Chapter 2 Turbulent Jet and Plumes

Objectives:

- Study buoyant jets and plumes, strong man-induced flow patterns used to achieve rapid initial dilutions for water quality control
- Understand the theory of jets and plumes before considering the special type of discharge structure for diluted wastes
- Give the design engineer a firm background in the fundamentals of the theory essential to the prediction of how a given discharge system will perform

2.3 Environmental Parameters

• Effects of jet parameters on jet dilution and mechanics → Sec. 2.2

- jet momentum
- buoyancy
- angle of discharge

• Effects of environmental factors → Sec. 2.3

- density stratification
- ambient currents
- ambient turbulence

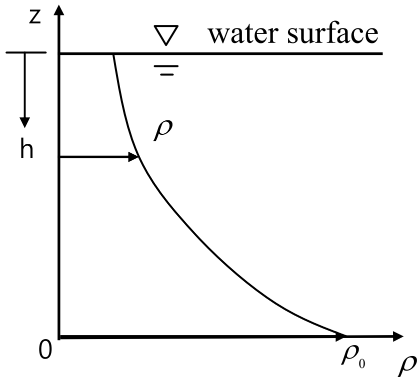
→ consider the effect of each of these factors acting alone on pure jet, pure plumes and buoyant jets

→ decide the ranges of the salient parameters over which each of the factors may predominate

2.3.1 Ambient Density Stratification

In the ocean, stratification arises from variations in salinity and temperature of different water masses.

(1) Vertical density distribution



$$\rho = \rho_o(1-\varepsilon(z)) \qquad (2.55)$$

where ρ_o = ambient density at $z = 0$

$\varepsilon(z)$ = density anomaly

Eq. (2.55) becomes

$$\varepsilon(z) = 1 - \frac{\rho}{\rho_o}$$

Differentiate once w.r.t. z

$$\varepsilon'(z) = \frac{d\varepsilon}{dz} = -\frac{1}{\rho_o}\frac{d\rho}{dz} \tag{2.56}$$

For statically stable environment, $\frac{d\rho}{dz} < 0 \quad \rightarrow \quad \frac{d\varepsilon}{dz} > 0$

→ Density decreases with *z* increasing in the upward direction.

For ocean and lake, $\varepsilon'(z) = 10^{-4} \sim 10^{-5}$ $1/m$ [L^{-1}]

• $\frac{1}{\varepsilon'(z)} = 1 \Big/ \frac{d\varepsilon}{dz}$ [L]

→ characteristic length which is measure of the intensity of stratification

→ The longer the length, the weaker the intensity of stratification.

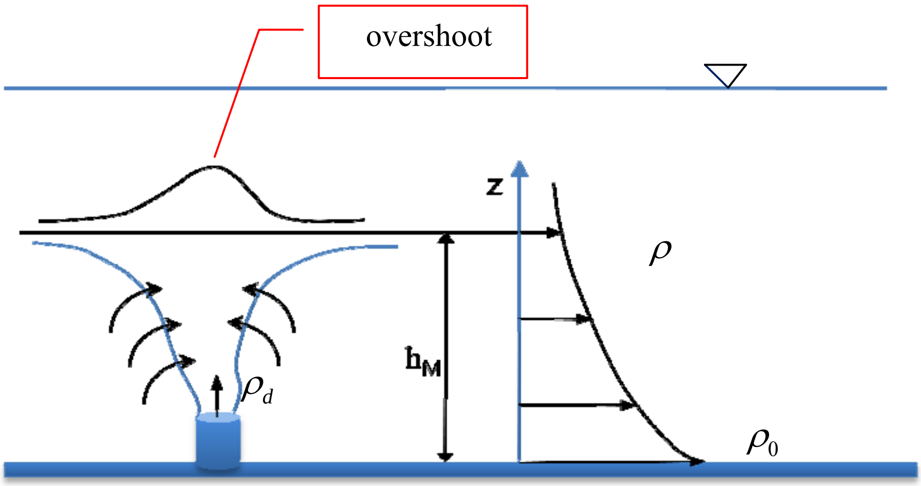
(2) Terminal height of rise

Consider point source of momentum directed vertically up in a stratified fluid for which

$\varepsilon' = const.$

→ The effect of momentum flux is to carry entrained dense fluid to where the ambient fluid is less dense.

→ Thus, such jet will have a terminal height of rise.



Let initial specific momentum flux $= M$

terminal height of rise $= h_M$

Then, h_M will depend only on M, ε', g

Effect of buoyancy is to modify gravity.

$\rightarrow$ ε' and g must be combined as $g\varepsilon'$

Eq. (2.56) becomes

$$-\varepsilon'(z) = \frac{1}{\rho_o}\frac{d\rho}{dz} \approx \frac{\Delta\rho}{\rho_o}\frac{1}{\Delta z}$$

$$-\varepsilon'(z)g = \frac{\Delta\rho}{\rho_o} g \frac{1}{\Delta z} = g_a' \frac{1}{\Delta z}$$

[Cf] $g_a' = \dfrac{\Delta\rho}{\rho_o} g = \dfrac{\rho_a - \rho_{a_o}}{\rho_{a_o}} g$

$$g_0' = \frac{\Delta\rho}{\rho_d} g = \frac{\rho_{a_o} - \rho_d}{\rho_d} g \qquad \rightarrow \qquad F_d = \frac{W}{\sqrt{g_o' D}}$$

i) h_M of a round simple momentum jet

Dimensional analysis gives

$$h_M = \phi(M, \varepsilon', g)$$

$$\phi_2\left(\frac{h_M}{\left(\frac{M}{\varepsilon' g}\right)^{1/4}}\right) = 0$$

$$h_M = const.\left(\frac{M}{\varepsilon' g}\right)^{1/4} \qquad (2.57)$$

Experiments by Fan (1967) and Fax (1970)

$$h_M = 3.8\left(\frac{M}{g\varepsilon'}\right)^{1/4} \qquad (2.57a)$$

[Re] Buckingham π theorem

$$\phi_1(h_M, M, \varepsilon' g) = 0$$

$$M^a(\varepsilon' g)^b h_M = M^0 L^0 T^0$$

$$\left[L^4 t^{-2}\right]^a \left[L^{-1} L t^{-2}\right]^b [L] = M^0 L^0 t^0$$

$$L: \ 4a + 1 = 0 \qquad a = -\frac{1}{4}$$

$$t: \ -2a - 2b = 0 \qquad b = -a = \frac{1}{4}$$

$$\therefore \ \phi_2\left(M^{-1/4}(\varepsilon' g)^{1/4} h_M\right) = 0$$

ii) h_B of a round simple plume

Dimensional analysis gives

$$h_B = 3.8 \frac{B^{1/4}}{(g\varepsilon')^{3/8}} \tag{2.58}$$

$B =$ specific buoyancy flux

Data by Crawford & Leonard (1962), Morton et al. (1956), Briggs (1965)

(3) Asymptotic solutions for buoyant jet

- Stratification parameter

$$\left(\frac{M}{g\varepsilon'}\right)^{1/4} \text{ and } \frac{B^{1/4}}{(g\varepsilon')^{3/8}} \text{ are characteristic length scales.}$$

$$l_j = \left(\frac{M}{g\varepsilon'}\right)^{1/4}, \quad l_p = \frac{B^{1/4}}{(g\varepsilon')^{3/8}}$$

The ratio of two length scales yields a stratification parameter.

$$\text{Let } N = \left(\frac{l_i}{l_p}\right)^8 = \frac{M^2/(g\varepsilon')^2}{B^2/(g\varepsilon')^3} = \frac{M^2 g\varepsilon'}{B^2}$$

$$N = \frac{M^2 g\varepsilon'}{B^2}$$

Two parameters to define buoyant jets in a linearly density-stratified environment:

- stratification parameter, N
- initial jet densimetric Froude number (Richardson number), F_d

- Terminal height of rise

From these results asymptotic functional relationships for the terminal height of rise, ζ_T is given as below.

i) Jet behavior: M - high, B - low $\rightarrow$ $N \gg 1$

Recall $\zeta = \dfrac{c_p}{R_p} \dfrac{z}{l_M}$ (A)

Substitute $l_M = \dfrac{M^{3/4}}{B^{1/2}}$, $z = h_M = 3.8 \left(\dfrac{M}{\varepsilon' g} \right)^{1/4}$ into (A)

Then, terminal height of rise is given as

$$\zeta_T = \frac{c_p}{R_p} \frac{3.8 \left(\dfrac{M}{\varepsilon' g} \right)^{1/4}}{\dfrac{M^{3/4}}{B^{1/2}}} = \frac{0.25}{0.557} (3.8) \frac{B^{1/2}}{M^{1/2} g^{1/4} \varepsilon'^{1/4}}$$

$$= 1.7 \left(\frac{M^2 \varepsilon' g}{B^2} \right)^{-1/4} = 1.7\, N^{-1/4}$$

ii) Plume behavior: M - low, B - high $\rightarrow$ $N \ll 1$

Substitute $h_B = 3.8 \dfrac{B^{1/4}}{(\varepsilon' g)^{3/8}}$ into (A),

$$\zeta_T = \frac{0.25}{0.557} \frac{3.8 \dfrac{B^{1/4}}{(\varepsilon' g)^{3/8}}}{\dfrac{M^{3/4}}{B^{1/2}}} = \frac{0.25}{0.557} (3.8) \frac{B^{3/4}}{M^{3/4} (\varepsilon' g)^{3/8}}$$

$$= 1.7 \left(\frac{M^2 \varepsilon' g}{B^2} \right)^{-3/8} = 1.7\, N^{-3/8}$$

$$\zeta_T = 1.7\, N^{-1/4}\,, \qquad N \gg 1 \text{ (jet-like)} \qquad (2.60a)$$

$$\zeta_T = 1.7\, N^{-3/8}\,, \qquad N \ll 1 \text{ (plume-like)} \qquad (2.60b)$$

- Mean dilution at the terminal level

Simple dimensional analysis gives the asymptotic equations for mean dilution at the terminal level, $\bar{\mu}_T$.

i) Jet, $N \gg 1$

$$\bar{\mu}_T = 1.2\, N^{-1/4} \qquad (2.61a)$$

ii) Plume, $N \ll 1$

$$\bar{\mu}_T = 1.5\, N^{-5/8} \qquad (2.61b)$$

Combining (2.60) and (2.61) gives

Jetlike, $$\bar{\mu}_T = 1.2\, N^{-1/4} = 1.2\left(\frac{1}{1.7}\zeta_T\right) = 0.73\,\zeta_T$$

Plumelike, $$\bar{\mu}_T = 1.5\, N^{-5/8} = 1.5\left(\frac{1}{1.7}N^{-8/3}\right)^{-5/8} = 0.6\,\zeta_T^{\,5/3}$$

[Cf] Mean dilution of round buoyant jets for <u>unstratified ambient</u>

Jetlike, $$\bar{\mu} = \zeta\,, \qquad \zeta \ll 1 \qquad (2.43)$$

Plumelike, $$\bar{\mu} = \zeta^{5/3}\,, \qquad \zeta \gg 1 \qquad (2.44)$$

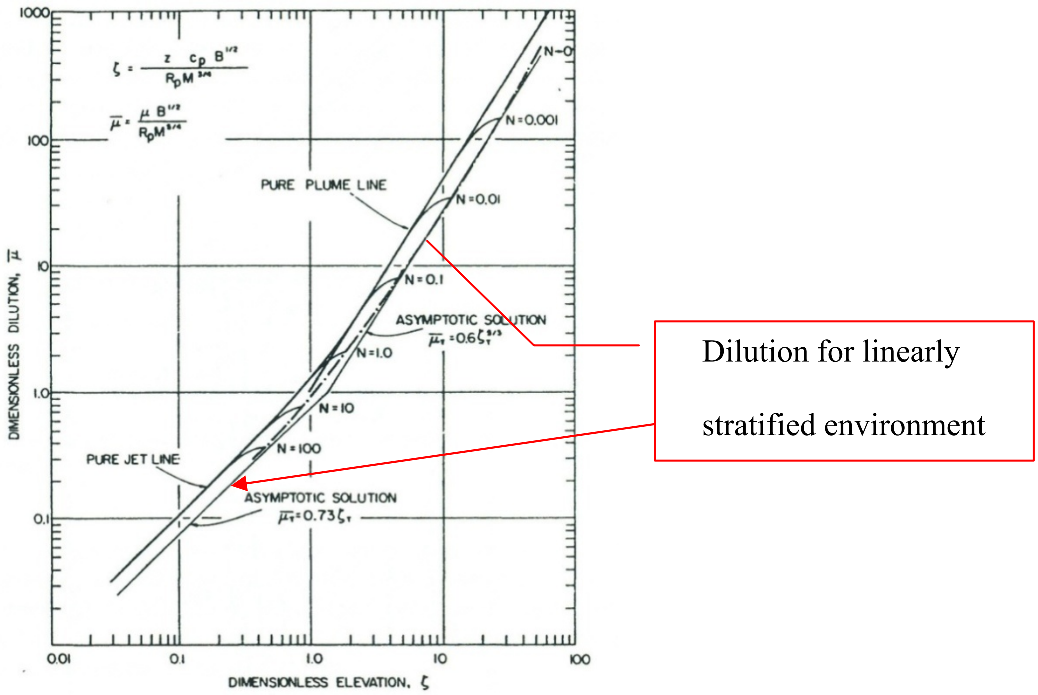


Fig. 2.26 Dilution in turbulent buoyant jets with a linearly stratified environment

