converges $\Rightarrow \displaystyle\lim_{n\to\infty} z_n = 0$] $\not\Leftarrow$ ex. $\left(\frac{1}{n}\right)$

If $z_1 + z_2 + \cdots$ converges, with the sum S, $z_m = S_m - S_{m-1}$

$\displaystyle\lim_{m\to\infty} z_m = \lim_{m\to\infty}(S_m - S_{m-1}) = \lim_{m\to\infty} S_m - \lim_{m\to\infty} S_{m-1} = S - S = 0$

Theorem 4. Cauchy's Convergence Principle for series.

series $z_1 + z_2 + \cdots$ is convergent $\longleftrightarrow$

(If a series is absolutely convergent, if $\displaystyle\sum_{m=1}^{\infty} |z_m|$ is convergent) it is convergent.

If $z_1 + z_2 + \cdots$ converges, but $\sum |z_m|$ is divergent, then the series $z_1 + z_2 \cdots$ is conditionally convergent.

Geometric series $\displaystyle\sum_{m=0}^{\infty} q^m = 1 + q + q^2 + \cdots$ converges to $\dfrac{1}{1-q}$ if $|q| < 1$

diverges if $|q| \geq 1$

① Ratio Test

<u>Th 7</u> If a series $z_1 + z_2 \cdots$ with $z_n \neq 0$ $(n = 1, 2 \ldots)$

$$\left| \frac{z_{n+1}}{z_n} \right| \leq q < 1 \longrightarrow$$ Series Converges.

Th8 $\displaystyle \lim_{n\to\infty} \left| \frac{z_{n+1}}{z_n} \right| = L$

$L < 1$ Converges

$L > 1$ diverges

$L = 1$. depends.

Ex 4) $\displaystyle \sum_{m=0}^{\infty} \frac{(100+75i)^n}{n!}$ $\quad \left| \frac{z_{n+1}}{z_n} \right| = \frac{|100+75i|}{n+1} = \frac{125}{n+1} \rightarrow L = 0.$

② Roots Test

$\sqrt[n]{|z_n|} \leq q < 1$. converges.

$q > 1$ divergees

15.2. Power Series

$\displaystyle \sum_{n=0}^{\infty} a_n (z - z_0)^n$: power series in power of $(z - z_0)$

convergence behavior of power series.

Ex1) $\displaystyle \sum_{n=0}^{\infty} z^n \longrightarrow$ converges absolutely if $|z| < 1$

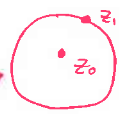
diverges if $|z| \geq 1$.

Ex2) $\displaystyle \sum_{n=0}^{\infty} \frac{z^n}{n!} = e^z =$ ratio test $\dfrac{|z|}{n+1} \rightarrow 0$ as $n \rightarrow \infty$

for any z. $(z \neq 0)$

<u>Theorem 1</u> convergence of a power series. $\left(\displaystyle\sum_{n=0}^{\infty} a_n (z - z_0)^n \right.$ —(1)

(a) Every power series converges at the center $z_0 \rightarrow$ to a_0



(b) If (1) converges at a point $z = z_1 \neq z_0$, it converges absolutely for every z closer to z_0 than z_1, that is,

$\boxed{|z - z_0| < |z_1 - z_0|} = r$

z_1

$a_n (z_1 - z_0)^n \rightarrow 0$ as $n \rightarrow \infty$

boundedness in absolute value.

$|a_n (z_1 - z_0)^n| < M$ for every $n = 0, 1 \ldots$

• $|a_n (z - z_0)^n| = \left| a_n \left(\dfrac{z - z_0}{z_1 - z_0} \right)^n (z_1 - z_0)^n \right|$

$\leq M \left| \dfrac{z - z_0}{z_1 - z_0} \right|^n$

Summation over n.

$$\sum_{n=1}^{\infty} |a_n (z - z_0)^n| \leq M \underbrace{\sum_{n=1}^{\infty} \left| \frac{z - z_0}{z_1 - z_0} \right|^n}_{\text{convergent}}$$

$\left| \dfrac{z - z_0}{z_1 - z_0} \right| < 1$

(C) If (1) diverges at $z = z_2$, it diverges for every z farther away from z_0 than z_2. $\longrightarrow$ proof) 거짓이라면, z_0 보다 z_2에서 먼 z_3에 수렴해야 한다. 그럼. (b) 에 의해 z_2에서 수렴해야 한다. 이것은 가정에 위배됨. 따라서

Radius of Convergence of a Power Series : Smallest circle with center z_0 that includes all the points at which a given power series converges.

$|z-z_0| < R$ Circle of Convergence

$\longrightarrow$ radius of convergence.

