Chapter 8 Origin of Turbulence and Turbulent Shear Stress

8.1 Introduction

8.1.1 Definition

• Hinze (1975): Turbulent fluid motion is an irregular condition of flow in which the various quantities show a random variation with time and space coordinates, so that statistically distinct average values can be discerned.

statistically distinct average values: mean flow, primary motion

random fluctuations: non-periodic, secondary motion,

instantaneously unsteady, varies w.r.t. time and space

$$u = \bar{u} + u'$$

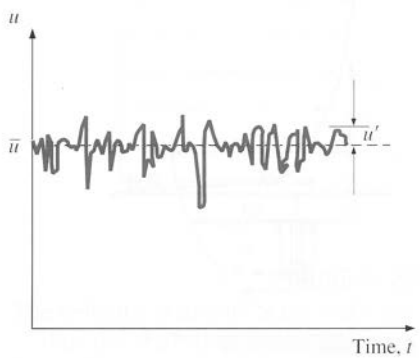


FIGURE 8–20
Fluctuations of the velocity component u with time at a specified location in turbulent flow.

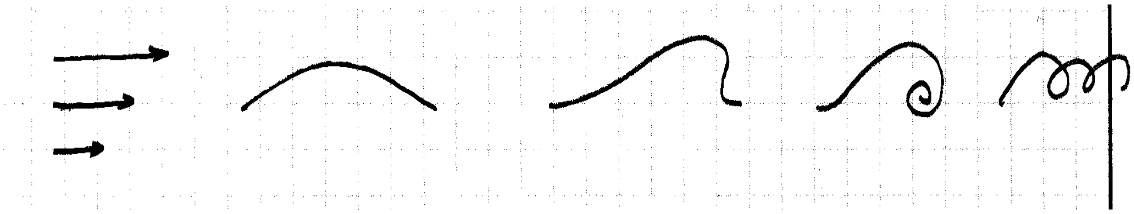
• Types of turbulence

Wall turbulence: turbulence generated and continuously affected by actual physical boundary such as solid walls

Free turbulence: absence of direct effect of walls, turbulent jet → AEH II

8.1.2 Origin of turbulence

(1) Shear flow instability



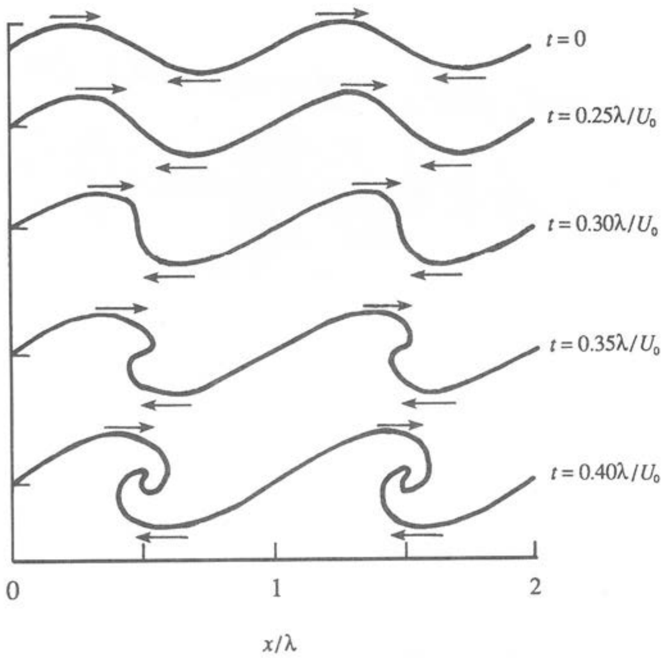


Figure 12.18 Nonlinear numerical calculation of the evolution of a vortex sheet that has been given a small sinusoidal displacement of wavelength λ. The density difference across the interface is zero, and U_0 is the velocity difference across the sheet. J. S. Turner, *Buoyancy Effects in Fluids*, 1973 and reprinted with the permission of Cambridge University Press.

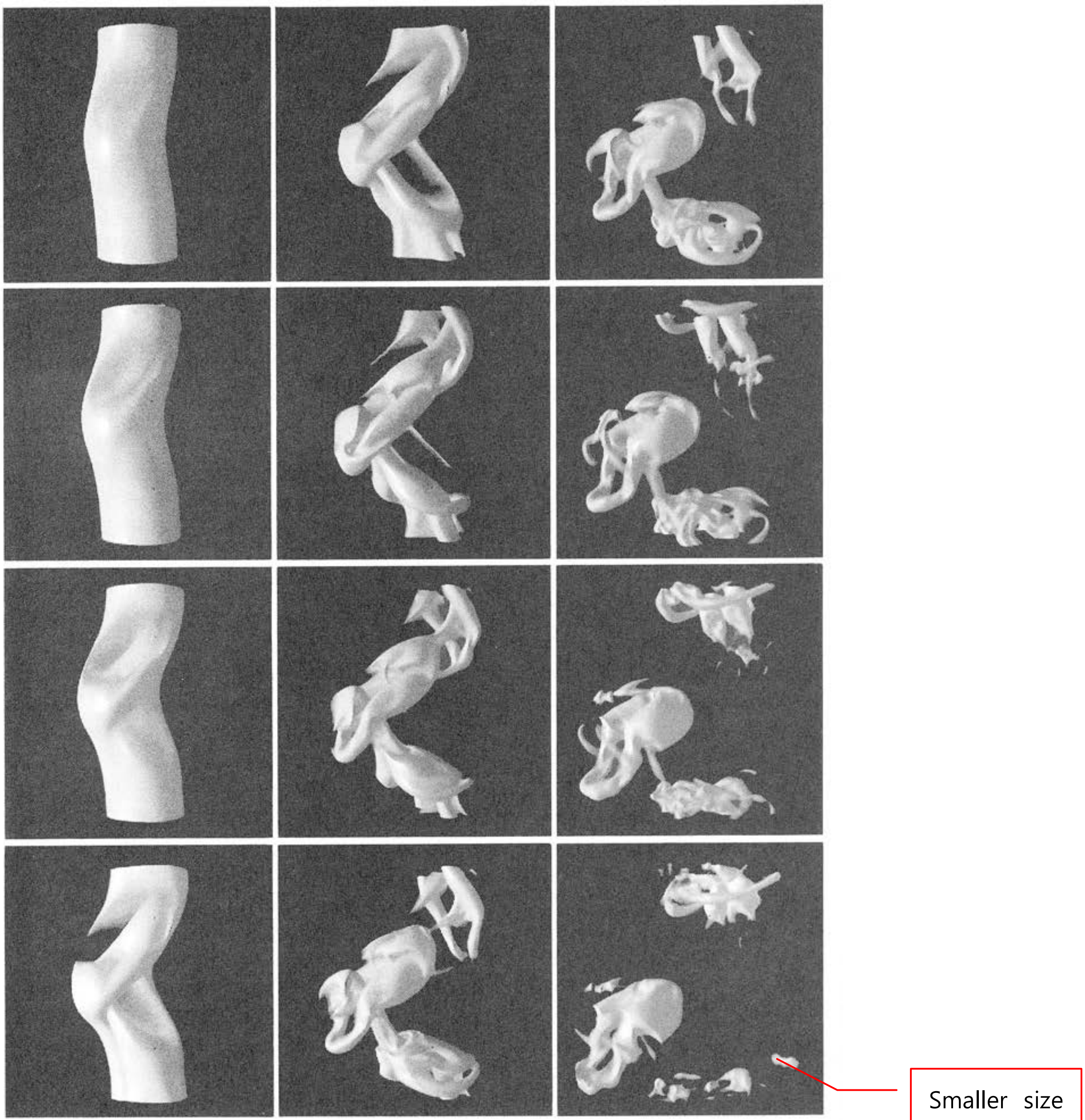


FIGURE 5.7. Vortex stretching, folding, sheetification. [Reprinted with permission from J. Bell and D. Marcus, *Comm. Math. Phys.* **147**, 371–394 (1992).]

(2) Boundary-wall-generated turbulence

~ wall turbulence

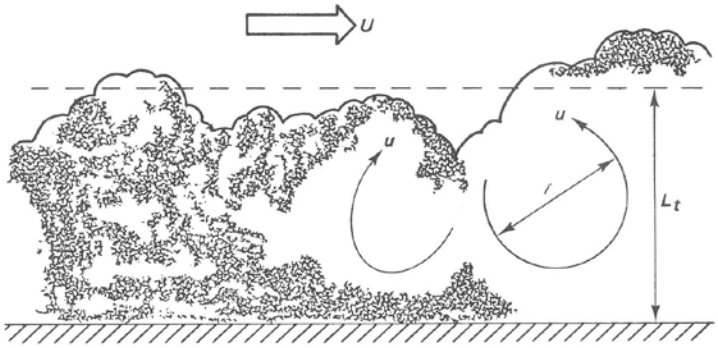


Figure 1.1. Large eddies in a turbulent boundary layer. The flow above the boundary layer has a velocity U; the eddies have velocities u. The largest eddy size (l) is comparable to the boundary-layer thickness (L_t). The interface between the turbulence and the flow above the boundary layer is quite sharp (Corrsin and Kistler, 1954).