- Plane jets and plumes

Terminal height of rise and dilution for plane jets & plumes

→ Table 2.4

Based on experimental data by Brooks (1973), and Bardey (1977)

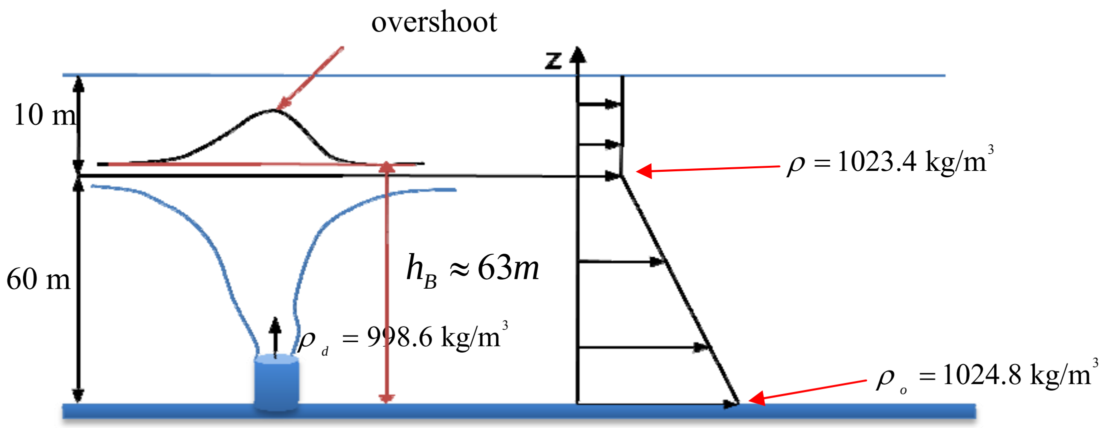
Jet, $N \gg 1$	Plume, $N \ll 1$
$h_M = 4.0\left(\dfrac{M}{\varepsilon' g}\right)^{1/3}$	$h_B = 2.8\dfrac{B^{1/3}}{(\varepsilon' g)^{1/2}}$
$\zeta_T = 1.6\, N^{-1/3}$	$\zeta_T = 1.1\, N^{-1/2}$
$\bar{\mu}_T = k\, N^{-1/6}$	$\bar{\mu}_T = 1.0\, N^{-1/2}$

[Example 2.4]

Suppose that a uniform temperature gradient exists over the lower 60 m of ocean in Example 2.3.

T = 11.1 °C at z = 0 m; T = 17.8 °C at z = 60 m

Discharge: freshwater; T_d = 17.8 °C at z = 0 m; Q = 1 m^3/s (no momentum → plume)



$$\varepsilon' = \frac{d\varepsilon}{dz} = -\frac{1}{\rho_o}\frac{d\rho_a}{dz} \approx -\frac{\Delta\rho_a}{\rho_o\,\Delta z} = -\frac{(\rho_a-\rho_o)}{\rho_o\,\Delta z}$$

$$= -\frac{(1023.4-1024.8)}{1024.8\,(60-0)} = 2.28\times10^{-5}\ \ 1/\mathrm{m}$$

$$g\varepsilon' = 9.81\,(\mathrm{m/s^2})\times2.28\times10^{-5}\ (1/\mathrm{m}) = 2.23\times10^{-4}\ \ 1/\mathrm{s}^2$$

$$g_o{}' = g\frac{\Delta\rho_0}{\rho_d} = 9.81\,(1024.8-998.6)/998.6 = 0.257\ \ \mathrm{m/s^2}$$

Buoyancy flux; $B = g_o{}'Q = (0.257)\,(1) = 0.257\ \ \mathrm{m^4/s^3}$

Terminal height; $h_B = 3.8\dfrac{B^{1/4}}{(g\varepsilon')^{3/8}} = 3.8\dfrac{(0.257)^{1/4}}{(2.23\times10^{-4})^{3/8}} = 63\,\mathrm{m}$

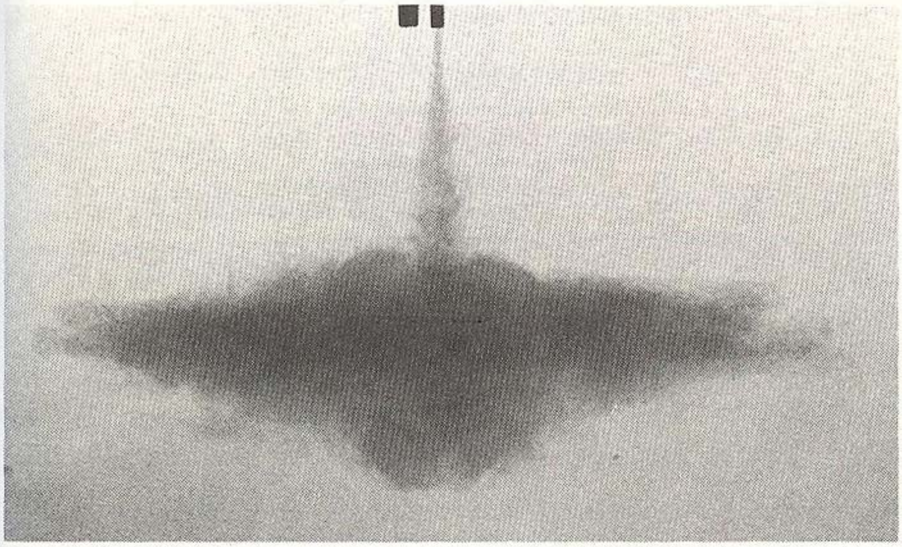
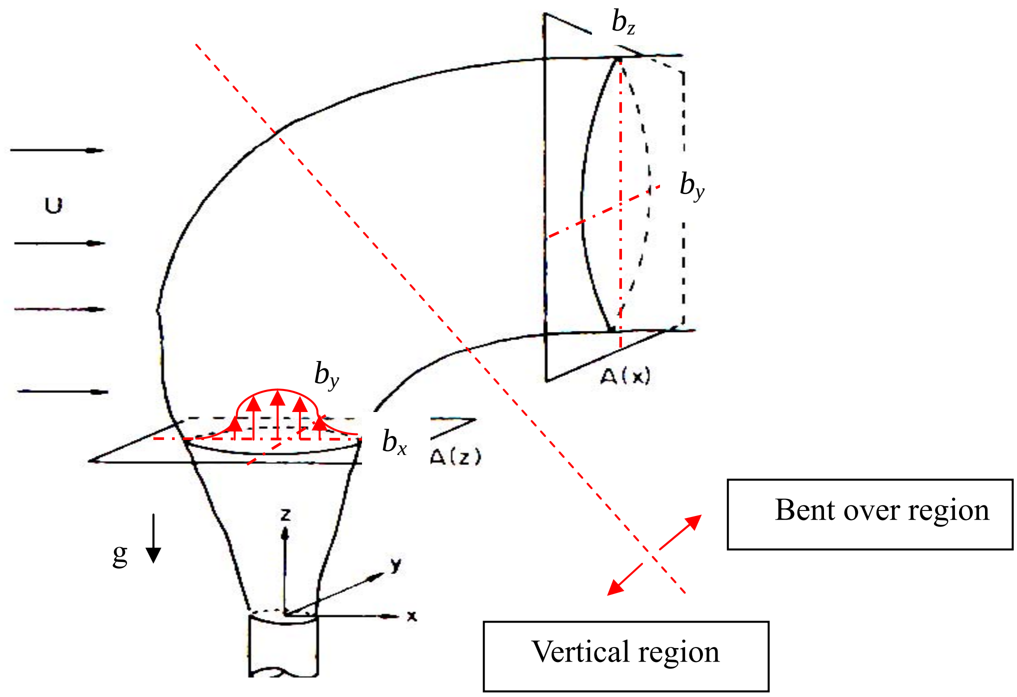


Figure 9.14 A vertical negatively buoyant jet descending in a stagnant, linearly stratified environment $R_0 = 0.052$, $N = 3.2$. [From Fan (1967).]

2.3.2 Ambient Crossflows



Ambient flow:

- shear flow → Fig. 2.15
- uniform flow → U (Fig. 2.16)

• Characteristic length scale

For momentum jets in a uniform crossflow U

$$z_m = \frac{M^{1/2}}{U}$$

• Asymptotic states of jet

(i) $\frac{z}{z_m} \ll 1$: M dominates ~ jet is unaffected by crossflow

(ii) $\frac{z}{z_m} \gg 1$: U dominates ~ jet is dominated by crossflow

Asymptotic solution:

$$\frac{w_m}{U} = f\left(\frac{z}{z_m}\right)$$

• Solution developed from the equation of motion

- Boussinesq approximation:

So far as the inertia of the flow is concerned, density difference between ambient fluid and outflow can be neglected, except when multiplied by *g*.

$$\rho_{a_o} \approx \rho_a \approx \overline{\rho_d} \qquad (1)$$

ρ_{a_o} = reference density at $z = 0$

ρ_a = ambient fluid density at $z = z$

$\overline{\rho_d}$ = jet fluid density

The lateral entrainment by $\bar{v}$ is very small

1) Continuity eq.

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} + \frac{\partial \bar{w}}{\partial z} = 0 \qquad (2)$$

2) 3D time-averaged momentum eq. for steady flow (Reynolds eq.)

$$x-\text{dir.}: \; \rho\left(\frac{\partial \bar{u}}{\partial t} + \bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} + \bar{w}\frac{\partial \bar{u}}{\partial z}\right)$$

$\bar{v} \approx 0$

Steady flow

$$= \rho g_x - \frac{\partial \bar{p}}{\partial x} + \mu \nabla^2 \bar{u} - \rho\left(\frac{\partial \overline{u'^2}}{\partial x} + \frac{\partial \overline{u'v'}}{\partial y} + \frac{\partial \overline{u'w'}}{\partial z}\right) \qquad (3)$$

Viscous stress is neglected

$$\therefore \text{ L.H.S } = \bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{w}\frac{\partial \bar{u}}{\partial z} = \frac{1}{2}\frac{\partial \bar{u}^2}{\partial x} + \frac{\partial}{\partial z}(\bar{u}\,\bar{w}) - \bar{u}\frac{\partial \bar{w}}{\partial z} \qquad (A)$$

Substitute continuity eq. (2) into (A): $\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{u}\frac{\partial \bar{w}}{\partial z} = 0$

Then, L.H.S $= \frac{1}{2}\frac{\partial \bar{u}^2}{\partial x} + \frac{\partial}{\partial z}(\bar{u}\,\bar{w}) + \underbrace{\bar{u}\frac{\partial \bar{u}}{\partial x}}_{\frac{1}{2}\frac{\partial \bar{u}^2}{\partial x}} = \frac{\partial \bar{u}^2}{\partial x} + \frac{\partial}{\partial z}(\bar{u}\,\bar{w})$

Therefore, (3) becomes

$$\frac{\partial}{\partial x}\left(\bar{u}^2 + \overline{u'^2} + \frac{\bar{p}}{\rho_o}\right) + \frac{\partial}{\partial y}(\overline{u'v'}) + \frac{\partial}{\partial z}(\bar{u}\,\bar{w} + \overline{u'w'}) = 0 \qquad (2.63)$$

$$z-\text{dir.: } \rho\left(\frac{\partial \bar{w}}{\partial t} + \bar{u}\frac{\partial \bar{w}}{\partial x} + \bar{v}\frac{\partial \bar{w}}{\partial y} + \underbrace{\bar{w}\frac{\partial \bar{w}}{\partial z}}_{\frac{1}{2}\frac{\partial \bar{w}^2}{\partial z}}\right)$$

$$= \rho g_z - \frac{\partial \bar{p}}{\partial z} + \mu \nabla^2 \bar{w} - \rho\left(\frac{\partial \overline{w'u'}}{\partial x} + \frac{\partial \overline{w'v'}}{\partial y} + \frac{\partial \overline{w'^2}}{\partial z}\right) \qquad (4)$$

Continuity eq.