Ex) $R=\infty$ if the series (1) converges for all z. $\sum \frac{z^n}{n!} = e^z$

Ex) $R=0$ if the series (1) converges only at the center $z=z_0$ $\Rightarrow \sum_{n=0}^{\infty} n!\, z^n$

Th2) Radius of Convergence R.

sequence $\left|\frac{a_{n+1}}{a_n}\right|$ converged with limit L^*

if $L^*=0$. then $R=\infty$

$$R = \frac{1}{L^*} = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$$

Cauchy – Hadamard formula.

If $\left|\frac{a_{n+1}}{a_n}\right| \to \infty$ then $R=0$ (convergence only at the center z_0).

Ex5) $\sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2}(z-3i)^n$ $\quad |a_n/a_{n+1}|$

$$R = \lim_{n\to\infty}\left|\frac{(2n)!}{(n!)^2} \Big/ \frac{(2n+2)!}{((n+1)!)^2}\right| = \lim_{n\to\infty} \frac{(n+1)^2}{(2n+2)(2n+1)} = \frac{1}{4}$$

Converges in the open disk $|z-3i| < \frac{1}{4}$

Ex6) $\sum_{n=0}^{\infty}\left[1+(-1)^n+\frac{1}{2^n}\right]z^n = 3+\frac{1}{2}z+\left(2+\frac{1}{4}\right)z^2+\frac{1}{8}z^3+\left(2+\frac{1}{16}\right)z^4$

$\longrightarrow$ $\left|\frac{a_{n+1}}{a_n}\right|$ not converges $\quad \frac{1}{6}$, $2\left(2+\frac{1}{4}\right)$, $\frac{1}{8}\left(2+\frac{1}{4}\right)$ . . .

수렴 안함.

$\sqrt[n]{|a_n|}$ $\Big<$ for odd n $\quad \frac{1}{2}$

for even n $\quad \sqrt[n]{2+\frac{1}{2^n}} = 1$ as $n\to\infty$

Two limit points.

$\longrightarrow R = \frac{1}{\tilde{l}}$, $\tilde{l}$ greatest limit point of the sequence $\sqrt[n]{|a_n|}$

$\therefore \tilde{l}=1$. $\therefore R=1$.

15.3 Functions given by Power series.

Question
(power series represent analytic functions ?)

$z_0 = 0 \longrightarrow \sum_{n=0}^{\infty} a_n z^n$

ex).

$f(z) = \sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + \cdots \quad (|z| < R)$

$\hookrightarrow$ $f(z)$ is represented by the power series ✓

or it is developed in the power series ✓

$f(z) = \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$ represents the function in the interior $|z|<1$ of unit circle

Th 1 : Continuity of the sum of a power series.

If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ with $|z| < R$, then $f(z)$ is continuous at $z=0$

Th 2 : Identity Theorem for power series. Uniqueness.

$a_0 + a_1 z + a_2 z^2 + \cdots$, $b_0 + b_1 z + b_2 z^2 + \cdots$.

✓ convergent for $|z| < R$

✓ same sum for all these z.

Then series are identical. $(a_0 = b_0,\ a_1 = b_1,\ a_2 = b_2 \ldots)$

$\boxed{\text{Operation on Power series}}$

i) Termwise addition and subtraction

ii) Termwise multiplication.

$f(z) \cdot g(z) = \left(\sum_{k=0}^{\infty} a_k z^k\right)\left(\sum_{m=0}^{\infty} b_m z^m\right)$

$= a_0 b_0 + (a_0 b_1 + a_1 b_0) z + (a_0 b_2 + a_1 b_1 + a_2 b_0) z^2 + \cdots$

Cauchy product.

iii) Termwise Differentiation and integration.

$\sum_{n=1}^{\infty} n a_n z^{n-1} = a_1 + 2 a_2 z + 3 a_3 z^2 + \cdots$ $\boxed{(\text{same } R)}$

iv) Termwise Integration of Power series.

$\sum_{n=0}^{\infty} \frac{a_n}{n+1} z^{n+1} = a_0 z + \frac{a_1}{2} z^2 + \frac{a_2}{3} z^3 + \cdots$ $\boxed{(\text{same } R)}$

< power series represent analytic functions >

Th5 : Analytic functions. Their Derivatives.

(powers series and its derivative) $\Rightarrow$ analytic function

(power series with a nonzero ~~represen~~ R) $\longrightarrow$ analytic function.

its derivative

?