(3) Free-shear-layer-generated turbulence

~ free turbulence

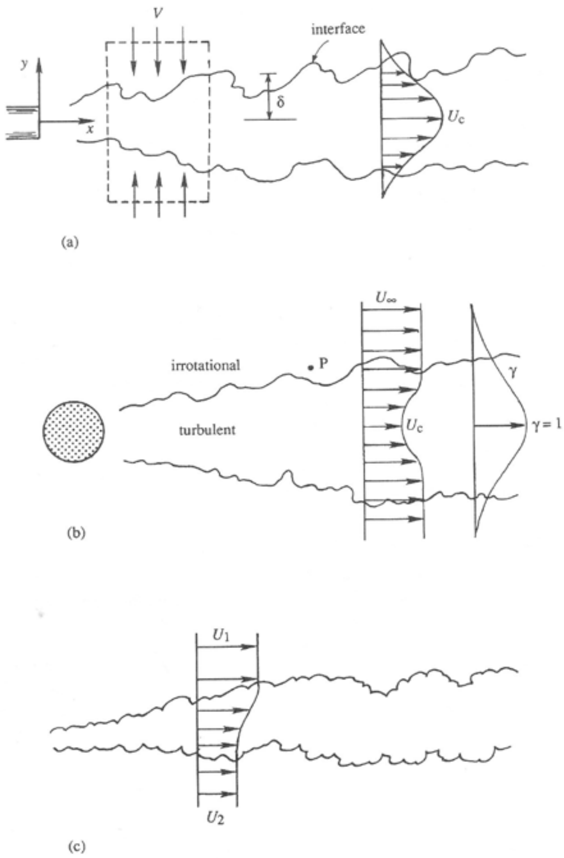
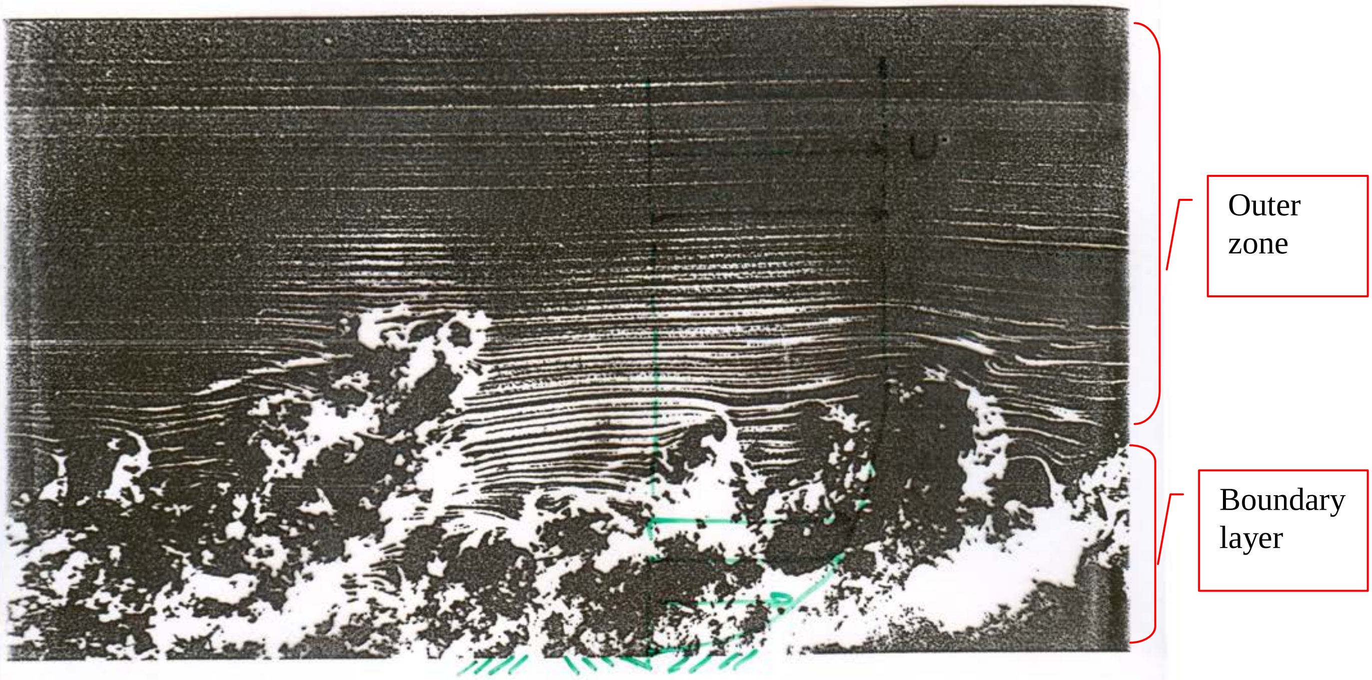


Figure 13.13 Three types of wall-free turbulent flows: (a) jet; (b) wake; and (c) shear layer.



157. Side view of a turbulent boundary layer. Here a turbulent boundary layer develops naturally on a flat plate 3.3 m long suspended in a wind tunnel. Streaklines from a smoke wire near the sharp leading edge are illuminated by a vertical slice of light. The Reynolds number is 3500 based on the momentum thickness. The intermittent nature of the outer part of the layer is evident. *Photograph by Thomas Corke, Y. Guezennec, and Hassan Nagib.*

158. Turbulent boundary layer on a wall. A fog of tiny oil droplets is introduced into the laminar boundary layer on the test-section floor of a wind tunnel, and the layer then tripped to become turbulent. A vertical sheet of light shows the flow pattern 5.8 m downstream, where the Reynolds number based on momentum thickness is about 4000. *Falco 1977*

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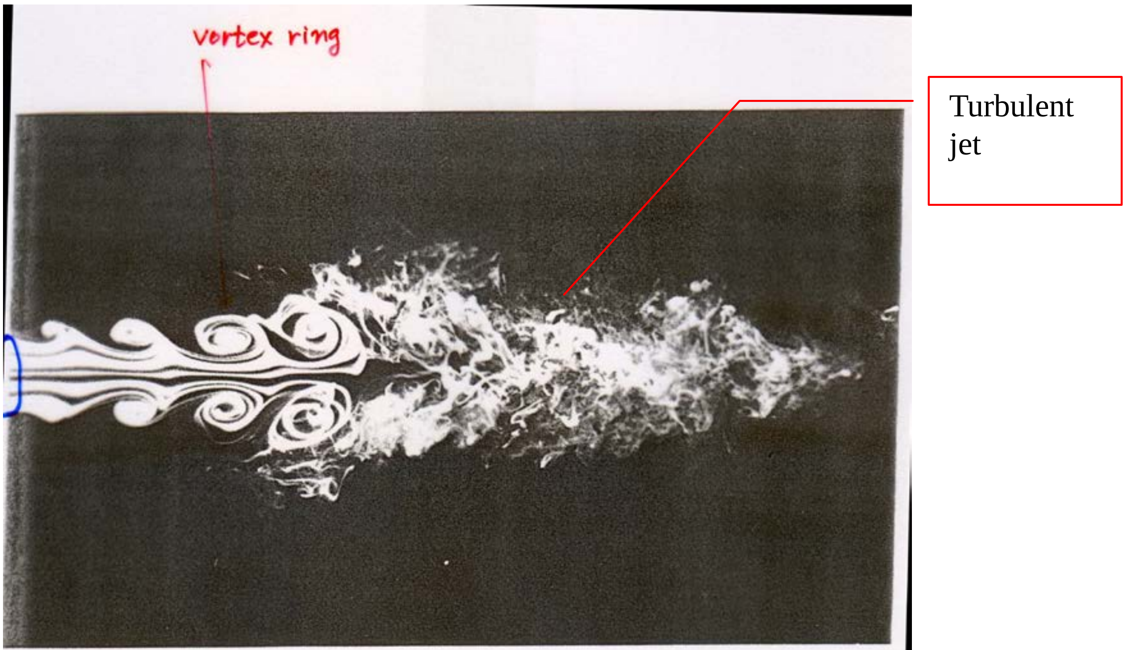
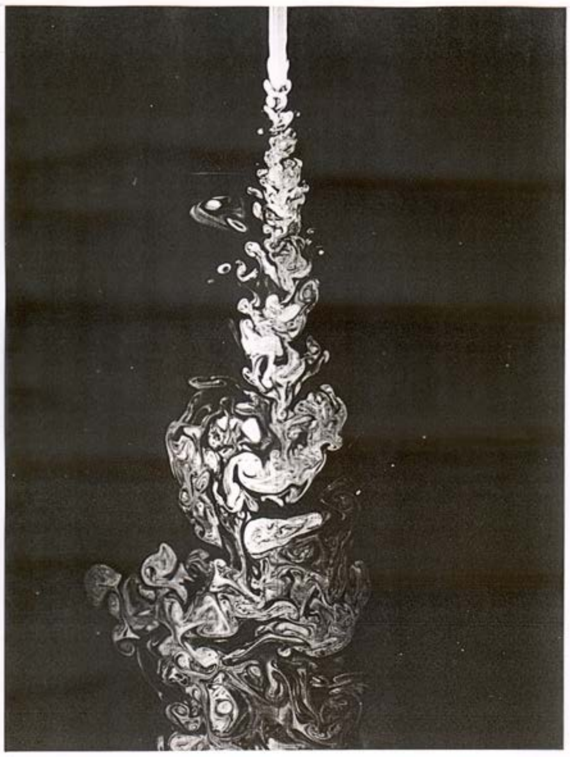


102. Instability of an axisymmetric jet. A laminar stream of air flows from a circular tube at Reynolds number 10,000 and is made visible by a smoke wire. The edge of the jet develops axisymmetric oscillations, rolls up into vortex rings, and then abruptly becomes turbulent. *Photograph by Robert Drubka and Hassan Nagib*



166. Turbulent water jet. Laser-induced fluorescence shows the concentration of jet fluid in the plane of symmetry of an axisymmetric jet of water directed downward into water. The Reynolds number is approximately 2300. The spatial resolution is adequate to resolve the Kolmogorov scale in the downstream half of the photograph. *Dimotakis, Lye & Papantoniou 1981*

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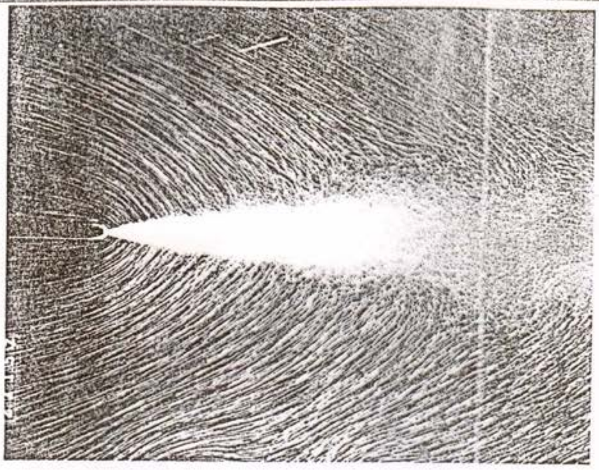
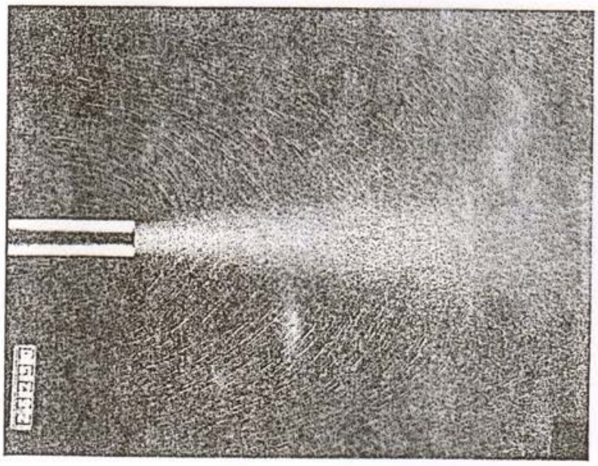
An Album of
Fluid Motion pp.99-101