By the way, $\bar{u}\frac{\partial \bar{w}}{\partial x} = \frac{\partial}{\partial x}(\bar{u}\,\bar{w}) - \bar{w}\frac{\partial \bar{u}}{\partial x} = \frac{\partial}{\partial x}(\bar{u}\,\bar{w}) + \underbrace{\bar{w}\frac{\partial \bar{w}}{\partial z}}_{\frac{1}{2}\frac{\partial \bar{w}^2}{\partial z}}$

Therefore, (4) becomes

$$\frac{\partial}{\partial x}(\overline{u}\ \overline{w}+\overline{u'w'})+\frac{\partial}{\partial y}(\overline{w'v'})+\frac{\partial}{\partial z}\left(\overline{w}^{\,2}+\overline{w'^{\,2}}+\frac{\overline{p}}{\rho_o}\right) = g'_z$$

$$= \frac{\Delta\rho}{\rho}g = \frac{\rho_a-\overline{\rho}}{\rho_o}g \qquad (2.64)$$

(1) <u>Vertical region</u>

a. Integrate Eq. (2.64) over jet cross section $A(z)$

$$\underbrace{\int_{A(z)}\frac{\partial}{\partial x}(\overline{u}\ \overline{w}+\overline{u'w'})\,dx\,dy}_{=I_A}+\underbrace{\int_{A(z)}\frac{\partial}{\partial y}(\overline{w'v'})\,dx\,dy}_{=I_B}$$

$$+\int_{A(z)}\frac{\partial}{\partial z}(\overline{w}^{\,2}+\overline{w'^{\,2}}+\frac{\overline{p}}{\rho_o})\,dx\,dy = \int_{A(z)}\left(\frac{\rho_a-\overline{\rho}}{\rho_o}\right)g\,dx\,dy \qquad (2.67)$$

$$I_A = \int_{A(z)}\frac{\partial}{\partial x}(\overline{u}\ \overline{w}+\overline{u'w'})\,dx\,dy = \int_{-b_y}^{b_y}\int_{-b_x}^{b_x}\frac{\partial}{\partial x}(\overline{u}\,\overline{v}+\overline{u'w'})\,dx\,dy$$

$$= \int_{-b_y}^{b_y}[(\overline{u}\,\overline{w}+\overline{u'w'})]_{-b_x}^{b_x}\,dy = 0$$

($\because$ $w = 0$, $w' \approx 0$ at both edges)

$$I_B = \int_{A(z)}\frac{\partial}{\partial y}(\overline{w'v'})\,dx\,dy = \int_{-b_x}^{b_x}\int_{-b_y}^{b_y}\frac{\partial}{\partial y}(\overline{w'v'})\,dy\,dx$$

$$= \int_{-b_x}^{b_x}[\overline{w'v'}]_{-b_y}^{b_y}\,dx = 0$$

($\because$ $w'v' \approx 0$ at both edges)

[Re] Here, we define the boundary of the jet as the perimeter of the jet beyond which mean vertical velocities and jet-induced turbulent stresses vanish.

Eq. (2.67) becomes

$$\int_{A(z)} \frac{\partial}{\partial z}(\overline{w}^{2} + \cancel{\overline{w'^{2}}} + \cancel{\frac{\overline{p}}{\rho_o}})\, dx\,dy = \int_{A(z)} \left(\frac{\rho_a - \overline{\rho}}{\rho_o}\right) g\, dx\,dy \qquad (2.68)$$

Miller and Comings (1957) have shown that, in 2-D jet $\overline{w'^2}$ and $\overline{p}/\rho$ are small and are opposite in sign. Thus, Eq. (2.68) reduces to

$$\int_{A(z)} \frac{\partial}{\partial z}\overline{w}^{2}\, dx\,dy = \int_{A(z)} \left(\frac{\rho_a - \overline{\rho}}{\rho_o}\right) g\, dx\,dy \qquad (2.70)$$

→ Rate of change of vertical flow force in vertical direction is equal to buoyancy force.

[Re] Derivation of Eq. (2.66)

Apply conservation of mass for <u>both jet flow and ambient flow</u>

(i) Jet: $\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0$ (1)

Introduce decomposition of variables

$$\rho = \overline{\rho} + \rho' \qquad (2a)$$

$$u = \overline{u} + u'\,;\ v = v'\ (\text{since } \overline{v} \approx 0);\ w = \overline{w} + w'$$

Substitute (2) into (1) and average over time and apply Reynolds rule of average

$$\therefore\ \frac{\partial}{\partial x}(\overline{\rho}\,\overline{u} + \overline{\rho' u'}) + \frac{\partial}{\partial y}(\overline{\rho' v'}) + \frac{\partial}{\partial z}(\overline{\rho}\,\overline{w} + \overline{\rho' w'}) = 0 \qquad (3)$$

(ii) Ambient flow: $\rho_a = const.\ \ \overline{v} = 0$

$$\frac{\partial}{\partial x}(\rho_a \overline{u}) + \frac{\partial}{\partial z}(\rho_a \overline{w}) = 0 \qquad (4)$$

Subtract (4) from (3)

$$\frac{\partial}{\partial x}\left[\overline{u}(\overline{\rho} - \rho_a) + \overline{u'\rho'}\right] + \frac{\partial}{\partial y}(\overline{v'\rho'}) + \frac{\partial}{\partial z}[\overline{w}(\overline{\rho} - \rho_a) + \overline{w'\rho'}] = 0 \qquad (2.66)$$

(1) (2) (3)

where $\overline{u'\rho'}$, $\overline{v'\rho'}$, $\overline{w'\rho'}$ = turbulent transport terms

b. Now integrate Eq. (2.66) over $A(z)$

$\overline{u} = U \pm$ entrainment at edge;

$\overline{\rho} \cong \rho_a$ at edge

$$(1) = \int_{-b_y}^{b_y} [\overline{u}(\overline{\rho} - \rho_a) + \overline{u'\rho'}]_{-b_x}^{b_x} dy = 0$$

$\overline{u'\rho'}$ is the same value at both edges.

$$(2) = \int_{-b_x}^{b_x} [\overline{v'\rho'}]_{-b_x}^{b_x} dx = 0 \quad (\because \text{ same values at both edges})$$

$$(3) = \int_{A(z)} \frac{\partial}{\partial z}(\overline{w'\rho'})\, dx\, dy = 0 \quad (\because \text{ ignore turbulent transports})$$

Thus, Eq. (2.66) becomes

$$\int_{A(z)} \frac{\partial}{\partial z}[\overline{w}(\rho_a - \overline{\rho})]\, dx\, dy = 0 \qquad (2.71)$$

→ Vertical flux of buoyancy is conserved.

$$\frac{\partial}{\partial z}\int_{A(z)} [\overline{w}(\rho_a - \overline{\rho})]\, dx\, dy = 0 \Rightarrow \int_{A(z)} [\overline{w}(\rho_a - \overline{\rho})]\, dx\, dy = const.$$

(2) <u>Bent over region</u>

Integrate of Eq. (2.64) and Eq. (2.66) across a vertical plane, $A(x)$ with making the same kind simplifications. Then we get

$$\int_{A(x)} \frac{\partial}{\partial x}(\bar{u}\,\bar{w})\,dy\,dz = \int_{A(x)} \left(\frac{\rho_a - \bar{\rho}}{\rho_o}\right) g\,dy\,dz \quad (2.72)$$

$$\int_{A(x)} \frac{\partial}{\partial x}[\bar{u}\,(\rho_a - \bar{\rho})]\,dy\,dz = 0 \quad (2.73)$$

Eq.(2.72): Horizontal flux of vertical momentum is the same as the buoyancy force acting in a vertical plane.

Eq.(2.73): Horizontal flux of buoyancy is conserved.

[Cf] Vertical region:

$$\int_{A(z)} \frac{\partial}{\partial z}(\bar{w}^2)\,dx\,dy = \int_{A(z)} \left(\frac{\rho_a - \bar{\rho}}{\rho_o}\right) g\,dx\,dy \quad (2.70)$$

$$\int_{A(z)} \frac{\partial}{\partial z}[\bar{w}(\rho_a - \bar{\rho})]\,dx\,dy = 0 \quad (2.71)$$

I. Jet behavior in a crossflow

I-1. Jet vertical region (J.V.)

① Maximum (centerline) velocity, $w_m(\bar{z})$

For $z \gg l_Q$, consider only Q, M and neglect buoyancy B (or g')

Then, Eq. (2.70) becomes

$$\int_{A(z)} \frac{\partial \bar{w}^2}{\partial z} dx\,dy = 0 \qquad \text{(A)}$$

Assume that velocity and tracer concentration profiles are similar in ZEF.

→ use similarity solution

$$\frac{\bar{w}(x, y, z)}{w_m(\bar{z})} = \phi\left(\frac{x}{\bar{z}}, \frac{y}{\bar{z}}\right) \qquad (2.74)$$

$$\frac{(\rho_a - \bar{\rho})/\rho_o}{\theta(\bar{z})} = \psi\left(\frac{x}{\bar{z}}, \frac{y}{\bar{z}}\right) \qquad (2.75)$$

Undefined functions describing the lateral distribution of velocity and tracer concentration

where $\bar{z} = z$ coordinate of the jet axis;

$\rho_a - \bar{\rho}$ = "excess concentration" of tracer material on the jet at point (x, y, z)

$\theta(z)$ = maximum concentration (dimensionless) $= \dfrac{(\rho_a - \bar{\rho})_{\max}}{\rho_a}$

Substitute Eq. (2.74) into (A)

$$\int_{A(z)} \frac{\partial}{\partial z}[\bar{z}^2 w_m^2(\bar{z})\phi^2] d(\frac{x}{\bar{z}}) d(\frac{y}{\bar{z}}) = 0 \qquad \text{(B)}$$

Apply Leibnitz rule:

$$\frac{d}{d\alpha}\int_{u_0(\alpha)}^{u_1(\alpha)} f(x,\alpha)\,dx = f(u_1,\alpha)\frac{du_1}{d\alpha} - f(u_0,\alpha)\frac{du_0}{d\alpha} + \underbrace{\int_{u_0}^{u_1} f_\alpha(x,\alpha)\,dx}_{\int_{u_0}^{u_1}\frac{\partial f}{\partial \alpha}dx}$$

$$\frac{d}{dz}\int_{A(z)} \bar{z}^2 w_m^2(\bar{z})\phi^2 d(\frac{x}{\bar{z}}) d(\frac{y}{\bar{z}})$$

$$= 0\frac{db}{dz} - 0\frac{db}{dz} + \int_{A(z)} \frac{\partial}{\partial z}[\bar{z}^2 w_m^2(\bar{z})\phi^2] d\left(\frac{x}{\bar{z}}\right) d\left(\frac{y}{\bar{z}}\right) \qquad \text{(C)}$$

Thus, (A) becomes

$$\frac{d}{dz}\int_{A(z)} \bar{z}^2 w_m^2(\bar{z})\phi^2 d\left(\frac{x}{\bar{z}}\right) d\left(\frac{y}{\bar{z}}\right) = 0 \qquad (2.76)$$

Dimension of (2.76) is $[L^4 T^{-2}] = [M]$

Therefore, Eq. (2.76) $\rightarrow$ $\frac{dM}{dz} = 0$

Since w_m and $\bar{z}$ don't vary over $A(z)$ at a particular $\bar{z}$ position, (2.76) becomes

$$\frac{d}{dz}\left\{ \bar{z}^2 w_m^2(\bar{z}) \underbrace{\int_{A(z)} \phi^2 d\left(\frac{x}{\bar{z}}\right) d\left(\frac{y}{\bar{z}}\right)} \right\} = 0$$

= constant which is determined by shape of similarity profile

Recall I_1 & I_2 by Albertson et al.

$$\therefore \quad \frac{d}{dz}\left\{ \bar{z}^{\,2} w_m^2(\bar{z}) \right\} = 0 \qquad \text{(D)}$$

$$\bar{z}^{\,2} w_m^2(\bar{z}) = const.$$

$$\bar{z}^{\,2} w_m^2(\bar{z}) \sim M$$

$$\therefore \quad w_m(\bar{z}) = c\frac{M^{1/2}}{\bar{z}} \qquad \text{(E)}$$

Divide each side by U

$$\frac{w_m(\bar{z})}{U} = c\frac{M^{1/2}}{\bar{z}\,U} \qquad \text{(F)}$$

Let $z_m = \dfrac{M^{1/2}}{U} \rightarrow \dfrac{[L^2 T^{-1}]}{[L T^{-1}]} = [L]$

Then, (F) becomes

$$\frac{w_m(\bar{z})}{U} = c\frac{z_m}{\bar{z}} \qquad \text{(2.82)}$$

$$\boxed{\frac{w_m(\bar{z})}{U} = c\left(\frac{\bar{z}}{z_m}\right)^{-1}}$$

② Centerline concentration

Substitute Eq. (2.74) and Eq. (2.75) into Eq. (2.71)

$$\int_{A(z)} \frac{\partial}{\partial z}\left[\rho_o \bar{z}^2 w_m(\bar{z})\,\theta(\bar{z})\,\phi\psi\right] d\left(\frac{x}{\bar{z}}\right) d\left(\frac{y}{\bar{z}}\right) = 0 \tag{2.77}$$

Applying Leibnitz rule into (2.77) yields

$$\frac{d}{dz}\int_{A(z)}\left[\bar{z}^2 w_m(\bar{z})\theta(\bar{z})\,\phi\psi\right] d\left(\frac{x}{\bar{z}}\right) d\left(\frac{y}{\bar{z}}\right) = 0$$

$$\frac{d}{dz}\left\{\bar{z}^2 w_m(\bar{z})\theta(\bar{z})\int_{A(z)} \phi\psi\, d\left(\frac{x}{\bar{z}}\right) d\left(\frac{y}{\bar{z}}\right)\right\} = 0 \qquad \text{(a)}$$

By the way, $\int_{A(z)} \phi\psi\, d\left(\frac{x}{\bar{z}}\right) d\left(\frac{y}{\bar{z}}\right) = const.$

Thus, (a) becomes

$$\frac{d}{dz}\left\{\bar{z}^2 w_m(\bar{z})\theta(\bar{z})\right\} = 0$$

Integration gives

$$\bar{z}^2 w_m(\bar{z})\theta(\bar{z}) = const. \qquad \text{(b)}$$

$$\rightarrow\ [L^3 T^{-1}] = [Q] \text{(volume flux)}$$

By the way, at outlet, $\frac{B}{g} = \frac{(\Delta\rho_o / \rho)gQ}{g} = \frac{\Delta\rho_o}{\rho}Q \qquad [L^3 T^{-1}]$

Thus, incorporating this into (b) gives

$$\bar{z}^2 w_m(\bar{z})\theta(\bar{z}) = const.\frac{B}{g} \tag{2.79}$$

→ Eq. (2.79) retains B for dimensional reason even though B is neglected for jet.

