



9 Review of Linear Systems, Sturm-Liouville Problems

9.1 Review of Linear Systems

- rank $A = \max \# \text{ of. lin. indept cols of } A = \dim \text{ of } \mathcal{C}(A)$
 $= \max \# \text{ of. lin. indept rows of } A = \dim \text{ of } \mathcal{R}(A)$
- Homogeneous lin.sys $Ax = 0, A \in \mathbb{R}^{m \times n}, x \in \mathbb{R}^n$ has nontrivial solutions iff rank $A < n$. These solutions with $x = 0$ form a vector space of dim ' $n - \text{rank } A$ ', called $\mathcal{N}(A)$
- Nonhomogeneous lin.sys $Ax = b : x = x_p + x_h$. $Ax_p = b, x_h$ runs through all the sols of the homo sys.
- $\det(A - \lambda I) = 0$, eigvals/eigvecs
- orthogonal/sym matrices.

9.2 Sturm-Liouville Problems

Recall from Eng Math 1.

- Sturm-Liouville eqn and famous examples

Sturm-Liouville Eqn: $(r(x)y')' + (q(x) + \lambda p(x))y = 0 \quad (1)$

ex. Bessel: $(xy')' + (-n^2/x + \lambda x)y = 0$
ex. Legendre: $((1 - x^2)y')' + \lambda y = 0 .$

- If y_m, y_n are eigfns of the S-L prob (??) with $p(x) > 0$ on $a \leq x \leq b$ corresponding to different eigvals λ_m, λ_n , then they are orthogonal w.r.t. $p(x)$, i.e.,

$$(y_m, y_n) \triangleq \int_a^b p(x)y_m(x)y_n(x) dx = 0 . \quad (2)$$

- **Definition.** [Orthonormal ftns]

$$(y_m, y_n) = \begin{cases} 0 & \text{if } m \neq n \\ 1 & \text{if } m = n \end{cases}$$

- Let $f(x) = \sum_{m=0}^{\infty} a_m y_m(x)$ on $a \leq x \leq b$.

$$\Rightarrow (f, y_n) \triangleq \int_a^b p(x)f(x)y_n(x) dx = a_n \|y_n\|^2$$
$$\therefore a_n = \frac{(f, y_n)}{\|y_n\|^2} = \frac{1}{\|y_n\|^2} \int_a^b p(x)f(x)y_n(x) dx . \quad (3)$$

Example .[S-L prob & Fourier series] $y'' + \lambda y = 0$, $y(\pi) = y(-\pi)$, $y'(\pi) = y'(-\pi)$

Solution.

If $\lambda < 0$, then trivial. For $\lambda > 0$, gen. sol. $y = A \cos kx + B \sin kx$ with $k = \sqrt{\lambda}$, and B.C. gives

$$\sin k\pi = 0 \Rightarrow k = n, \quad \lambda = n^2 = 0, 1, 4, 9, \dots$$

This is S-L prob with $p(x) = 1$, $a = -\pi$, $b = \pi$, thus eigtns $1, \cos x, \sin x, \cos 2x, \sin 2x, \dots$ are orthogonal by (??). This can be also proven by direct integration.

Let $f(x) = a_0 + \sum_{m=1}^{\infty} (a_m \cos mx + b_m \sin mx)$ on $-\pi \leq x \leq \pi$. \rightsquigarrow Fourier series

Since

$$\begin{aligned} \|1\| &= \sqrt{\int_{-\pi}^{\pi} 1^2 dx} = \sqrt{2\pi} \\ \|\cos nx\| &= \sqrt{\int_{-\pi}^{\pi} \cos^2 nx dx} = \sqrt{\pi} \\ \|\sin nx\| &= \sqrt{\int_{-\pi}^{\pi} \sin^2 nx dx} = \sqrt{\pi}, \end{aligned}$$

(??) becomes Euler formulas:

$$\left. \begin{array}{lcl} a_0 & = & \frac{(f, 1)}{\|1\|^2} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx \\ a_n & = & \frac{(f, \cos nx)}{\|\cos nx\|^2} = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx \\ b_n & = & \frac{(f, \sin nx)}{\|\sin nx\|^2} = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx. \end{array} \right\} \rightsquigarrow \text{Fourier coeff.} \quad \blacksquare$$