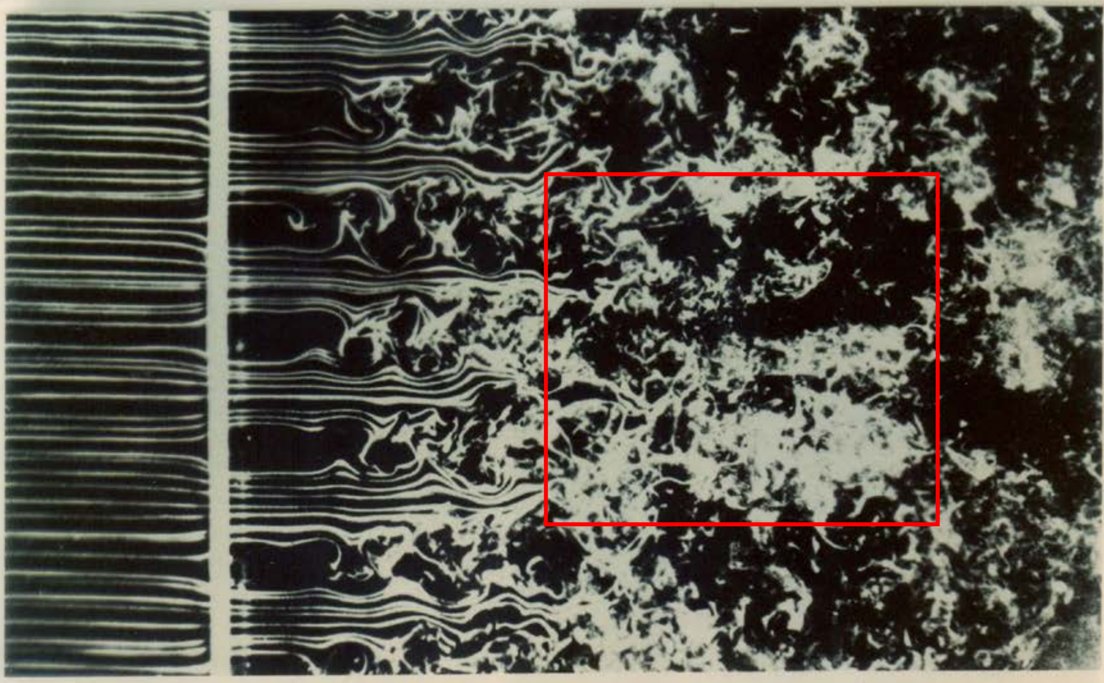


169. Entrainment by a plane turbulent jet. A time exposure shows the mean flow of a plane jet of colored water issuing into ambient water at 100 cm/s. Tiny air bubbles mark the streamlines of the slow motion induced in the surrounding water. *ONERA photograph, Werlé 1974*



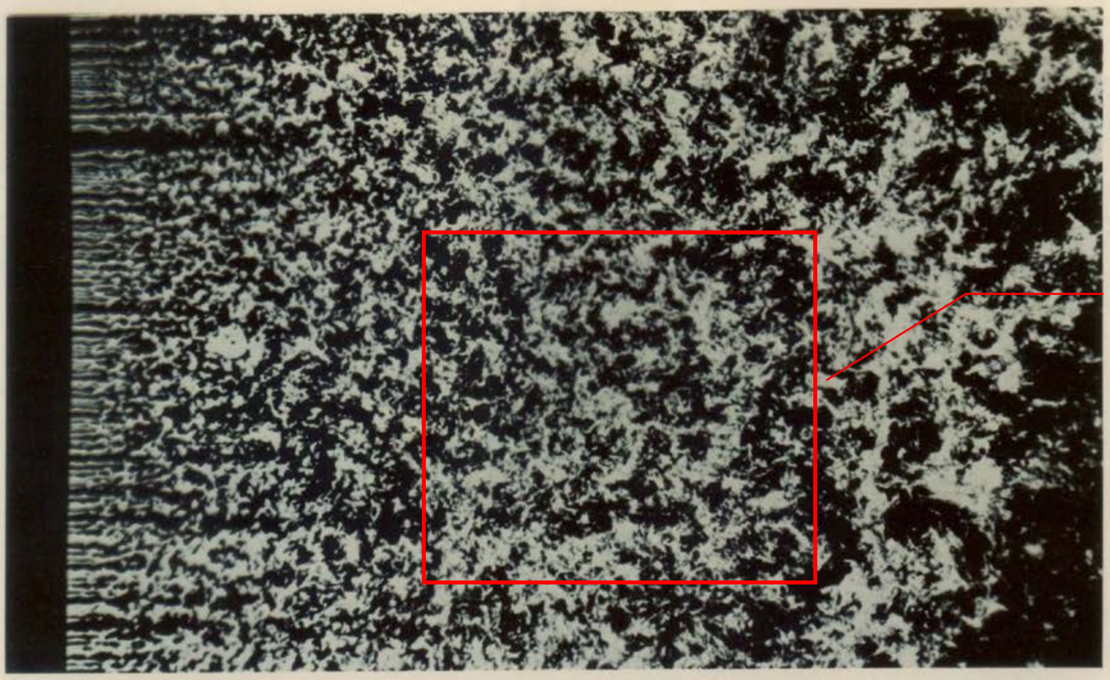
170. Entrainment by an axisymmetric turbulent jet. A jet of colored turbulent water flows from a tube of 9 mm diameter at 200 cm/s. According to boundary-layer theory the streamlines shown by air bubbles in the water outside the jet are paraboloids of revolution, and parabolas in the plane case above. *ONERA photograph, Werlé 1974*

174. Turbulent wake of a cylinder. A sheet of laser light slices through the wake of a circular cylinder at a Reynolds number of 1770. Oil fog shows the instantaneous flow pattern, covering 40 diameters centered 50 diameters downstream. *Photograph by R. E. Falco*



152. Generation of turbulence by a grid. Smoke wires show a uniform laminar stream passing through a $^{1}/_{16}$-inch plate with ¾-inch square perforations. The Reynolds number is 1500 based on the 1-inch mesh size. Instability of the shear layers leads to turbulent flow downstream. *Photograph by Thomas Corke and Hassan Nagib*

grid turbulence



Isotropic turbulence

153. Homogeneous turbulence behind a grid. Behind a finer grid than above, the merging unstable wakes quickly form a homogeneous field. As it decays downstream, it provides a useful approximation to the idealization of isotropic turbulence. *Photograph by Thomas Corke and Hassan Nagib*

8.1.3 Nature of turbulence

(1) Irregularity

~ randomness

~ need to use statistical methods to turbulence problems

~ Turbulent motion can also be described by Navier-Stokes Eq.

[Cf] Coherent structure

(2) Diffusivity

~ causes rapid mixing and increased rates of momentum, heat, and mass transfer

~ exhibit spreading of velocity fluctuations through surrounding fluid

~ the most important feature as far as practical applications are concerned; it increases heat transfer rates in machinery, it increases mass transfer in water

(3) Large Reynolds numbers

~ occur at high Reynolds numbers

~ Turbulence originates as an instability of laminar flows if Re becomes too large.

pipe flow	$\mathrm{Re}_c = 2{,}100$
boundary layer	$\mathrm{Re}_c = \dfrac{U\delta^*}{\nu} = 600$
free shear flow	$\mathrm{Re}_c \sim low$

(4) Three-dimensional vorticity fluctuations

~ Turbulence is rotational and three-dimensional.

~ high levels of fluctuating vorticity

~ need to use vorticity dynamics

~ tend to be isotropic

[Cf] The 2-D flows like cyclones, random (irrotational) waves in the ocean are not turbulent motions.

(5) Dissipations

~ dissipative

~ deformation work increases the internal energy of the fluid while dissipating kinetic energy of the turbulence

~ needs a continuous supply of energy to make up for viscous losses.

~ main energy supply comes from mean flow by interaction of shear stress and velocity gradient

~ If no energy is supplied, turbulence decays rapidly.

[Re] Energy cascade

main flow → large scale turbulence → small scale turbulence→ heat

(6) Continuum

~ continuum phenomenon

~ governed by the equation of fluid mechanics: Navier-Stokes Eq. + Continuity Eq.

~ larger than any molecular length scale

(7) Flow feature

~ feature of fluid flows not fluid itself

~ Most of the dynamics of turbulence is the same in all fluids.

~ Major characteristics of turbulent flows are not controlled by the molecular properties of the fluid.

8.1.4 Description of turbulence problems

(1) Turbulence modeling

• Time-averaged Navier-Stokes Eq. → Reynolds Equations

→ No. of unknowns {mean values ($\bar{u}, \bar{v}, \bar{w}, \bar{p}$) + Reynolds stress components

$(\sigma_{ij} = -\rho < u_i', u_j' >)$} > No. of equations

→ Closure problem:

~ The gap (deficiency of equations) can be closed only with models and estimates based on intuition and experience.

(2) Methods of analysis

1) Phynomenological concepts of turbulence

~ based on a superficial <u>resemblance between molecular motion and turbulent motion</u>

~ crucial assumptions at a early stage in the analysis

• <u>Eddy viscosity model</u>

~ turbulence-generated viscosity is modeled using analogy with molecular viscosity

~ characteristics of flow

• Mixing length model

~ analogy with mean free path of molecules in the kinetic theory of gases

2) Dimensional analysis

~ one of the most powerful tools

~ result in the relation between the dependent and independent variables

[Ex] form of the spectrum of turbulent kinetic energy

3) Asymptotic theory

~ based on asymptotic invariance

~ exploit asymptotic properties of turbulent flows as *Re* approaches infinity (or very high).