Eq. (2.79) becomes

$$\theta(z) = const.\frac{B}{g\bar{z}^2 w_m} = const.\frac{B}{g\bar{z}^2 c\frac{M^{1/2}}{\bar{z}}}$$

$$\therefore \quad \theta(z) = const.\frac{B}{g M^{1/2}\bar{z}}$$

$$\rightarrow \quad \frac{g}{B}\theta(\bar{z}) = const.\frac{1}{M^{1/2}}\frac{1}{z} \tag{c}$$

Dimension of each side $\rightarrow$ $[L^3 T^{-1}]$

In order to non-dimensionalize multiply (c) by $\frac{M}{U}$

$$\bar{z}_m = \frac{M^{1/2}}{U}$$

$$\frac{M\,g}{U\,B}\theta(z) = const.\frac{M^{1/2}}{U\bar{z}} = const.\frac{\bar{z}_m}{\bar{z}}$$

$$\frac{M\,g}{U\,B}\theta(z) = D_1\frac{z_m}{\bar{z}} \tag{2.83}$$

$$\frac{M\,g}{U\,B}\theta(z) = D_1\left(\frac{\bar{z}}{z_m}\right)^{-1}$$

③ Jet trajectory

Eq. (2.82) & (2.83) are valid for $w_m(z) \gg U$ or $\overline{z} \ll z_m$ $\left(= \dfrac{M^{1/2}}{U}\right)$

We can interpret $\overline{z}$ as vertical height at which vertical velocity in the jet has decayed to the order of the crossflow velocity.

For a jet in a crossflow, the slope of the jet trajectory is

$$\frac{d\overline{z}}{dx} = \frac{w_m(\overline{z})}{U} \tag{2.84}$$

Substituting (2.84) into (2.82) yields

$$\frac{d\overline{z}}{dx} = const.\frac{z_m}{\overline{z}}$$

$$\overline{z}\, dz = const.\, z_m\, dx \rightarrow \frac{1}{2} d(\overline{z}^{\,2}) = const.\, z_m\, dx$$

Integrate once

$$\int \frac{1}{2} d(\overline{z}^{\,2}) = const.\, z_m \int dx$$

$$\frac{1}{2}\overline{z}^2 = const.\, z_m\, x + const.$$

$$\frac{\overline{z}^{\,2}}{z_m^{\,2}} = const.\frac{x}{z_m}$$

$$\frac{\overline{z}}{z_m} = C_1 \left(\frac{x}{z_m}\right)^{1/2} \tag{2.85}$$

• Summary for J.V. region

→ momentum-dominated jet with a weak crossflow

① Maximum vertical velocity

$$\frac{w_m(\bar{z})}{U} = const. \cdot \frac{z_m}{\bar{z}} \qquad (2.82)$$

② Tracer concentration

$$\frac{M\, g}{U\, B}\,\theta(\bar{z}) = D_1 \frac{z_m}{\bar{z}} \qquad (2.83)$$

③ Jet trajectory

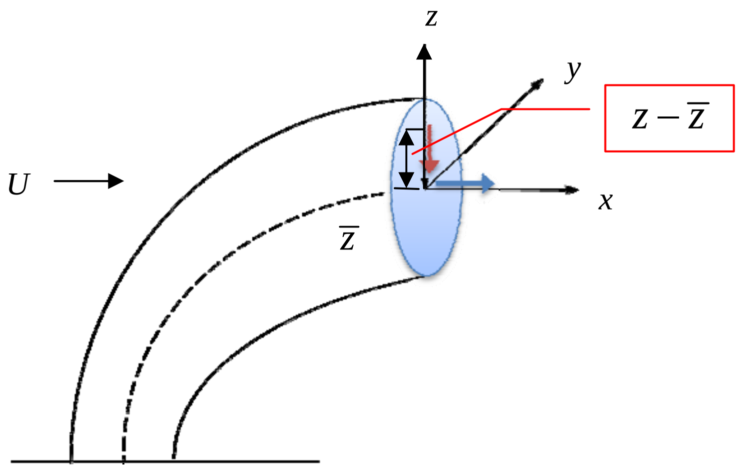
$$\frac{\bar{z}}{z_m} = C_1 \left(\frac{x}{z_m} \right)^{1/2} \qquad (2.85)$$

Table 2.8 Constants used in asymptotic trajectory and dilution laws for a buoyant jet

Investigator(s)	Constant C_1
Hoult *et al.* (1969)	1.8–2.5
Wright (1977)	1.8–2.3
	Constant C_2
Briggs[a] (1975)	1.8–2.1
Wright (1977)	1.6–2.1
Chu and Goldberg (1974)	1.44
	Constant C_3
Wright (1977)	1.4–1.8
	Constant C_4
Briggs[a] (1975)	1.1 (0.82–1.3)
Wright (1977)	$(0.85–1.4)(z_M/z_B)^2$
Chu and Goldberg (1974)	1.14
	Constants $D_1 - D_4$
Wright (1977)	~2.4

[a] Summary of 14 investigations.

I-2. Jet Bent Over region (JBO)



$\overline{z}$ = z-coordinate of jet centerline

$$\int_{A(x)} \frac{\partial}{\partial x}(\overline{u}\,\overline{w})\,dy\,dz = \int_{A(x)} \left(\frac{\rho_a - \overline{\rho}}{\rho_o}\right) g\,dy\,dz \tag{2.72}$$

(neglect buoyancy)

$$\int_{A(x)} \frac{\partial}{\partial x}[\,\overline{u}\,(\rho_a - \overline{\rho})\,]\,dy\,dz = 0 \tag{2.73}$$

Assume self-similarity

$$\frac{\overline{w}}{w_m(\overline{z})} = \phi\left(\frac{z-\overline{z}}{\overline{z}}, \frac{y}{\overline{z}}\right) \tag{2.86}$$

$$\frac{(\rho_a - \overline{\rho})/\rho_o}{\theta(\overline{z})} = \psi\left(\frac{z-\overline{z}}{\overline{z}}, \frac{y}{\overline{z}}\right) \tag{2.87}$$

$$\overline{u} \approx U \tag{2.88}$$

As done in JV, substitute Eqs. (2.86)~(2.88) into Eqs. (2.72) & (2.73)

$$\frac{d}{dx}\left\{\overline{z}^{\,2} U\, w_m \underbrace{\int_{A(x)} \phi \, d\left(\frac{z-\overline{z}}{\overline{z}}\right) d\left(\frac{y}{\overline{z}}\right)}_{=\text{ const.}}\right\} = 0 \tag{2.89}$$

$$\frac{d}{dx}\left\{\overline{z}^{\,2} U\, \theta \underbrace{\int_{A(x)} \psi \, d\left(\frac{z-\overline{z}}{\overline{z}}\right) d\left(\frac{y}{\overline{z}}\right)}_{=\text{ const.}}\right\} = 0 \tag{2.90}$$

① Centerline velocity

Eq.(2.89): $\frac{d}{dx}(\overline{z}^{\,2} U\, w_m) = 0$

Integrate once

$$\overline{z}^{\,2} U\, w_m = const.\, M$$

$$\frac{w_m}{U} = const.\frac{M}{U^2}\frac{1}{\overline{z}^{\,2}}$$

$$\frac{M}{U^{2}} = z_m^{\,2}$$

$$\frac{w_m(\overline{z})}{U} = const.\left(\frac{z_m}{\overline{z}}\right)^2, \qquad (\overline{z} \gg z_m) \tag{2.91}$$

② Centerline concentration

Eq. (2.90): $\frac{d}{dx}(\overline{z}^{\,2} U\, \theta(\overline{z})) = 0$

Integrate

$$\overline{z}^{\,2} U\, \theta(\overline{z}) = const.$$

$$\overline{z}^{\,2} U\, \theta(\overline{z}) = const.\frac{B}{g}$$

Incorporated due to dimensional reason

$$\theta(\bar{z}) = const.\frac{B}{gU}\frac{1}{\bar{z}^2} \qquad \text{(a)}$$

Multiply (a) by $\dfrac{M\,g}{U\,B}$

$$\frac{M\,g}{U\,B}\theta(\bar{z}) = const.\frac{M\,g}{U\,B}\frac{B}{gU}\frac{1}{\bar{z}^2}$$

$$= const.\frac{M}{U^2}\frac{1}{\bar{z}^2}$$

$$\boxed{\frac{M\,g}{U\,B}\theta(\bar{z}) = D_2\left(\frac{z_m}{\bar{z}}\right)^2,} \qquad \bar{z} \gg z_m \qquad (2.92)$$

③ Jet trajectory

Combine Eq. (2.84) and Eq. (2.91)

$$\frac{d\bar{z}}{dx} = \frac{w_m(\bar{z})}{U} \qquad (2.84)$$

$$\frac{d\bar{z}}{dx} = const.\frac{z_m^{\ 2}}{\bar{z}^2}$$

$$d\bar{z}\,(\bar{z})^2 = const.\,z_m^{\ 2}\,dx$$

Integrate once

$$\frac{\bar{z}^3}{3} = const.\,z_m^{\ 2}\,x$$

$$\frac{\bar{z}^3}{z_m^{\ 3}} = const.\frac{x}{z_m}$$

$$\boxed{\frac{\bar{z}}{z_m} = C_2\left(\frac{x}{z_m}\right)^{1/3},} \qquad \bar{z} \gg z_m \qquad (2.93)$$

• Summary for J.B.O. region

→ momentum-dominated jet with a weak crossflow

① Maximum jet axis velocity

$$\frac{w_m(z)}{U} = const.\left(\frac{z_m}{\bar{z}}\right)^2 \tag{2.91}$$

$$= const.\left(\frac{\bar{z}}{z_m}\right)^{-2}$$

② Tracer concentration

$$\frac{M\,g}{U\,B}\theta(z) = D_2\left(\frac{z_m}{\bar{z}}\right)^2 \tag{2.92}$$

$$= D_2\left(\frac{\bar{z}}{z_m}\right)^{-2}$$

③ Jet trajectory

$$\frac{\bar{z}}{z_m} = C_2\left(\frac{x}{z_m}\right)^{1/3} \tag{2.93}$$

→ Table 2.8

$C_2 = 1.8 \sim 2.1$ (Hoult et al., 1969)

$D_2 = 2.4$ (Wright, 1977)

II. Plume behavior in a crossflow

~ Flow is produced solely by a source of buoyancy flux B.

~ R.H.S. of Eqs. (2.70) & (2.72) is not zero

II-1. Plume vertical region (P.V.)