(next section) Every given analytic function $f(z)$ can be represented by power series? Taylor series

15.4 Taylor and Maclaurin series. : Every analytic function $f(z)$ can be represented by power series.

$(z_0 = 0)$

$$f(z) = \sum_{n=1}^{\infty} a_n (z - z_0)^n \quad \text{where} \quad a_n = \frac{1}{n!} f^{(n)}(z_0)$$

real Taylor series

$$= \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(z_0)(z - z_0)^n \qquad |z - z_0| < R.$$

or

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z^*)}{(z^* - z_0)^{n+1}} dz^*$$

(from 14.4 Derivative of analytic functions)

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z^*)}{(z^* - z_0)^{n+1}} dz^*$$

(C : simple closed path that contains z_0 in its interior
$f(z)$ analytic in a domain containing C)

Maclaurin series : Taylor series with center $z_0 = 0$.

$$f(z) = f(z_0) + \frac{z - z_0}{1!} f'(z_0) + \frac{(z - z_0)^2}{2!} f''(z_0) + \cdots + \frac{(z - z_0)^n}{n!} f^{(n)}(z_0) + R_n(z)$$

$$\text{where } R_n(z) = \frac{(z - z_0)^{n+1}}{2\pi i} \oint_C \frac{f(z^*)}{(z^* - z_0)^{n+1}(z^* - z)} dz^*$$

remainder.

Th1. Taylor's Theorem.

Let $f(z)$: analytic in a domain D.
$z = z_0$ any point in D.
$\longrightarrow$ Taylor series exists.

$R_n(z)$.

$$|a_n| \le \frac{M}{r^n} \quad : \quad M : \text{maximum of } |f(z)| \text{ on a circle } |z - z_0| = r \text{ in } D$$

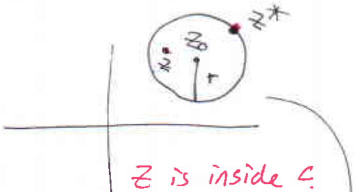
Beginning from

proof) Cauchy's integral formula.

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0} dz$$

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(z^*)}{z^* - z} dz^* \leftarrow$$

develop $\frac{1}{z^* - z}$ in powers of $z - z_0$.



z is inside C

$$\therefore \frac{1}{z^* - z} = \frac{1}{z^* - z_0 - (z - z_0)} = \frac{1}{(z^* - z_0)\left(1 - \frac{z - z_0}{z^* - z_0}\right)}$$

Consider z^* and z_0.

$$\left|\frac{z - z_0}{z^* - z_0}\right| < 1$$

from geometric sum,

$$1 + q + q^2 + \cdots + q^n = \frac{1 - q^{n+1}}{1 - q} = \frac{1}{1 - q} - \frac{q^{n+1}}{1 - q} \qquad (q \ne 1)$$

entering

$$\therefore \frac{1}{1 - q} = 1 + q + q^2 + \cdots + q^n + \frac{q^{n+1}}{1 - q} \Rightarrow \left(q = \frac{z - z_0}{z^* - z_0}\right)$$

$$\frac{1}{z^*-z} = \frac{1}{z^*-z_0}\left[\left(1+\frac{z-z_0}{z^*-z_0}+\left(\frac{z-z_0}{z^*-z_0}\right)^2+\cdots+\left(\frac{z-z_0}{z^*-z_0}\right)^n\right]+\frac{1}{z^*-z}\left(\frac{z-z_0}{z^*-z_0}\right)^{n+1}\frac{1}{(z^*-z_0)}$$

$(z^*-z_0)(1-g)$

entering $f(z) = \frac{1}{2\pi i}\oint_C \frac{f(z^*)}{z^*-z}dz^*$

$$= \frac{1}{2\pi i}\oint_C \frac{f(z^*)}{z^*-z_0}dz^* + \frac{(z-z_0)}{2\pi i}\oint_C \frac{f(z^*)}{(z^*-z_0)^2}dz^* + \frac{(z-z_0)^2}{2\pi i}\oint_C \frac{f(z^*)}{(z^*-z_0)^3}dz^*$$

$$+\cdots+\frac{(z-z_0)^n}{2\pi i}\oint_C \frac{f(z^*)}{(z^*-z_0)^{n+1}}dz^* + R_n(z)$$

$$R_n(z) = \frac{(z-z_0)^{n+1}}{2\pi i}\oint_C \frac{f(z^*)}{(z^*-z_0)^{n+1}(z^*-z)}dz^*$$

So that The integrals are those in (2) $a_n = \frac{1}{2\pi i}\oint \frac{f(z^*)\,dz^*}{(z^*-z_0)^{n+1}}$

so that we have proved the Taylor formula

end of proof

$\boxed{\text{as } n\to\infty,\quad R_n(z)\to 0.}$ ✓

Then $f(z) = \sum a_n(z-z_0)^n$ converges.

Singularity. Radius of Convergence

↳ on the circle of convergence, there is at least one singular point of $f(z)$, a point $z=c$ at which $f(z)$ is not analytic.

→ $f(z)$ is singular at c or has a singularity at c.