[Ex)] Theory of turbulent boundary layers

Reynolds-number similarity

4) Deterministic approach

Large Eddy Simulation (L.E.S)

~ model only large fluctuations

5) Stochastic approach

8.2 Sources of Turbulence

8.2.1 Source of turbulence

(1) Surfaces of flow discontinuity (velocity discontinuity)

1) tip of sharp projections - a

2) the trailing edges of air foils and guide vanes - c

3) zones of boundary-layer separation - d

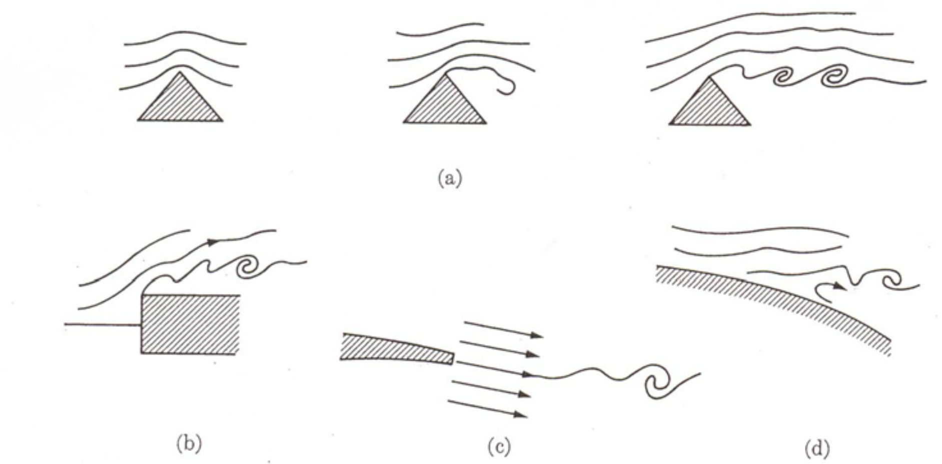


FIG. 11-1. Eddy formation at velocity discontinuity surfaces: (a) sharp projection; (b) bluff body; (c) trailing edge; (d) boundary-layer separation.

At surfaces of flow discontinuity,

→ tendency for waviness to develop by accident from external cause or from disturbance transported by the fluid.

→ waviness tends to be unstable

→ amplify (grow in amplitude)

→ curl over

→ break into separate eddies

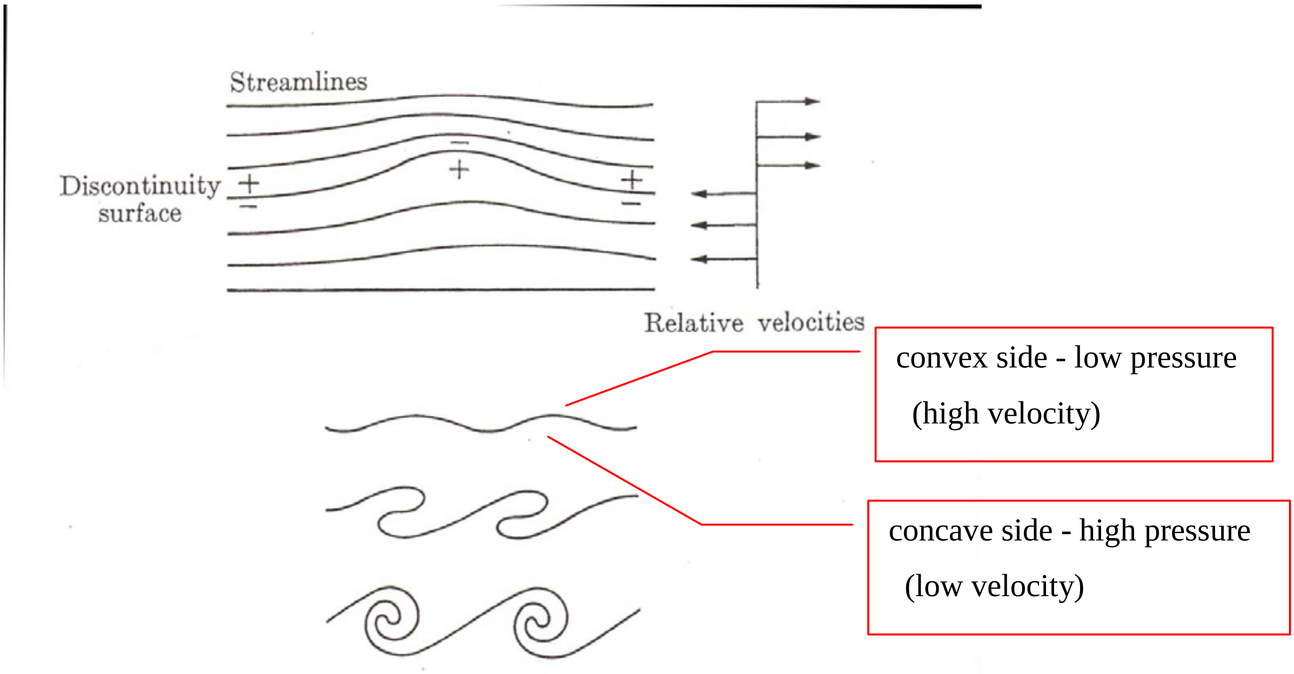


FIG. 11-3. Eddies arising from waves at a surface of discontinuity.

(2) Shear flows where velocity gradient occurs w/o an abrupt discontinuity

~ Shear flow is becoming unstable and degenerating into turbulence.

[Ex] Reynolds' experiment with a dye-streak in a glass tube

[Re] How turbulence arises in a flow

1) Presence of boundaries as obstacles creates vorticity inside a flow which was initially irrotational (vorticity, $\vec{\omega} = \nabla \times \vec{u}$).

2) Vorticity produced in the proximity of the boundary will diffuse throughout the flow which will become turbulent in the rotational regions.

3) Production of vorticity will then be increased due to vortex filaments stretching mechanism.

[Re] Grid turbulence = turbulence created behind a fixed grid in a wind tunnel

8.2.2 Mechanisms of instability

• Tollmien-Schlichting's small perturbation theory

~ Disturbance are composed of oscillations of a range of frequencies which can be selectively amplified by the hydrodynamic flow field.

$Re < Re_{crit} \rightarrow$ all disturbances will be damped

$Re > Re_{crit} \rightarrow$ disturbances of certain frequencies will be amplified and others damped

• Tollmien-Schlichting stability diagram

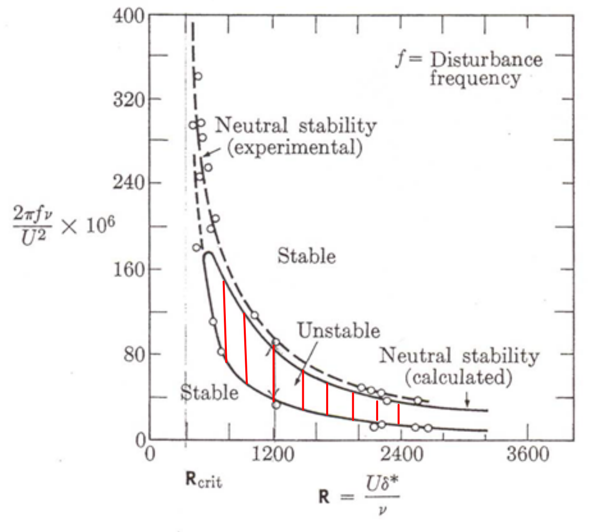


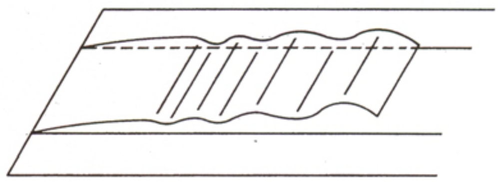
FIG. 11–4. Tollmien-Schlichting stability diagram for a laminar boundary layer. The ordinate is a dimensionless frequency function. Above R_{crit}, disturbances of the frequencies falling within the neutral stability loop are unstable and will amplify. Disturbances of all other frequencies will be stable [2].

• Disturbances appear in spots

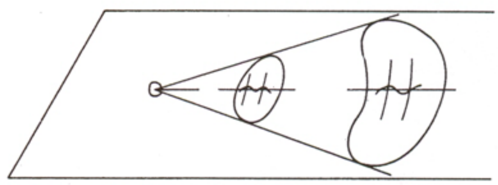
~ These spots grow as they are swept downstream.

~ spread and amplification of a spot disturbance into a turbulence patch

~ Mechanism of the initiation of spots of turbulence is related to what happens when the small disturbance, whose amplification is predicted by the small-perturbation theory, become large.



(a)



(b)

FIG. 11–5. Two-dimensional versus localized spot disturbance in a laminar boundary layer: (a) two-dimensional wave amplification by Tollmien-Schlichting hypothesis; (b) observed spread and amplification of a spot disturbance into a turbulence patch.