Substitute self-similarity for w and $\Delta\rho/\rho_o$, and integrate

Eq. (2.70) → $$\frac{d}{dz}\left[\bar{z}^2 w_m^2(\bar{z})\right] \sim g\bar{z}^2\theta(\bar{z}) \qquad (2.94)$$

Eq. (2.71) → $$\frac{d}{dz}\left[\bar{z}^2 w_m(\bar{z})\theta(\bar{z})\right] = 0 \qquad (2.95)$$

Eq. (2.84): $$\frac{w_m(\bar{z})}{U} = \frac{d\bar{z}}{dx}$$

① Maximum jet axis velocity

Eq.(2.94): $$\frac{d}{dz}\left[\bar{z}^2 w_m^2(\bar{z})\right] \sim g\bar{z}^2\theta(\bar{z})$$

Integrate

$$\bar{z}^2 w_m^2(\bar{z}) \sim \frac{g}{3}\bar{z}^3\theta(\bar{z}) \qquad \text{(a)}$$

$$\theta(\bar{z}) = const.\ \frac{B}{g w_m(\bar{z})}\frac{1}{\bar{z}^2}$$

Substitute (1) into (a)

$$w_m^2(\bar{z}) \sim \frac{g}{3}\bar{z}\left(const.\ \frac{B}{g w_m(\bar{z})}\frac{1}{\bar{z}^2}\right)$$

$$w_m^3(\bar{z}) \sim const.\frac{B}{\bar{z}}$$

$$\frac{w_m^{\ 3}(\overline{z})}{U^{\ 3}} \sim const. \frac{1}{\overline{z}} \frac{B}{U^{\ 3}}$$

Let $z_B = \dfrac{B}{U^{\ 3}}$ → characteristic length scale of the ratio of buoyancy and crossflow

$$\frac{w_m(\overline{z})}{U} \sim \left(\frac{z_B}{\overline{z}} \right)^{1/3} , \qquad \overline{z} \ll z_B \qquad (2.96)$$

② Tracer concentration

Eq.(2.95): $\dfrac{d}{dz}\left[\overline{z}^{\ 2} w_m(\overline{z}) \theta(\overline{z}) \right] = 0$

Integrate

$$\overline{z}^{\ 2} w_m(\overline{z})\, \theta(\overline{z}) = const. \frac{B}{g}$$

$$\theta(\overline{z}) = const. \frac{B}{g\, w_m(\overline{z})} \frac{1}{\overline{z}^{\ 2}} \qquad (1)$$

Substitute Eq. (2.96) into (1)

$$\theta(\overline{z}) = const. \frac{B}{g} \frac{1}{U\, (z_B / \overline{z})^{1/3}} \frac{1}{\overline{z}^{\ 2}} \qquad (2)$$

Multiply (2) by $\dfrac{g\, M}{U\, B}$

$$\frac{g\, M}{U\, B} \theta(\overline{z}) = const. \frac{B}{g} \frac{1}{U} \frac{1}{(z_B / \overline{z})^{1/3}} \frac{1}{\overline{z}^{\ 2}} \frac{g\, M}{U\, B}$$

$$= const. \frac{1}{(z_B / \overline{z})^{1/3}} \frac{1}{\overline{z}^2} \frac{M}{U^2}$$

$$z_M = \frac{M^{1/2}}{U}$$

$$= const. \frac{1}{(z_B / \overline{z})^{1/3}} \frac{1}{\overline{z}^2} z_M^2 \qquad (3)$$

Multiply (3) by z_B^2 / z_M^2

$$\left(\frac{z_B}{z_M}\right)^2 \frac{g M}{U B} \theta(\overline{z}) = D_3 \left(\frac{z_B}{\overline{z}}\right)^{5/3}, \quad \overline{z} \ll z_B \qquad (2.97)$$

③ Jet trajectory

Eq. (2.84): $\frac{d\overline{z}}{dx} = \frac{w_m(\overline{z})}{U}$ (i)

Substitute Eq. (2.96) into (i)

$$\frac{d\overline{z}}{dx} = const. \left(\frac{z_B}{\overline{z}}\right)^{1/3}$$

$$d\overline{z}\, \overline{z}^{1/3} = const.\, z_B^{1/3} dx$$

Integrate

$$\frac{3}{4} \overline{z}^{4/3} = const.\, z_B^{1/3} x$$

$$\overline{z} = const.\ z_B^{1/4} x^{3/4}$$

$$\frac{\overline{z}}{z_B} = C_3 \left(\frac{x}{z_B}\right)^{3/4}, \quad \overline{z} \ll z_B \qquad (2.98)$$

• Summary for P.V. region

→ buoyancy-dominated plume with a weak crossflow

$$\frac{w_m(\bar{z})}{U} \sim \left(\frac{z_B}{\bar{z}}\right)^{1/3}, \quad \bar{z} \ll z_B \tag{2.96}$$

$$\left(\frac{z_B}{z_M}\right)^2 \frac{g\,M\,\theta(\bar{z})}{U\,B} = D_3\left(\frac{z_B}{\bar{z}}\right)^{5/3}, \quad \bar{z} \ll z_B \tag{2.97}$$

$$\frac{\bar{z}}{z_B} = C_3\left(\frac{x}{z_B}\right)^{3/4}, \quad \bar{z} \ll z_B \tag{2.98}$$

where $z_B = \dfrac{B}{U^3}$: characteristic length scale

= vertical distance along the jet trajectory where the vertical velocity of the plume decays to the order of the crossflow velocity

$$C_3 = 1.4 \sim 1.8$$

$$D_3 = 2.4$$

II-2. Plume Bent-Over Region (P.B.O.)

Start with Eqs. (2.72) & (2.73)

Eq. (2.72): $$\int_{A(x)} \frac{\partial}{\partial x}(\bar{u}\,\bar{w})\,dy\,dz = \int_{A(x)} \left(\frac{\rho_a - \bar{\rho}}{\rho_o}\right) g\,dy\,dz$$

Eq. (2.73): $$\int_{A(x)} \frac{\partial}{\partial x}[\,\bar{u}\,(\rho_a - \bar{\rho})\,]\,dy\,dz = 0$$

Substitute self-similarity [Eqs. (2.86) ~ (2.88)] into Eqs. (2.72) & (2.73)

$$\frac{d}{dx}\left[\,\bar{z}^{\,2} U\, w_m(\bar{z})\,\right] \sim g\,\bar{z}^{\,2}\,\theta(\bar{z}) \tag{2.99}$$

$$\frac{d}{dx}\left[\,\bar{z}^{\,2} U\,\theta(\bar{z})\,\right] = 0 \tag{2.100}$$

Eqs. (2.99) & (2.100) becomes

$$\frac{w_m(\bar{z})}{U} \sim \left(\frac{z_B}{\bar{z}}\right)^{1/2}, \qquad \bar{z} \gg z_B \tag{2.101}$$

$$\left(\frac{z_B}{z_M}\right)^2 \frac{g\,M\,\theta(\bar{z})}{U\,B} = D_4\left(\frac{z_B}{\bar{z}}\right)^2, \qquad \bar{z} \gg z_B \tag{2.102}$$

Substitute Eq. (2.101) into Eq. (2.84)

$$\frac{\bar{z}}{z_B} = C_4\left(\frac{x}{z_B}\right)^{2/3}, \qquad \bar{z} \gg z_B \tag{2.103}$$

• **Summary**

	JV	JBO	PV	PBO
$\dfrac{w_m(z)}{U}$	$c\,\dfrac{z_M}{\bar{z}}$	$c\left(\dfrac{z_M}{\bar{z}}\right)^2$	$c\left(\dfrac{z_B}{\bar{z}}\right)^{1/3}$	$c\left(\dfrac{z_B}{\bar{z}}\right)^{1/2}$
$\dfrac{M\,g}{U\,B}\theta(z)$	$D_1\,\dfrac{z_M}{\bar{z}}$	$D_1\left(\dfrac{z_M}{\bar{z}}\right)^2$	$D_3\left(\dfrac{z_B}{\bar{z}}\right)^{5/3}$	$D_4\left(\dfrac{z_B}{\bar{z}}\right)^2$
$\dfrac{\bar{z}}{z_m}$ $\dfrac{\bar{z}}{z_B}$	$C_1\left(\dfrac{x}{z_M}\right)^{1/2}$	$C_2\left(\dfrac{x}{z_M}\right)^{1/3}$	$C_3\left(\dfrac{x}{z_B}\right)^{3/4}$	$C_4\left(\dfrac{x}{z_B}\right)^{2/3}$

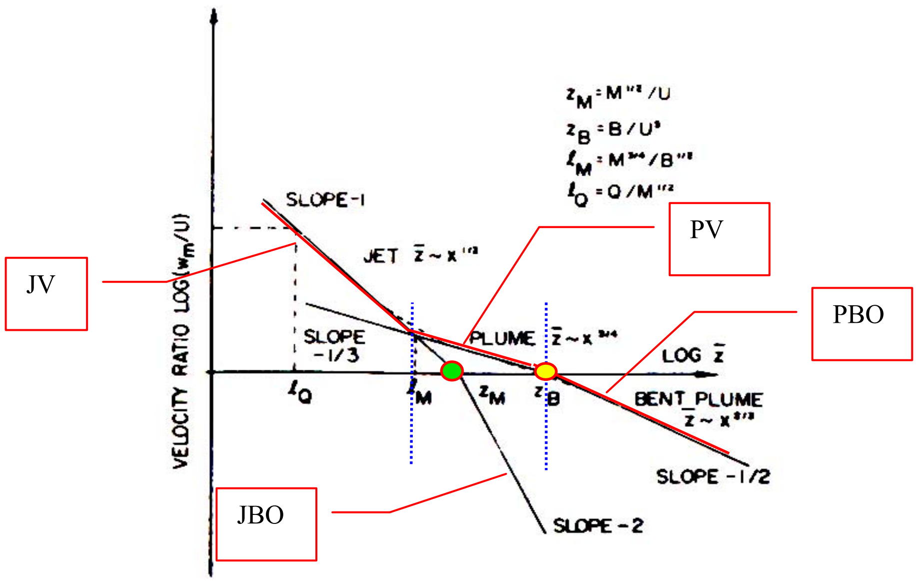
• Behavior of buoyant jet in crossflow

$$l_Q = \frac{Q}{M^{1/2}};\; l_M = \frac{M^{3/4}}{B^{1/2}};\; z_M = M^{1/2}/U;\; z_B = B/U^3$$

Jet behavior at the beginning

(i) $z_M < z_B$

→ Buoyancy flux is strong: J.V.(MDNF) → P.V.(BDNF) → P.B.O.(BDFF)

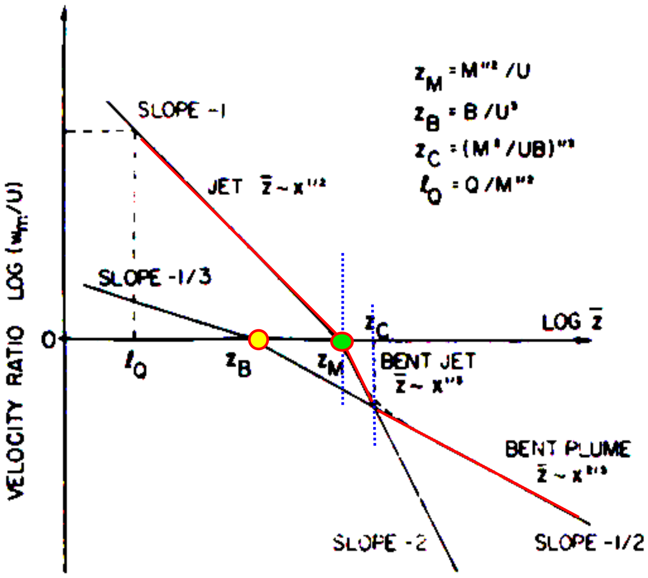


(ii) $z_M > z_B$

→ Buoyancy flux is weak: J.V. → J.B.O. → P.B.O.

$$z_C = \left(\frac{M^2}{U\,B} \right)^{1/3}$$

Ultimately transforms to a plume



• Normalized descriptions of jets in crossflows

For design purposes normalized form is suitable to determine the sensitivity of a design to changes in parameters.