→ 11/28 (木)

Theorem 2

or

Power series with a nonzero radius of convergence

→ Taylor series of its sum

$f(z)=(1-z)^{-1}$ $\quad f'(z)=(1-z)^{-2}$ $\quad f''(z)=2!\,(1-z)^{-3}$ $\quad f^{(n)}(z)=n!\,(1-z)^{\cdots}$

① Ex 1 $\quad f(z)=\frac{1}{1-z} \qquad f^{(n)}(z)=\frac{n!}{(1-z)^{n+1}} \to f^{(n)}(0)=n! \to a_n=\frac{n!}{n!}=1$

$$\frac{1}{1-z}=\sum_{n=0}^{\infty} z^n = 1+z+z^2+\cdots \qquad f(z) \text{ is singular at } z=1.$$

$|z|<1$

singular point is located on the circle of convergence.

④ Ex 2. $\quad e^z=\sum_{n=0}^{\infty}\frac{z^n}{n!}$ $\quad$ real → yes! $\quad$ imaginary → ?

↳ $z=iy$

even $\qquad$ odd

$$e^{iy}=\sum_{n=0}^{\infty}\frac{(iy)^n}{n!}=\underbrace{\sum_{k=0}^{\infty}(-1)^k\frac{y^{2k}}{(2k)!}}_{\cos y}+i\underbrace{\sum_{k=0}^{\infty}(-1)^k\frac{y^{2k+1}}{(2k+1)!}}_{\sin y}$$

$\cos y + i \sin y$

$$\cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} \qquad \text{from} \quad \cos iz = \cosh z$$

$$\sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} \qquad \sin iz = i \sinh z$$

5 Logarithm

$$\mathrm{Ln}(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - + \cdots$$

$\downarrow (z \to -z)$ (−) both sides

$$+ \quad -\mathrm{Ln}(1-z) = z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots = \mathrm{Ln}\frac{1}{1-z}$$

$$\mathrm{Ln}\frac{(1+z)}{1-z} = 2\left(z + \frac{z^3}{3} + \frac{z^5}{5} + \cdots\right)$$

2 Ex 5.

$$f(z) = \frac{1}{z^2+1} \qquad \left(\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n\right)$$

$\therefore z \to -z^2$ substitution

$$\frac{1}{1+z^2} = \sum_{n=0}^{\infty} (-z^2)^n = \sum_{n=0}^{\infty} (-1)^n z^{2n} \qquad |z|<1$$

3 Ex 6

$$f(z) = \arctan z$$

integration

$$f'(z) = \frac{1}{1+z^2} = \sum_{n=0}^{\infty} (-1)^n z^{2n}$$

integration term by term

$$\arctan z = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} z^{2n+1} \qquad |z|<1$$

6 Ex 7. Develop $\frac{1}{c-z}$ in powers of $z-z_0$

$$\frac{1}{c-z} = \frac{1}{c-z_0-(z-z_0)} = \frac{1}{(c-z_0)\left(1-\frac{z-z_0}{c-z_0}\right)} = \frac{1}{c-z_0}\sum_{n=0}^{\infty}\left(\frac{z-z_0}{c-z_0}\right)^n$$

for only $\left|\frac{z-z_0}{c-z_0}\right| < 1$ or $|z-z_0| < |c-z_0|$

7 Ex 8. Binomial series.

$$f(z) = \frac{2z^2+9z+5}{z^3+z^2-8z-12}$$

Find the Taylor series with center ($z_0 = 1$)

n 개 $\frac{(-m)(-m-1)\cdots(-m-n+1)}{n!}$

$$\frac{1}{(1+z)^m} = (1+z)^{-m} = \sum_{n=0}^{\infty}\binom{-m}{n} z^n = 1 - mz + \frac{m(m+1)}{2!}z^2 - \frac{m(m+1)(m+2)}{3!}z^3 + - \cdots$$

$$\binom{r}{n} = \frac{r(r-1)\cdots(r-n+1)}{1\cdot 2\cdots n}$$

$$f(z) = \frac{1}{(z+2)^2} + \frac{2}{z-3} = \frac{1}{(3+z-1)^2} - \frac{2}{2-(z-1)} = \frac{1}{9}\left(\frac{1}{(1+\frac{1}{3}(z-1))^2}\right) - \frac{1}{1-\frac{1}{2}(z-1)}$$

$$= \frac{1}{9}\sum_{n=0}^{\infty}\binom{-2}{n}\left(\frac{z-1}{3}\right)^n - \sum_{n=0}^{\infty}\left(\frac{z-1}{2}\right)^n = \sum_{n=0}^{\infty}\left[\frac{(-1)^n (n+1)}{3^{n+2}} - \frac{1}{2^n}\right](z-1)^n$$

$$\frac{(-2)(-3)\cdots(-2-n+1)}{1\cdot 2\cdots n} = \frac{2\cdots(n+1)(-1)^n}{1\cdot 2\cdots n} = (-1)^n(n+1)$$

$(|z-1|<3$ or $|z-1|<2)$