8.3 Velocities, Energies, and Continuity in Turbulence

8.3.1 Reynolds decomposition

(1) Velocity decomposition

$$u = \bar{u} + u'$$

$$v = \bar{v} + v'$$

$$w = \bar{w} + w' \qquad (8.1)$$

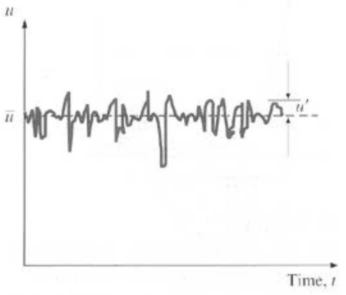


FIGURE 8–20
Fluctuations of the velocity component u with time at a specified location in turbulent flow.

u, v, w = instantaneous velocity

$\bar{u}, \bar{v}, \bar{w}$ = mean value = time-averaged value

u', v', w' = fluctuating components

$$\bar{u} = \frac{1}{T}\int_0^T u \, dt \qquad \left(\text{steady flow; } \frac{\partial \bar{u}}{\partial t} = 0\right) \qquad (8.2)$$

Pipe flow: 10^{-1}~10^{0} sec

Channel/River flow: 10^{0}~10^{1}

where T = long time compared to the time scale of the turbulence

$$\overline{u'} = \frac{1}{T}\int_0^T u' \, dt \equiv 0 \quad (\because \text{ fluctuations are both plus and minus}) \qquad (8.3)$$

$$\left(\frac{1}{T}\int_0^T (u - \bar{u}) dt = \bar{u} - \bar{u} = 0\right)$$

(2) Pressure and stress decomposition

$$p = \bar{p} + p'$$

$$\overline{p'} \equiv 0$$

$$\sigma_{ij} = \overline{\sigma_{ij}} + \sigma'_{ij}$$

$$\overline{\sigma'_{ij}} \equiv 0$$

• Mean stress tensor

$$\overline{\sigma_{ij}} = -\overline{p}\delta_{ij} + 2\mu\overline{S_{ij}}$$

$$\sigma_{ij}' = -p'\delta_{ij} + 2\mu S_{ij}'$$

in which $\overline{S_{ij}}$ = mean strain rate $\equiv \dfrac{1}{2}\left(\dfrac{\partial \overline{u_i}}{\partial x_j} + \dfrac{\partial \overline{u_j}}{\partial x_i}\right)$

S_{ij}' = strain-rate fluctuations $\equiv \dfrac{1}{2}\left(\dfrac{\partial u_i'}{\partial x_j} + \dfrac{\partial u_j'}{\partial x_i}\right)$

(3) Turbulence Intensity

→ root-mean-square (rms) = square root of variance = standard deviation average intensity of the turbulence = rms of u'

$$TI = \sqrt{\overline{u'^2}} = \left\{ \frac{1}{T}\int_0^T u'^2\, dt \right\}^{\frac{1}{2}} \qquad (8.4)$$

• Relative Turbulence Intensity (RTI) $= \dfrac{\sqrt{\overline{u'^2}}}{\overline{u}}$

(4) Average kinetic energy of turbulence per unit mass

~ average KE of turbulence / mass

$$= \frac{1}{2}\left(\overline{u'^2} + \overline{v'^2} + \overline{w'^2}\right)$$
$$= \frac{1}{2}\sum (intensity)^2 \tag{8.5}$$

(5) Energy density, $\phi(f)$

The kinetic energy is decomposed into an energy spectrum (density) vs. frequency.

≡ limit of average kinetic energy per unit mass divided by the bandwidth Δf

$$\phi(f) = \lim_{\Delta f \to 0} \frac{average\, KE \,/\, mass \;\; contained \;\; in \;\; \Delta f}{\Delta f} = \frac{\partial KE}{\partial f}$$

where f = ordinary frequency in cycles per second = $\dfrac{\omega}{2\pi}$

$\therefore$ average KE of turbulence / mass = $\displaystyle\int_0^\infty \phi(f)\,df = \frac{1}{2}\left(\overline{u'^2} + \overline{v'^2} + \overline{w'^2}\right)$

(6) Correlation between u', v', and w'

exact correlation = one-to-one correlation

zero correlation = completely independent

$$\overline{u'v'} = \frac{1}{T}\int_0^T u'v'\,dt \quad \left[\begin{array}{ll} \neq 0 & correlated \\ = 0 & uncorrelated \end{array}\right. \tag{8.6}$$

~ In a shear flow in an *xy*-plane, $\overline{u'v'}$ is finite, and it is related to the magnitude of the <u>turbulent shear stress</u> ($\tau = -\rho \overline{u'v'}$).

[Re] Correlated variables

1) Averages of products *u*

$$\begin{aligned}\overline{u_i u_j} &= \overline{(\overline{u_i} + u_i')(\overline{u_j} + u_j')} \\ &= \overline{\overline{u_i}\,\overline{u_j}} + \overline{u_i' u_j'} + \cancel{\overline{\overline{u_i} u_j'}} + \cancel{\overline{\overline{u_j} u_i'}} \\ &= \overline{u_i}\,\overline{u_j} + \overline{u_i' u_j'}\end{aligned}$$

If $\overline{u_i' u_j'} \neq 0 \rightarrow$ u_i' and u_j' are said to be correlated.

If $\overline{u_i' u_j'} = 0 \rightarrow$ u_i' and u_j' are uncorrelated.

2) Correlation coefficient

$$c_{ij} = \frac{\overline{u_i' u_j'}}{\left(\overline{u_i'^2} \cdot \overline{u_j'^2}\right)^{1/2}}$$

in which $\overline{u_i'^2}$, $\overline{u_j'^2}$ = variances

If $c_{ij} = \pm 1$ $\rightarrow$ perfect correlation

[Re] Classification of turbulence

1) General turbulence

$$\overline{u} \neq \overline{v} \neq \overline{w}$$

$$\overline{u'^2} \neq \overline{v'^2} \neq \overline{w'^2}$$

$$\overline{u'v'} \neq \overline{v'w'} \neq \overline{w'u'}$$

2) Homogeneous turbulence

~ statistically independent of the location

$$(\overline{u_i' u_j'})_a = (\overline{u_i' u_j'})_b$$

3) Isotropic turbulence

~ statistically independent of the orientation and location of the coordinate axes

$$\overline{u'^2} = \overline{v'^2} = \overline{w'^2} = \text{constant}$$

$$\overline{u'v'} = \overline{v'w'} = \overline{w'u'} = 0$$

~ uncorrelated

~ not coherent structures

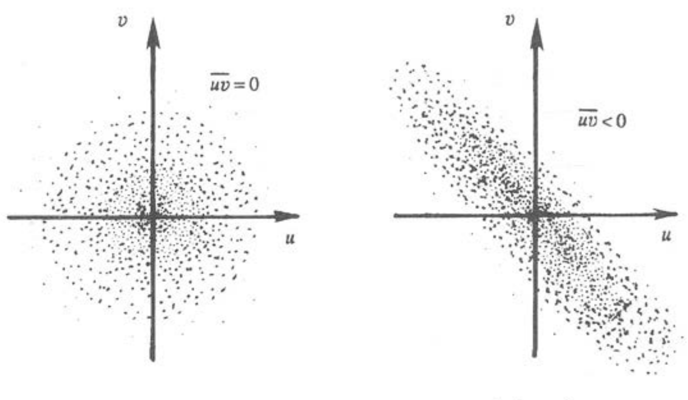


Figure 13.6 Isotropic and anisotropic turbulent fields. Each dot represents a uv-pair at a certain time.

8.3.2 Measurement of turbulence

~ measure turbulent fluctuations

(1) Hot-wire (hot-film) anemometer

~ Hot-film is usable in contaminated water.

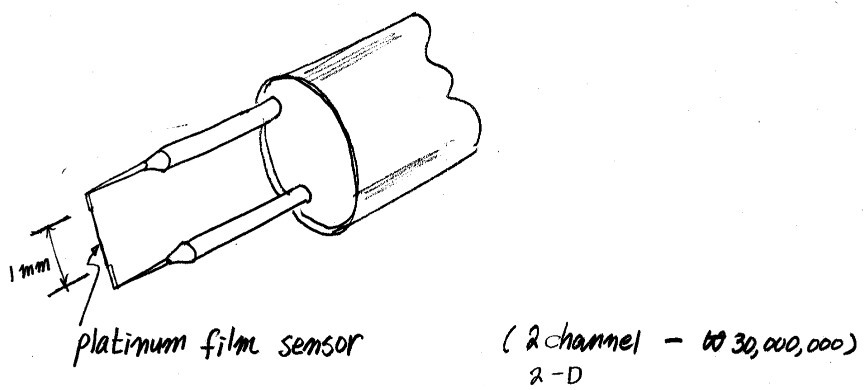
~ Change of temperature affects the electric current flow or voltage drop through wire. Fine platinum wire (film) is heated electrically by a circuit that maintains voltage drop constant.

~ When inserted into the stream, <u>the cooling, which is a function of the velocity</u>, can be detected as variations in voltage.

~ Use two or more wires at one point in the flow to make simultaneous measurements of different velocity components.

→ After subtracting mean value, rms-values, correlations, and energy spectra can be computed using fluctuation.

→ These operations can be performed electronically



(2) Laser Doppler Velocimeter (LDV)

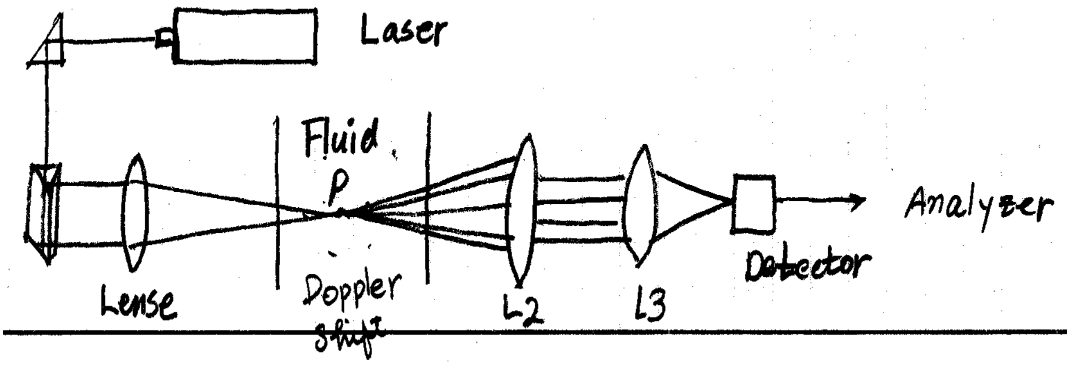
~ Doppler effects

~ An laser (ultrasonic) beam transmitted into the fluid will be reflected by impurities or bubbles in the fluid to a receiving sensor at a different frequency.

→ The transmitted and reflected signals are then compared by electronic means to calculate the Doppler shift which is proportional to the velocity.