Asymptotic solutions are useful for design purposes because they allow a rapid categorization of the type of problem under consideration.

i) Dimensionless variables for asymptotic solution $\rightarrow$ Table 2.5

Table 2.5 Dimensionless Variables for Asymptotic Solutions for a Turbulent Buoyant Jet

$z_M > z_B$	$z_M < z_B$
$\xi = \dfrac{x}{z_M}\left(\dfrac{C_1}{C_2}\right)^6$	$\hat{\xi} = \dfrac{x}{z_M}\left(\dfrac{z_B}{z_M}\right)\left(\dfrac{C_3}{C_1}\right)^4$
$\zeta = \dfrac{\overline{z}}{z_M}\dfrac{1}{C_1}\left(\dfrac{C_1}{C_2}\right)^3$	$\hat{\zeta} = \dfrac{z}{z_M}\left(\dfrac{z_B}{z_M}\right)\left(\dfrac{C_3}{C_1}\right)^4$
$S = \dfrac{\left(\dfrac{\mu U}{M}\right)}{\left(\dfrac{1}{D_1}\dfrac{C_2^{\,3}}{C_1^{\,2}}\right)}$	$\hat{S} = \dfrac{\left(\dfrac{\mu U}{M}\right)}{\left(\dfrac{1}{D_1}\left(\dfrac{z_M}{z_B}\right)^{1/2}\dfrac{C_1^{\,3}}{C_3^{\,2}}\right)}$

[Cf] mean dilution $= \dfrac{\mu}{Q}$

ii)Asymptotic solution for trajectories and mean dilutions for $z_M < z_B$

(buoyancy flux is strong) → Table 2.6

Table 2.6 Asymptotic Solutions for Trajectories and Mean Dilutions for a Vertical Turbulent Buoyant Jet in a uniform Crossflow

	$\hat{\xi} \ll 1$	$1 \ll \hat{\xi} \ll \hat{\xi}_c$	$\hat{\xi}_c \ll \hat{\xi}$
$\hat{\zeta}$	$\hat{\xi}^{1/2}$	$\hat{\xi}^{3/4}$	$\hat{k}\left(\frac{z_B}{z_M}\right)^{1/6}\hat{\xi}^{2/3}$
$\hat{S}$	$\hat{\xi}^{1/2}$	$\hat{\xi}^{5/4}$	$\left(\frac{z_M}{z_B}\right)^{1/6}\hat{\xi}^{4/3} \Big/ \hat{k}$

where $\hat{\xi}_c = \hat{k}^{1/2}\left(\frac{z_B}{z_M}\right)^2 \quad \leftarrow \quad l_M = \frac{M^{3/4}}{B^{1/2}}$

$$\hat{k} = \left(\frac{C_4}{C_3}\right)\left(\frac{C_3}{C_1}\right)^{1/3}$$

[Re] $\frac{1}{2} = \left(\frac{1}{2}\right)\times 1; \quad \frac{5}{4} = \left(\frac{3}{4}\right)\times\left(\frac{5}{3}\right); \quad \frac{4}{3} = \left(\frac{2}{3}\right)\times 2$

Dilution: $\hat{S} = \frac{C_o}{\theta}$

JV: $\hat{S} \sim \left(\frac{\bar{z}}{z_M}\right)^1 \sim \left[\left(\frac{x}{z_M}\right)^{1/2}\right]^1 \sim \left(\frac{x}{z_M}\right)^{1/2}$

PV: $\hat{S} \sim \left(\frac{\bar{z}}{z_B}\right)^{5/3} \sim \left[\left(\frac{x}{z_B}\right)^{3/4}\right]^{5/3} \sim \left(\frac{x}{z_B}\right)^{5/4}$

PBO: $\hat{S} \sim \left(\frac{\bar{z}}{z_B}\right)^2 \sim \left[\left(\frac{x}{z_B}\right)^{2/3}\right]^2 \sim \left(\frac{x}{z_B}\right)^{4/3}$

iii)Asymptotic solution for trajectories and mean dilutions for $z_M > z_B$

(buoyancy flux is weak) $\rightarrow$ Table 2.7

Table 2.7 Asymptotic Solutions for Trajectories and Mean Dilutions for a Vertical Turbulent Buoyant jet in a Uniform Crossflow

	$\xi \ll 1$	$1 \ll \xi \ll \xi_c$	$\xi_c \ll \xi$
ς	$\xi^{1/2}$	$\xi^{1/3}$	$k\left(\dfrac{z_B}{z_M}\right)^{1/3}\xi^{2/3}$
S	$\xi^{1/2}$	$\xi^{2/3}$	$k^2\left(\dfrac{z_B}{z_M}\right)^{2/3}\xi^{4/3}$

where $\xi_c = \left(\dfrac{1}{k^3}\right)\left(\dfrac{z_M}{z_B}\right)$

$$k = \left(\frac{C_4}{C_1}\right)\left(\frac{C_2}{C_1}\right)$$

[Re] $\dfrac{1}{2} = \left(\dfrac{1}{2}\right)\times 1$; $\dfrac{2}{3} = (2)\times\left(\dfrac{1}{3}\right)$; $\dfrac{4}{3} = \left(\dfrac{2}{3}\right)\times 2$

$$S = \frac{C_o}{\theta}$$

JV: $S \sim \left(\dfrac{\overline{z}}{z_M}\right)^1 \sim \left[\left(\dfrac{x}{z_M}\right)^{1/2}\right]^1 \sim \left(\dfrac{x}{z_M}\right)^{1/2}$

JBO: $S \sim \left(\dfrac{\overline{z}}{z_M}\right)^2 \sim \left[\left(\dfrac{x}{z_M}\right)^{1/3}\right]^2 \sim \left(\dfrac{x}{z_M}\right)^{2/3}$

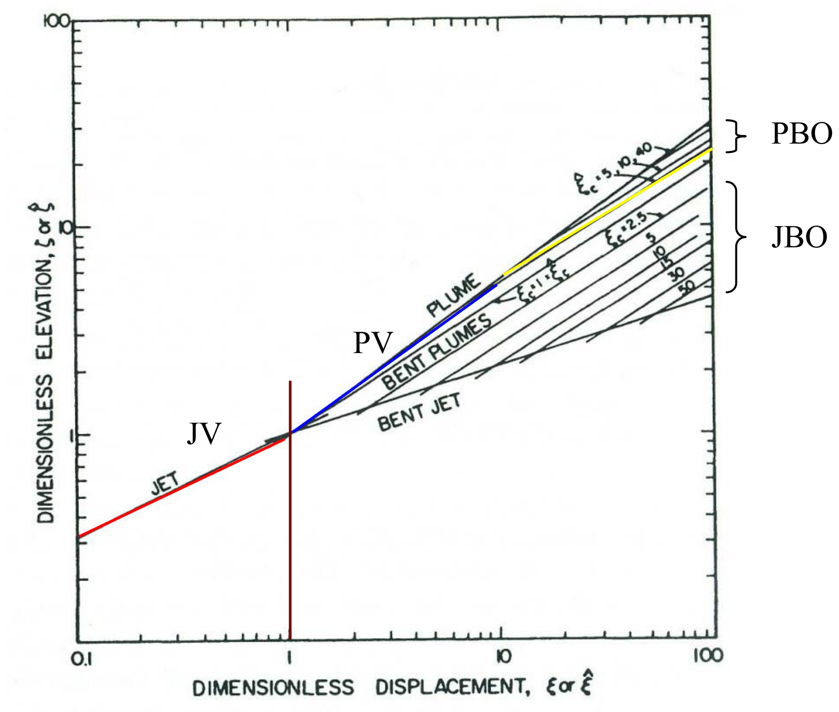


Fig. 2.21 Possible trajectories for round turbulent buoyant jets in a uniform crossflow

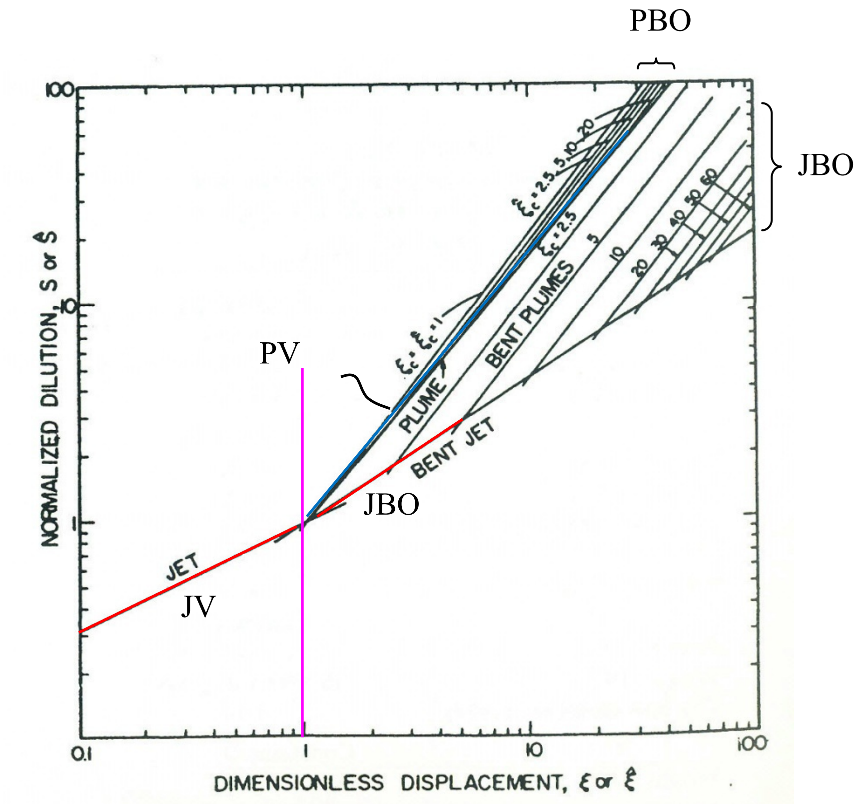


Fig. 2.22 Mean dilution in round turbulent buoyant jets in a uniform corssflow

▪Limitations of asymptotic solutions

~ provide order of magnitude estimates for trajectories and dilutions

→ There will be factors that will modify asymptotic theory to consider actual conditions such as velocity shear, localized density stratifications, and geometric influences.

•For a strongly buoyant plume in a crossflow, interaction of the crossflow and the discharge generate the a horseshoe vortex

→ bifurcation into two concentration maxima

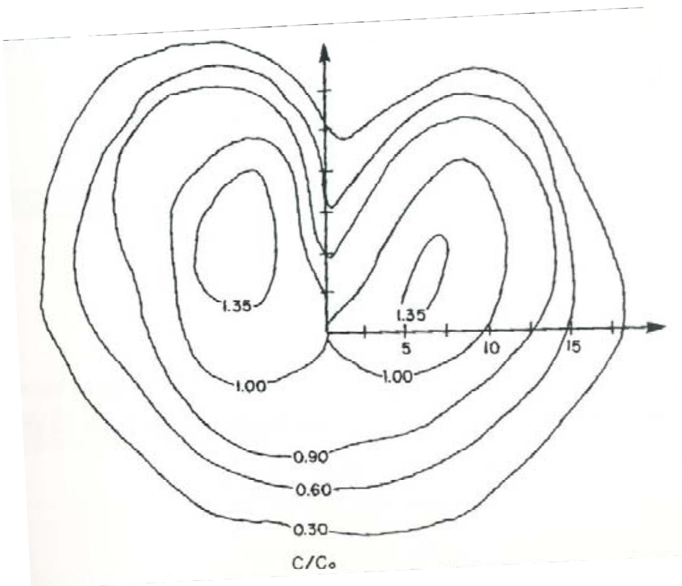


Fig. 2.23 Concentration isopleths showing bifurcation in a turbulent buoyant jet

•Negatively buoyant jets in uniform cross flows

~ crossflow may be parallel or transverse to the vertical plane containing jet axis

$$z_t = l_M f\left(\frac{z_M}{z_B}\right)$$

$$z_M = \frac{M^{1/2}}{U};\ z_B = \frac{B}{U^3}$$

Terminal height of rise

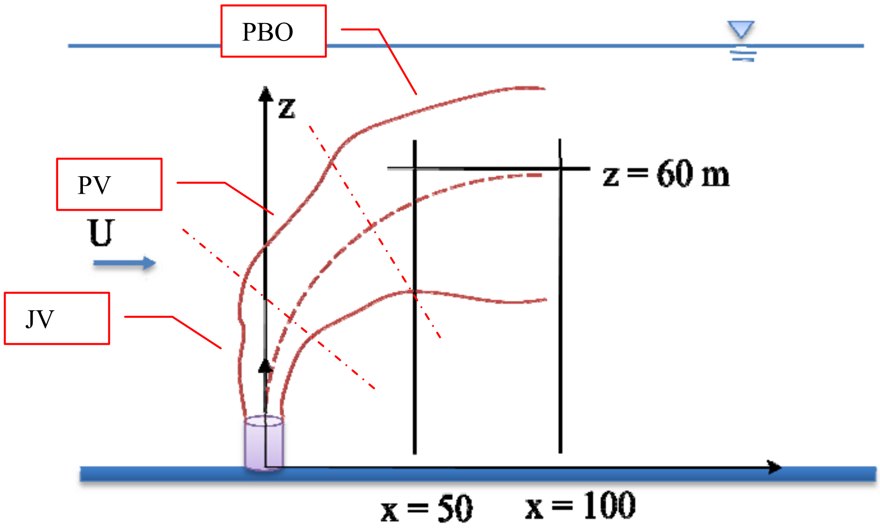
[Example 2.5] Vertical buoyant discharge into uniform crossflow

The ocean is homogeneous: $T = 11.1\,^{o}C$; Salinity $= 32.5\ ‰$

The discharge is fresh water: $T = 17.8^{\circ}\ \mathrm{C}$

$$Q = 1\,\mathrm{m^3/sec}\ ,\quad M = 3\,\mathrm{m^4/sec^2}\ ,\quad W = 3\,\mathrm{m/sec}$$

$$U = 0.25\,\mathrm{m/sec}\ ,\quad B = 0.257\,\mathrm{m^4/sec^3}$$



1) Compute characteristic length scales

$$l_Q = \frac{Q}{M^{1/2}} = \frac{1}{\sqrt{3}} = 0.58\,\mathrm{m}$$

$$l_M = \frac{M^{3/4}}{B^{1/2}} = \frac{(3)^{3/4}}{(0.257)^{1/2}} = 4.5\,\mathrm{m}$$

$$z_M = \frac{M^{1/2}}{U} = \frac{(3)^{1/2}}{0.25} = 6.9\,\mathrm{m}$$

$$z_B = \frac{B}{U^3} = \frac{0.257}{(0.25)^3} = 16.4\,\mathrm{m}$$

$\therefore\ l_M < z_M < z_B$

→ Buoyancy flux is strong.

→ Jet trajectory follows Fig. 2.19 (JV → PV → PBO)

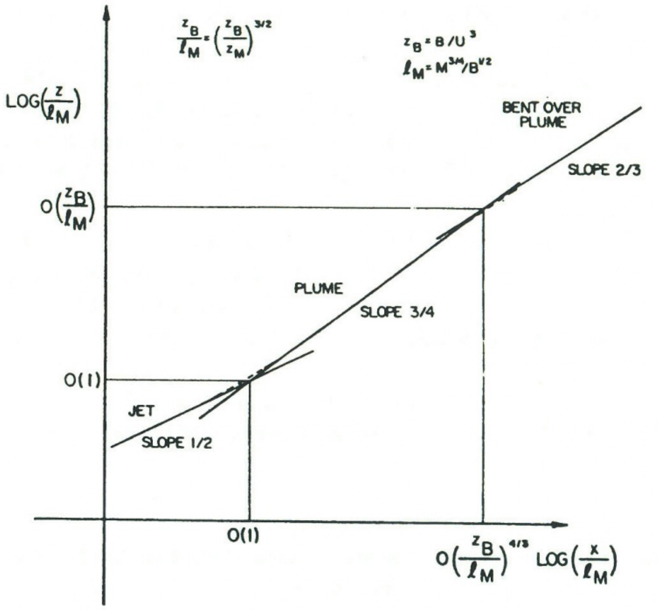


Fig. 2.19 Jet trajectory when $z_M < z_B$

i) $l_Q \ll \bar{z} \ll l_M$: $$\frac{\bar{z}}{z_M} = C_1\left(\frac{x}{z_M}\right)^{1/2}$$ J.V. (2.85)

ii) $l_M \ll \bar{z} \ll z_B$: $$\frac{\bar{z}}{z_B} = C_3\left(\frac{x}{z_B}\right)^{4/3}$$ P.V. (2.98)

iii) $\bar{z} \gg z_B$: $$\frac{\bar{z}}{z_B} = C_4\left(\frac{x}{z_B}\right)^{2/3}$$ P.B.O (2.103)

2) Find intersection of Eqs. (2.85) & (2.98) → JV & PV

$$\bar{z} = C_1 x_1^{1/2} z_M^{1/2} = C_3 x_1^{3/4} z_B^{1/4}$$

$$\therefore \quad x_1 = \frac{z_M^{\ 2}}{z_B}\left(\frac{C_1}{C_3}\right)^4$$

$$\therefore \quad \frac{\overline{z}_1}{z_M} = \frac{C_1}{z_M^{\ 1/2}}\left\{\frac{z_M^{\ 2}}{z_B}\left(\frac{C_1}{C_3}\right)^4\right\}^{1/2} = \frac{C_1^{\ 3}}{C_3^{\ 2}}\left(\frac{z_M}{z_B}\right)^{1/2}$$

Similarly, we get intersection of Eqs. (2.98) & (2.103) → PV & PBO

$$x_2 = z_B\left(\frac{C_4}{C_3}\right)^{12}$$

$$\overline{z}_2 = z_B\,\frac{C_4^{\ 9}}{C_3^{\ 8}}$$

From Table 2.8, use average values of constants

e.g. $C_1 = 2.1$, $C_3 = 1.6$, $C_4 = 1.1$

$$\therefore \quad x_1 = \frac{(6.9)^2}{16.4}\left(\frac{2.1}{1.6}\right)^4 = 8.6\text{ m}$$

$$\overline{z}_1 = \frac{(2.1)^3}{(1.6)^2}\frac{(6.9)^{3/2}}{(16.4)^{1/2}} = 16\text{ m}$$

$$x_2 = 16.4\left(\frac{1.1}{1.6}\right)^{12} = 0.2\text{ m}$$

$$\overline{z}_2 = 16.4\,\frac{(1.1)^9}{(1.6)^8} = 0.9\text{ m}$$

$\overline{z}_1 > \overline{z}_2$ .... wrong ← error in constants

[Cf] If we use $C_1 = 1.8$, $C_3 = 1.8$ $C_3 = 1.4$, $C_4 = 1.1$

Then, $x_1 = 2.9\,\text{m}$, $x_2 = 6.7\,\text{m}$, $\bar{z}_1 = 4.5\,\text{m}$, $\bar{z}_2 = 11.8\,\text{m}$

3) Now, assume plume solution

$$\frac{\bar{z}}{z_B} = C_3 \left(\frac{x}{z_B} \right)^{3/4}$$

For $\bar{z} = 60\,\text{m}$, $C_3 = 1.6$

$$\frac{60}{16.4} = 1.6 \left(\frac{x_{60}}{16.4} \right)^{3/4}$$

$$\therefore \; x_{60} = 49\,\text{m} \approx 50\,\text{m}$$

Now, consider plume bent-over solution for $\bar{z} = 60\,\text{m}$ with $C_4 = 1.1$

$$\frac{\bar{z}}{z_B} = C_4 \left(\frac{x}{z_B} \right)^{2/3}$$

$$\frac{60}{16.4} = 1.1 \left(\frac{x_{60}}{16.4} \right)^{2/3}$$

$$\therefore \; x_{60} = 99\,\text{m} \approx 100\,\text{m}$$

→ We can estimate the plume will be approximately 60 m from the bottom at between 50 and 100 m from the discharge point.

4) Calculate approximate dilution at $\overline{z} = 60\,\mathrm{m}$

From Table 2.5: Case $z_M < z_B$

$$\hat{\xi}_{60} = \left(\frac{\overline{z}}{z_M}\right)\left(\frac{z_B}{z_M}\right)^{1/2}\left(\frac{C_3}{C_1}\right)^2\frac{1}{C_1}$$

$$= \frac{60}{6.9}\left(\frac{16.4}{6.9}\right)^{1/2}\left(\frac{1.6}{2.0}\right)^2\left(\frac{1}{2.0}\right) = 4.3$$

$$\hat{\xi}_c = \left[\left(\frac{C_4}{C_3}\right)\left(\frac{C_3}{C_1}\right)^{1/6}\right]^{1/2}\left(\frac{z_B}{z_M}\right)^2$$

$$= \left[\left(\frac{1.1}{1.6}\right)\left(\frac{1.6}{2.0}\right)^{1/6}\right]^{1/2}\left(\frac{16.4}{6.9}\right)^2$$

$$= (0.8139)(5.649) \cong 4.6$$

From Fig. 2.21 → $\hat{\xi}_{60} \approx 9$ (PBO for $\hat{\xi}_c = 4.6$)

From Fig. 2.22 → $\hat{S}_{60} \approx 19$

From Table 2.5

$$\hat{S}_{60} = \left(\frac{\mu U}{M}\right)\Bigg/\left(\frac{1}{D_1}\left(\frac{z_M}{z_B}\right)^{1/2}\frac{C_2^{\,3}}{C_1^{\,2}}\right)$$

$$\therefore \left(\frac{\mu}{Q}\right)_{60} = \hat{S}_{60}\frac{M}{QU}\left(\frac{1}{D_1}\right)\left(\frac{z_M}{z_B}\right)^{1/2}\left(\frac{C_1^{\,3}}{C_3^{\,2}}\right)$$

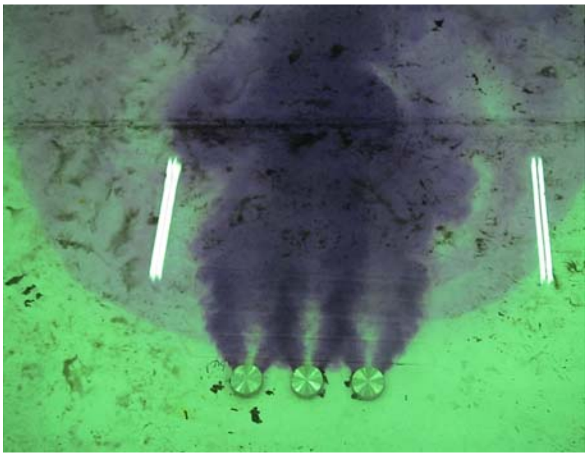
$$\therefore \left(\frac{\mu}{Q}\right)_{60} = 19\frac{3}{(1)(0.25)}\left(\frac{1}{2.4}\right)\left(\frac{6.9}{16.4}\right)^{1/2}\left(\frac{2.0^{3}}{1.6^{2}}\right) = 192$$

[Cf] Recall buoyant jets in still water, $\frac{\mu}{Q} = 88$ (plume solution)

Thus, crossflow increases dilution by increasing of <u>entrainment length</u> and by inducing the <u>forced entrainment</u> due to crossflow.