~ non-intrusive sensing (immersible LDA)

~ sampling frequency = 20,000 Hz

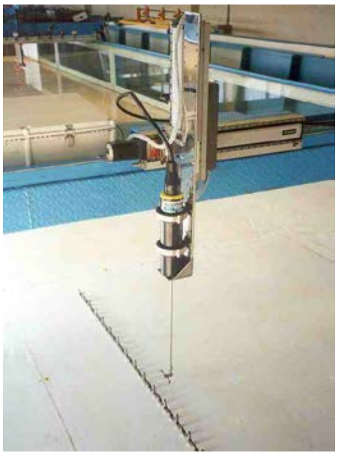


(3) Acoustic Doppler Velocimeter (ADV)

~ use Doppler effects

~ intrusive sensing

~ sampling frequency = 25-50 Hz



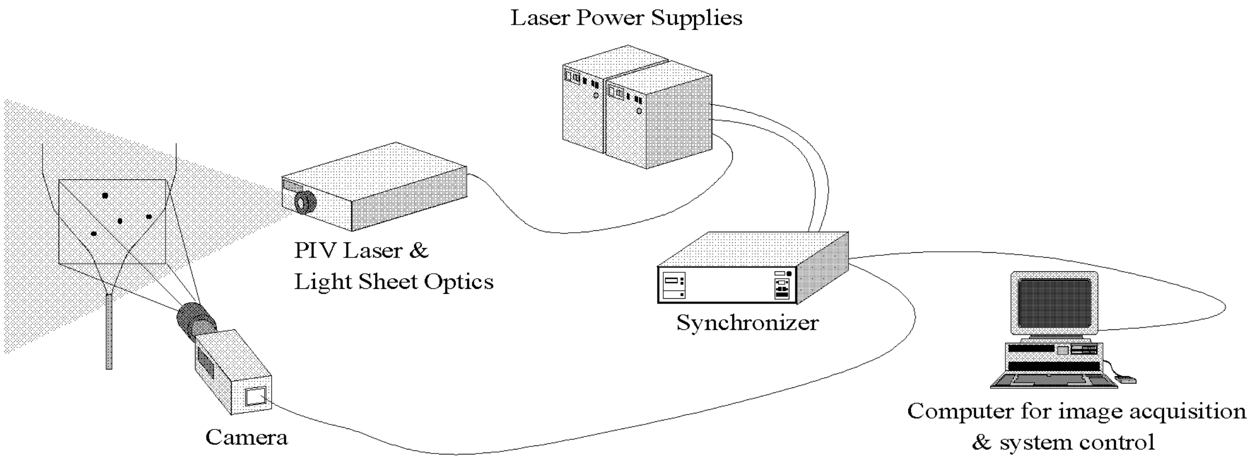
(4) Particle Image Velocimetry (PIV)

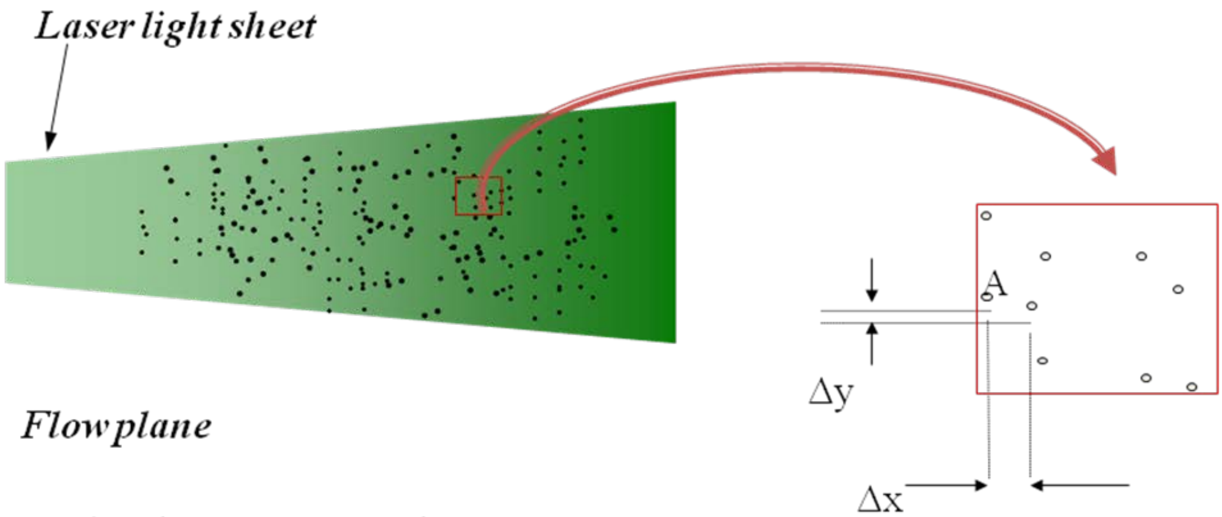
~ use Laser and CCD camera

~ measure flow field at once

~ sampling frequency = 30 Hz

PIV system



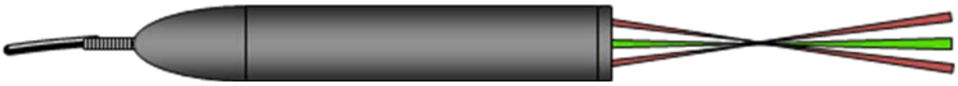


Δt - time between two pulses
Δx - particle displacement in x direction
Δy - particle displacement in y direction

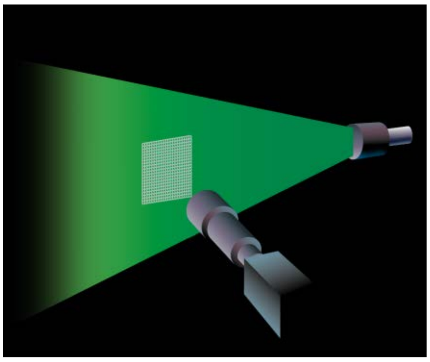
Velocity of particle A: $u_x = \dfrac{\Delta x}{\Delta t}\ as\ \Delta t \to 0$

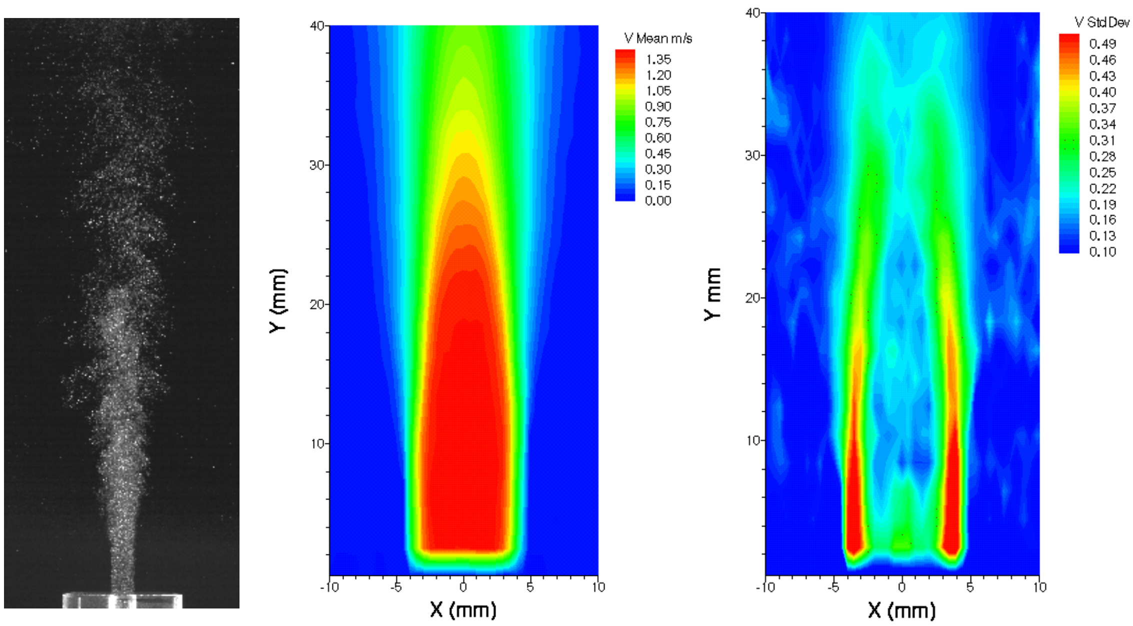
$$u_y = \frac{\Delta y}{\Delta t}\ as\ \Delta t \to 0$$

LDV: single point measurement



PIV: field measurement





a) Image b)Velocity c)Turbulence Intensity

Fig. 1 Jet Characteristics Measured by PIV (Seo et al., 2002)

[Re] Reynolds rules of averages: Schlichting (1979) Boundary-Layer Theory

Let *f* and *g* are two dependent variables whose time mean values are to be found. *s* is any one of the independent variables x, y, z, t.

$$\overline{\overline{f}} = \overline{f}$$

$$\overline{f + g} = \overline{f} + \overline{g}$$

$$\overline{f \cdot \overline{g}} = \overline{f} \cdot \overline{g}$$

$$\overline{\frac{\partial f}{\partial s}} = \frac{\partial \overline{f}}{\partial s} \rightarrow \left\{ \begin{array}{l} \text{since time averaging is carried out by integrating over a long} \\ \text{period of time,which commutes with differentiation with respect} \\ \text{to another independant variable} \end{array} \right.$$

$$\overline{\int f\, ds} = \int \overline{f}\, ds$$

8.3.3 Continuity for turbulent motion

Continuity equation for incompressible fluid

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad \text{(A)}$$

Substitute velocity decomposition into (A)

$$\frac{\partial(\overline{u} + u')}{\partial x} + \frac{\partial(\overline{v} + v')}{\partial y} + \frac{\partial(\overline{w} + w')}{\partial z} = 0 \qquad (8.7)$$

$$\frac{\partial \overline{u}}{\partial x} + \frac{\partial \overline{v}}{\partial y} + \frac{\partial \overline{w}}{\partial z} + \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0 \qquad \text{(B)}$$

Take time-averages of each term of (B)

$$\frac{\overline{\partial \bar{u}}}{\partial x}+\frac{\overline{\partial \bar{v}}}{\partial y}+\frac{\overline{\partial \bar{w}}}{\partial z}+\cancel{\frac{\overline{\partial u'}}{\partial x}}+\cancel{\frac{\overline{\partial v'}}{\partial y}}+\cancel{\frac{\overline{\partial w'}}{\partial z}}=0$$

$$\left(\because \frac{\overline{\partial u'}}{\partial x}=\frac{\partial(\overline{u'})}{\partial x}=0\right)$$

$$\boxed{\frac{\partial \bar{u}}{\partial x}+\frac{\partial \bar{v}}{\partial y}+\frac{\partial \bar{w}}{\partial z}=0} \qquad (8.8)$$

Substitute (8.8) into (B)

$$\boxed{\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}+\frac{\partial w'}{\partial z}=0} \qquad (8.9)$$

→ Both mean-motion components and the superposed turbulent-motion components must satisfy the continuity equation.