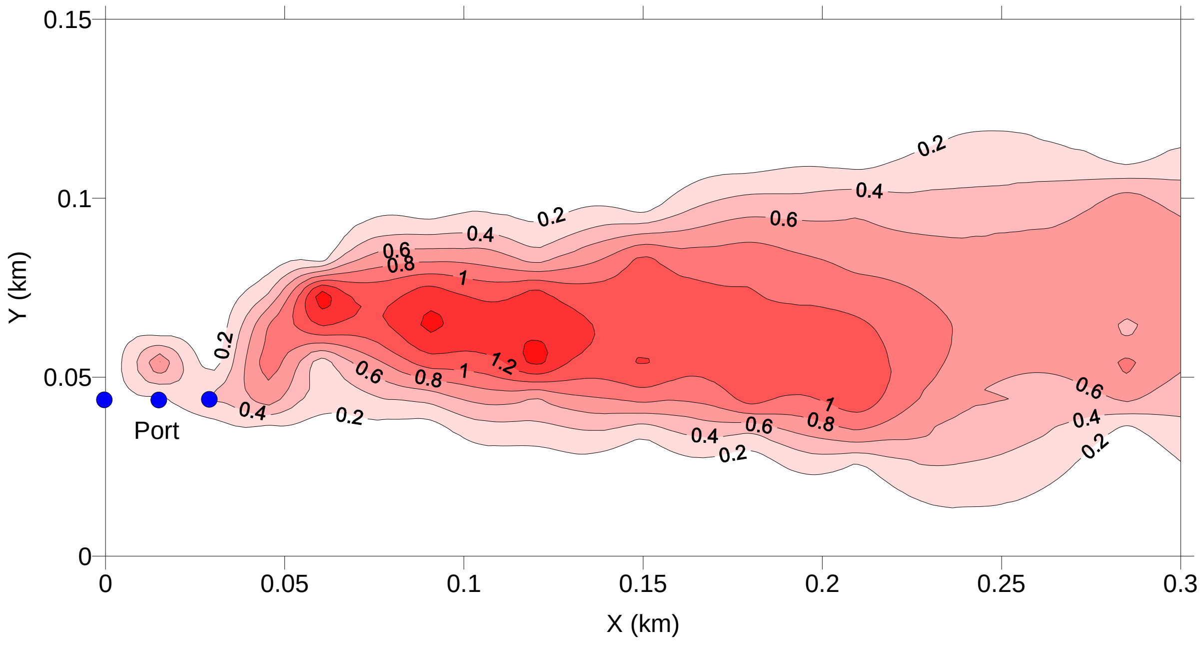
Model test of Tee diffuser

U = 0.84 m/s

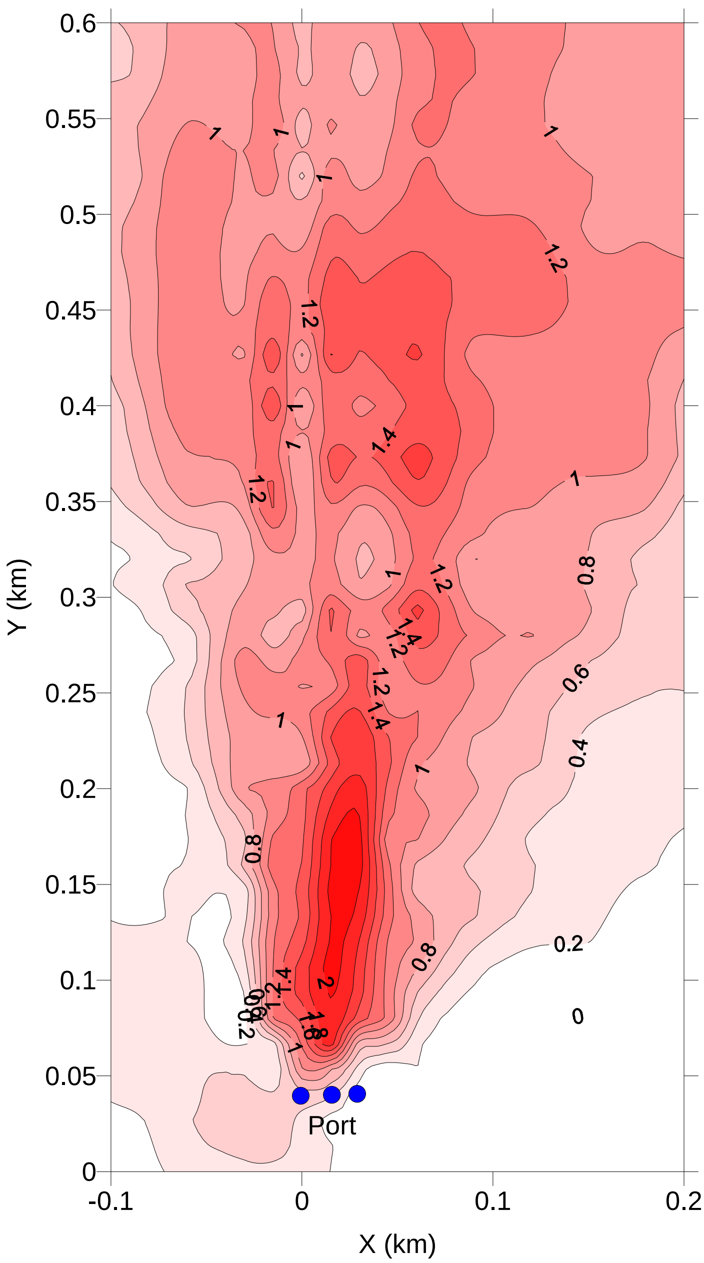


Stagnant Ambient

Excess temperature distribution (U = 0.84 m/s)



Excess temperature distribution (Stagnant Ambient)



2.3.3 Jets with Ambient Crossflow and Stratification

When density stratification exists along with a crossflow,

then the parameters involved are Q, M, B, $g\varepsilon'$, U.

→ 3 dimensionless parameters will govern the solutions.

→ length scales: l_Q, l_M, h_M, h_B, z_M, z_B

→ 3 dimensionless parameters can be represented by ratios of length scales:

1) Richardson number $= \dfrac{l_Q}{l_M} = R_o$ → fixes the origin (Fig.2.7)

2) Stratification parameter $= N = \left(\dfrac{h_M}{h_B}\right)^8 = \dfrac{M^2 \varepsilon' g}{B^2}$

3) Crossflow parameter $= \dfrac{z_M}{z_B} = \dfrac{M^{1/2} U^2}{B}$

• For jets in crossflow, use length scale that involves only the crossflow and stratification

$$\lambda = \frac{U}{(g\varepsilon')^{1/2}} \tag{2.104}$$

$(g\varepsilon')^{-1/2}$ ~ resonant period of oscillation of any particle located at a position of neutral density, [t]

λ ~ horizontal wavelength of vertical oscillation of the moving plume, [L]

→ wavelike nature of the plume oscillations - Fig. 2.24

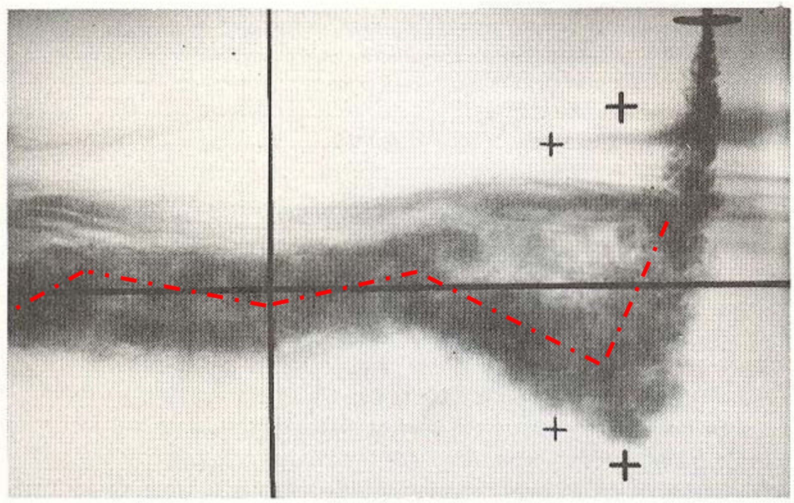


Figure 9.24 Turbulent negatively buoyant jet descending into a moving density-stratified environment $l_Q \ll z_M \ll z_B$. [From Wright (1977).]

Fig. 2.24 Turbulent negatively buoyant jet descending into a moving density-stratified environment $l_Q \ll z_M \ll z_B$

•The terminal height of rise for any asymptotic solution may be roughly specified by replacing *x* by λ in the trajectory equations.

→ In table 2.5 ~ 2.7, replace ξ by ξ_T and $\hat{\xi}$ by $\hat{\xi}_T$

where $\xi_T \sim \dfrac{\lambda}{z_M} = \dfrac{U^2}{(g\varepsilon' M)^{1/2}}$

$$\hat{\xi}_T \sim \left(\frac{\lambda}{z_M}\right)\left(\frac{z_B}{z_M}\right) = N^{-1/2} = \left(\frac{M^2 g\varepsilon'}{B^2}\right)^{1/2}$$

• Asymptotic solutions in a linearly stratified uniform crossflow

→ Table 2.9 , $E_1 = E_2 = E_3 = E_4 \approx 3.8$

i) $z_M > z_B$ → λ is critical parameter (jet behavior: momentum is dominant)

→ Jet w/o crossflow
- JV in stratified ambient
- JBO in stratified ambient

ii) $z_M < z_B$ → N is critical parameter (plume behavior: buoyancy is dominant)

→ Jet w/o crossflow
- PV in stratified ambient → plumelike reaches a terminal height of rise before being significant bent over
- PBO in stratified ambient

Table 2.9 Asymptotic heights of rise for a vertical turbulent buoyant jet

	Case	$\frac{U^2}{(g\varepsilon' M)^{1/2}} \ll 1$	$1 \ll \frac{U^2}{(g\varepsilon' M)^{1/2}} \ll \frac{z_M}{z_B}$ (JV)	$\frac{z_M}{z_B} \ll \frac{U^2}{(g\varepsilon' M)^{1/2}}$ (JBO)
Jet	$z_M > z_B$	$\frac{z_T}{h_M} = E_1$	$\frac{z_T}{z_M^{2/3}\lambda^{1/3}} = E_2$	$\frac{z_T}{z_B^{1/3}\lambda^{2/3}} = E_4$
	Case	$N^{-1/2} \ll 1$	$1 \ll N^{-1/2} \ll \left(\frac{z_B}{z_M}\right)^2$	$\left(\frac{z_B}{z_M}\right)^2 \ll N^{-1/2}$
Plume	$z_M < z_B$	$\frac{z_T}{h_M} = E_1$	$\frac{z_T}{h_B} = E_3$ (PV)	$\frac{z_T}{z_B^{1/3}\lambda^{2/3}} = E_4$ (PBO)

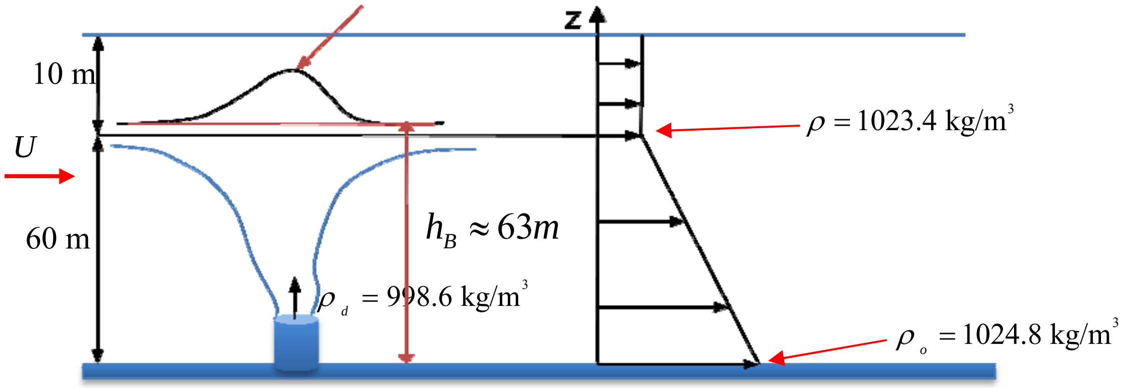
[a] See text for values of E_i.

[Example 2.6] Fresh water discharge as in Example 2.4

Suppose that a uniform temperature gradient exists over the lower 60 m of ocean:

$T = 11.1\ ^\circ\text{C}$ at $z = 0$ m; $T = 17.8\ ^\circ\text{C}$ at $z = 60$ m

Discharge: freshwater; $T_d = 17.8\ ^\circ\text{C}$ at $z = 0$ m; $Q = 1\ \text{m}^3/\text{s}$ (no momentum $\rightarrow$ plume)



$$\varepsilon' = 2.28\times10^{-5}\ 1/\text{m},\ U = 0.25\ \text{m/s},\ g\varepsilon' = 2.23\times10^{-4}\ 1/\text{s}^2$$

<Sol>

$$M = QW = 3\ \text{m}^4/\text{s}^2$$

$$B = g_o{}'Q = g\frac{\Delta\rho_o}{\rho}Q = (0.257\ \text{m/s}^2)(1\ \text{m}^2/\text{s}) = 0.257\ \text{m}^4/\text{s}^3$$

$$l_Q = 0.58\,\text{m},\ l_M = 4.5\,\text{m},\ z_M = 6.9\,\text{m},\ z_B = 16.4\,\text{m}$$

Since $z_M < z_B$: stratification is dominant $\rightarrow$ N is critical parameter

$$N = \frac{M^2 g\varepsilon'}{B^2} = \frac{(3\ \text{m}^4/\text{s}^2)^2(2.23\times01^{-4}\ 1/\text{s}^2)}{(0.257\ \text{m}^4/\text{s}^3)^2} = 0.03$$

$$N^{-1/2} = \frac{1}{(0.03)^{1/2}} = \underline{5.7} > 1$$

$$\left(\frac{z_B}{z_M}\right)^2 = \left(\frac{16.4}{6.9}\right)^2 = \underline{5.65}$$

$$N^{-1/2} \approx \left(\frac{z_B}{z_M}\right)^2$$

→ From Table 2.9, we may take either PV or PBO.

i) PV: $\frac{z_T}{h_B} = E_3 = 3.8$

$$\therefore \; z_T = 3.8\, h_B = 3.8 \frac{B^{1/4}}{(\varepsilon' g)^{3/8}} = 3.8 \frac{(0.257)^{1/4}}{(2.23 \times 10^{-4})^{3/8}}$$

$$= 3.8\,(16.7) = 63.3\,\text{m}$$

ii) PBO: $z_T = E_4 z_B^{1/3} \lambda^{2/3} = 3.8 \left(\frac{B}{U^3}\right)^{1/3} \left\{\frac{U}{(g\varepsilon')^{1/2}}\right\}^{2/3} = 3.8 \left(\frac{B}{U g \varepsilon'}\right)^{1/3}$

$$= 3.8 \left\{\frac{0.257}{0.25\,(2.23 \times 10^{-4})}\right\}^{1/3} = 3.8\,(16.6) = 63.1\,\text{m}$$

→ This result is identical to the result of Example 2.4.

9.3.4 Shear Flows and Ambient Turbulence

Researches on the influence of ambient shear flows and ambient turbulence on turbulent jet were focused on air pollution in atmospheric flows.

→ Slawson and Csanady (1971)

The effect of ambient shear flows and turbulence will be to increase dilutions.

→ Neglecting these effects would be a conservative design for the outfall structures.

The influence of density stratification on ambient turbulence levels and mixing should be investigated.

Homework Assignment #2-2

Due: 2 weeks from today

1. Derive Eq. (2.72) and Eq. (2.73)

2. Derive Eqs. (2.96) ~ (2.98) starting from Eqs. (2.70) ~ (2.71) and Eq. (2.84)

3. Derive Eqs. (2.101) ~ (2.103) starting from Eqs. (2.72) ~ (2.73) and Eq. (2.84)

Hint: Use self-similarity assumption for velocity and concentration profiles.

Homework Assignment #2-3

Due: 2 weeks from today

Investigate the behavior of the buoyant jet of which parameters are the same as given in Example 2.6.

1) Plot $\bar{z}$ vs x

2) Plot μ / Q vs s (s is coordinate of jet/plume centerline)

3) Plot w_m vs s

4) Plot 2-D ($x - z$ plane) contour of equi-concentration lines assuming Gaussian profile.