→ Continuity must be satisfied for both turbulent and laminar motions.

[Re] Continuity Eq. for compressible fluid

$$\frac{\partial \rho}{\partial t}+\frac{\partial \rho u_i}{\partial x_i}=0$$

$$\frac{\partial(\bar{\rho}+\rho')}{\partial t}+\frac{\partial\left\{(\bar{\rho}+\rho')(\bar{u}_i+u_i')\right\}}{\partial x_i}=0$$

Time averaging yields

$$\overline{\frac{\partial(\overline{\rho}+\rho')}{\partial t}}+\overline{\frac{\partial\{(\overline{\rho}+\rho')(\overline{u_i}+u_i')\}}{\partial x_i}}=0$$

$$\frac{\partial\overline{\overline{\rho}}}{\partial t}+\cancel{\frac{\partial\overline{\rho'}}{\partial t}}+\frac{\partial}{\partial x_i}\overline{(\overline{\rho}\overline{u_i}+\cancel{\rho'\overline{u_i}}+\cancel{\overline{\rho}u_i'}+\rho'u_i')}=0$$

$$\frac{\partial\overline{\rho}}{\partial t}+\frac{\partial}{\partial x_i}(\overline{\rho}\overline{u}_i+\overline{\rho'u_i'})=0$$

$$\frac{\partial\overline{\rho}}{\partial t}+\frac{\partial\overline{\rho}\overline{u}}{\partial x}+\frac{\partial\overline{\rho}\overline{v}}{\partial y}+\frac{\partial\overline{\rho}\overline{w}}{\partial z}+\frac{\partial}{\partial x}(\overline{\rho'u'})+\frac{\partial}{\partial y}(\overline{\rho'v'})+\frac{\partial}{\partial z}(\overline{\rho'w'})=0$$

8.4 Turbulent Shear Stress and Eddy Viscosities

8.4.1 Fall of pressure drop due to shear stress

shear stress = resistance to motion

→ dissipate flow energy → fall of pressure drop along a pipe → head loss ($h_L = \dfrac{\tau' l}{\gamma R}$)

laminar flow; $\dfrac{d(p+\gamma h)}{dz} \propto V_z$

turbulent flow: $\dfrac{d(p+\gamma h)}{dz} \propto V_z^n \quad (n \approx 2)$

where V_z = average velocity

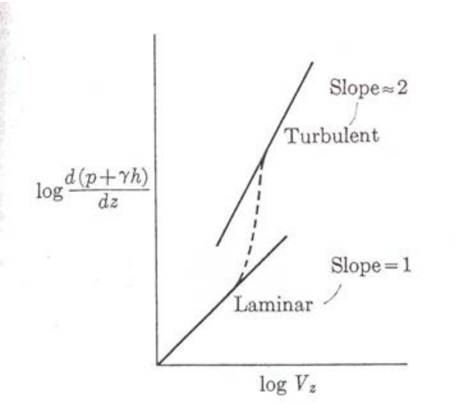


FIG. 11–6. Pressure gradient with laminar and turbulent flow in a conduit.

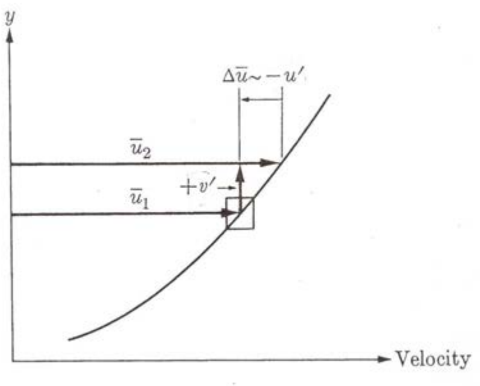


FIG. 11–7. Momentum transport by turbulent velocity fluctuation.

8.4.2 Shear stress resisting to motion

(1) Boussinesq's eddy viscosity concept

$$\tau_{total} = \mu \frac{d\bar{u}}{dy} + \eta \frac{d\bar{u}}{dy} \tag{8.10}$$

laminar flow

turbulent flow

where

$\bar{u}$ = mean local velocity (time - averaged)

μ = dynamic molecular viscosity → property of the fluid

η = dynamic eddy viscosity that depends on the state of the turbulent motion

($\varepsilon = \dfrac{\eta}{\rho}$ = kinematic eddy viscosity)

$\mu \dfrac{d\bar{u}}{dy}$ - apparent stress computed from the velocity gradient of mean motion.

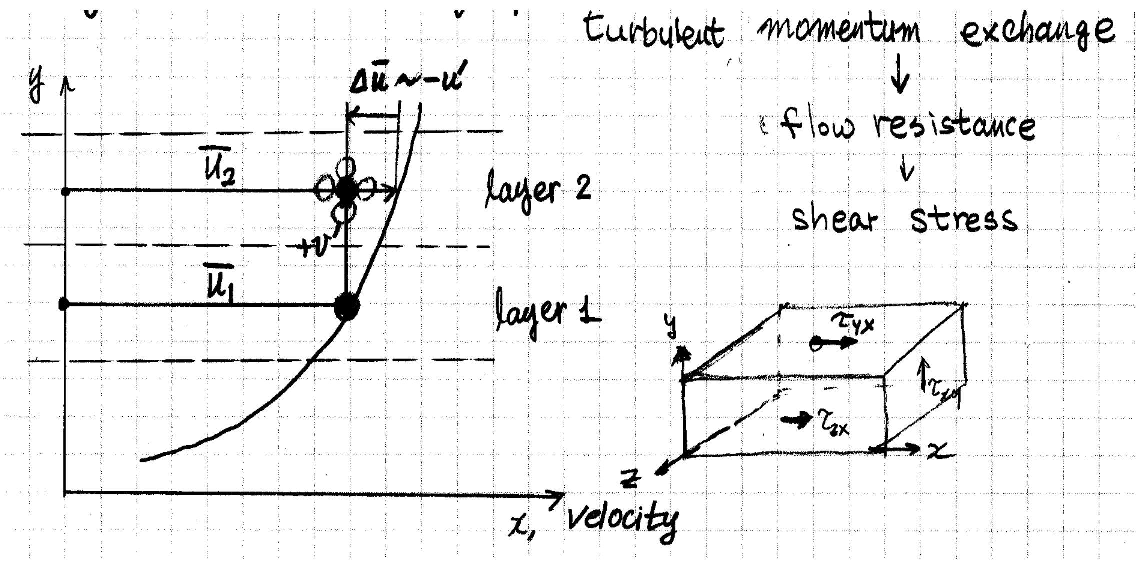
$\eta \dfrac{d\bar{u}}{dy}$ - additional apparent stress associated with the turbulence

For laminar flow, $\eta = 0$

For turbulent flow, $\eta \gg \mu \quad \rightarrow \quad \tau_{turb} > \tau_{lam}$

(2) Physical model of momentum transport (exchange)

~ momentum transport by turbulent velocity fluctuation



Step 1: lower-velocity fluid parcel in layer 1 fluctuates with a v'-velocity into layer 2

Step 2: its velocity in the direction of the stream is less than mean velocity of the layer 2 by an amount $-u'$

Step 3: drag of the faster moving surroundings accelerates the fluid element and increases its momentum

Step 4: The mass flux crossing from layer 1 to layer 2

$$= \rho v' \left(\frac{mass}{time \times area} \right)$$

Step 5: Flow-direction momentum change = mass × velocity

$$= \rho v' \times (-u') = -\rho u' v'$$

Step 6: Average over a time period

$$= -\rho \overline{u' v'}$$

= effective resistance to motion

= effective shearing stress

(3) Reynolds stress

$$= -\rho \overline{u' v'} \tag{8.11}$$

= time rate change of momentum per unit area

= effective resistance to motion

~ actually acceleration terms

~ instantaneous viscous stresses due to turbulent motion = $\eta \dfrac{d\bar{u}}{dy}$

$$\tau_{total} = \mu\left(\frac{d\bar{u}}{dy}\right) - \rho\overline{u'v'} = \tau_{yx} \tag{8.12}$$

shear stress due to transverse molecular momentum transport	shear stress due to transverse momentum transport of macroscopic fluid particles by turbulent motion

For fully developed turbulence,

$$\boxed{\tau_{yx} \cong \eta\left(\frac{d\bar{u}}{dy}\right) \approx -\rho\overline{u'v'}} \propto V_z^2 \tag{8.13}$$

[Re] $\boxed{\text{Reynolds stress} = -\rho\overline{u'v'}}$

~ If u' and v' are uncorrelated, there would be no turbulent momentum transport.

~ usually not zero (correlated)

~ may exchange momentum of mean motion

~ exchanges momentum between turbulence and mean flow

[Re] Effective addition to the normal pressure intensity acting in the flow direction

$$= -\overline{\rho u'u'} = -\overline{\rho u'^2} \tag{8.14}$$

[Re] Momentum transport

Eq. (3.2): $$\frac{d(\Delta mv)}{dt}\frac{1}{area} = K\frac{d}{dy}\left(\frac{\Delta mv}{vol}\right)$$

Newton's 2nd law of motion

$$F = ma = m\frac{dv}{dt} = \frac{d(mv)}{dt} \tag{A}$$

$$\frac{F}{area} = \frac{d(mv)}{dt}\frac{1}{area} \tag{B}$$

Assume only shear stresses exist,

Then LHS of (B) = τ

Combine (3. 2) & (B)

$$\tau = \eta\frac{d\bar{u}}{dy} \tag{C}$$

By the way, for the turbulent motion

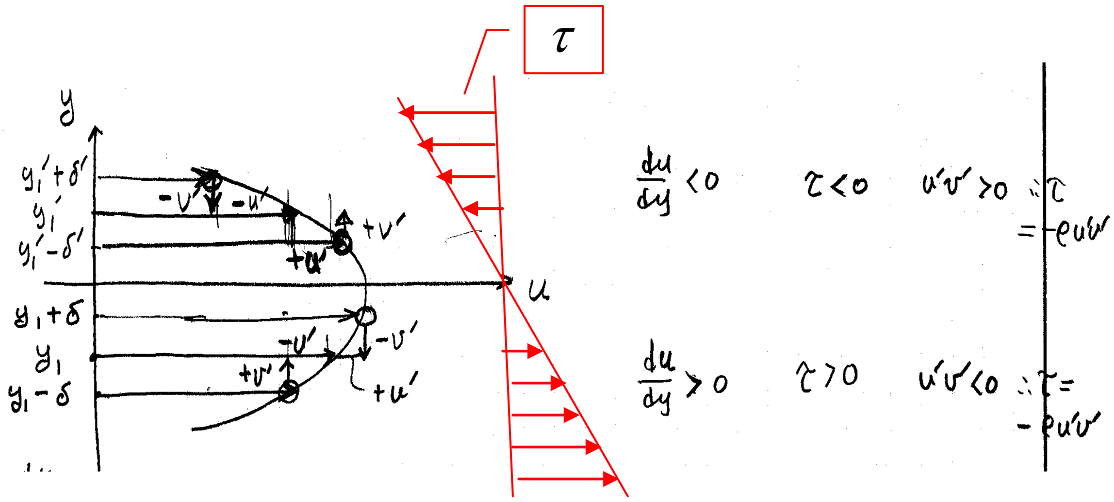
RHS of (B) = time rate change of momentum per unit area = $-\rho\overline{u'v'}$

$$\therefore\ -\rho\overline{u'v'} = \tau \tag{D}$$

Combine (C) and (D)

$$\tau_t = -\rho\overline{u'v'} = \eta\frac{d\bar{u}}{dy} \tag{E}$$

[Re] Shear stress for turbulent jet



Case I: $\dfrac{du}{dy} > 0$ → positive τ

1) $y_1 - \delta \rightarrow y_1$

mass flux $= \rho v'$

velocity change $= -u'$

$\therefore$ momentum change = $(\rho v') \times (-u') = -\rho u' v'$

$\tau = -\rho u' v'$ → + momentum change → positive

2) $y_1 + \delta \rightarrow y_1$

mass flax $= \rho(-v')$

velocity change = $+u'$

$\therefore$ momentum change = $(-\rho v') \times (u') = -\rho u' v'$

$\tau = -\rho u' v'$ → + momentum change → positive

Case II: $\frac{du}{dy} < 0 \rightarrow$ negative τ

1) $y_1' - \delta \rightarrow y_1'$

mass flax = $\rho v'$

velocity change = $+u'$

$\therefore$ momentum change = $(\rho v') \times (u') = \rho u' v'$

$\tau = -\rho u' v' \rightarrow$ - momentum change $\rightarrow$ negative

2) $y_1' + \delta \rightarrow y_1'$

mass flux = $\rho\ (-v')$

velocity change = $-u'$

$\therefore$ momentum change = $(-\rho v') \times (-u')\ = \rho u' v'$

$\tau = -\rho u' v' \rightarrow$ - momentum change $\rightarrow$ negative

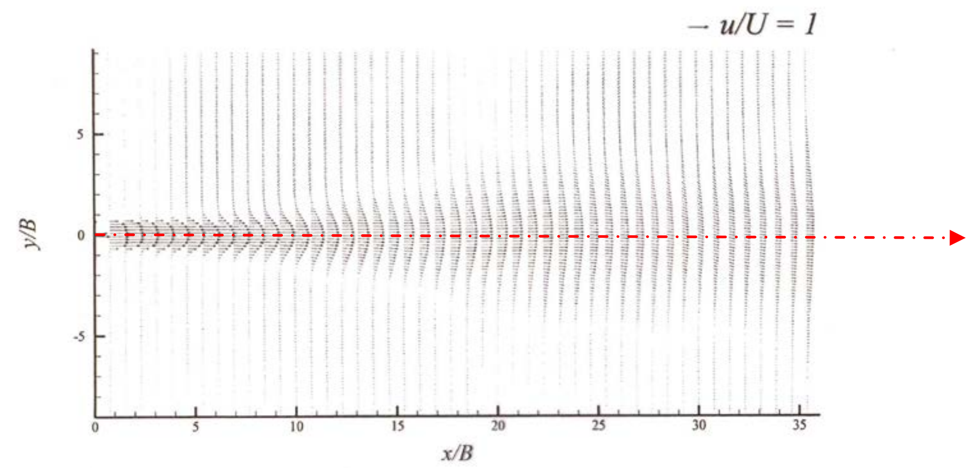


Fig. 4.6(c) Velocity Vector Fields for Case NFJ300 by PIV

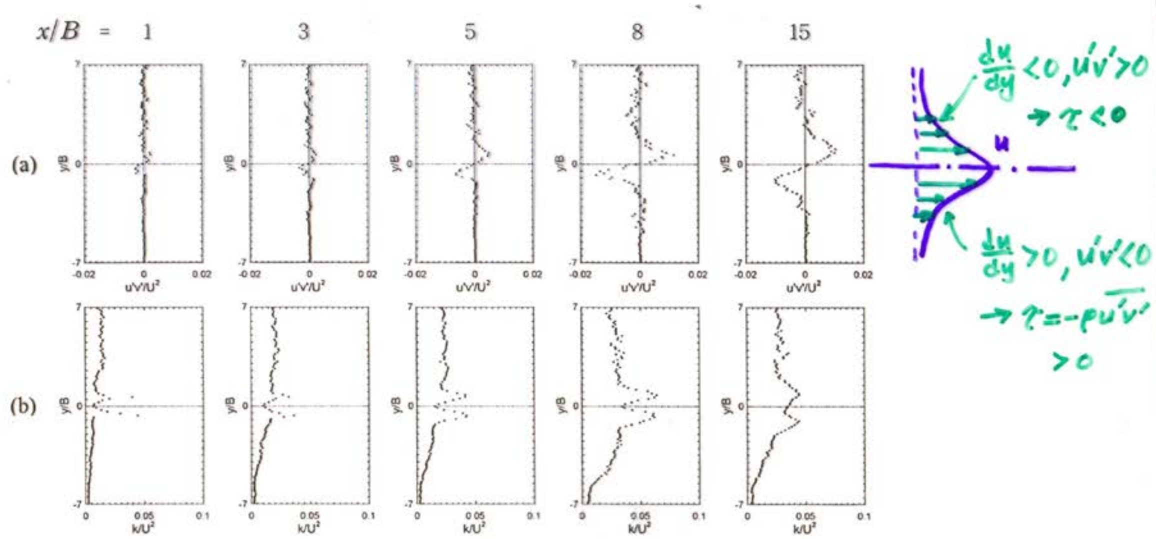


Fig. 4.15 Streamwise Development of Turbulent Characteristics for NFJ300 :
(a) $\overline{u'v'}/U^2$; (b) k/U^2

k = kinetic turbulent energy
$= \frac{1}{2}\left(\overline{u'^2} + \overline{v'^2} + \overline{w'^2}\right)$

8.5 Reynolds Equations for Incompressible Fluids

8.5.1 Reynolds Equation

Navier-Stokes Eq. = equations of motion of a viscous fluid

~ applicable to both turbulent and non-turbulent flows

~ very difficult to obtain exact solution because of complexity of turbulence

~ Alternative is to consider the pattern of the mean turbulent motion even through we cannot establish the true details of fluctuations.

→ average Navier-Stokes Eq. over time to derive Reynolds Eq.

N-S Eq. in x-dir.:

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=\rho g_x-\frac{\partial p_x}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}+\frac{\partial \tau_{zx}}{\partial z} \tag{8.15}$$

Continuity Eq. for incompressible fluid:

$$\rho\left(u\frac{\partial u}{\partial x}+u\frac{\partial v}{\partial y}+u\frac{\partial w}{\partial z}\right)=0 \tag{A}$$

Add (A) to (11.15), then LHS becomes

$$\text{LHS}=\frac{\partial u}{\partial t}+2u\frac{\partial u}{\partial x}+\left(v\frac{\partial u}{\partial y}+u\frac{\partial v}{\partial y}\right)+\left(w\frac{\partial u}{\partial z}+u\frac{\partial w}{\partial z}\right)=\frac{\partial u}{\partial t}+\frac{\partial u^2}{\partial x}+\frac{\partial uv}{\partial y}+\frac{\partial uw}{\partial z}$$

Whole equation is

$$\rho\left(\frac{\partial u}{\partial t}+\frac{\partial u^2}{\partial x}+\frac{\partial uv}{\partial y}+\frac{\partial uw}{\partial z}\right)=\rho g_x-\frac{\partial p_x}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}+\frac{\partial \tau_{zx}}{\partial z} \tag{8.16}$$

Decomposition:

$$
\begin{aligned}
u &= \bar{u} + u' \\
v &= \bar{v} + v' \\
w &= \bar{w} + w' \\
p_x &= \overline{p_x} + p_x'
\end{aligned}
\qquad (8.17)
$$

Substitute (8.17) into (8.16), and <u>average over time</u>

$$
\rho\left\{\frac{\overline{\partial(\bar{u}+u')}}{\partial t} + \frac{\overline{\partial(\bar{u}+u')^2}}{\partial x} + \frac{\overline{\partial(\bar{u}+u')(\bar{v}+v')}}{\partial y} + \frac{\overline{\partial(\bar{u}+u')(\bar{w}+w')}}{\partial z}\right\}
$$

$$
= \rho g_x - \frac{\overline{\partial(\overline{p_x}+p_x')}}{\partial x} + \frac{\overline{\partial \tau_{yx}}}{\partial y} + \frac{\overline{\partial \tau_{zx}}}{\partial z}
$$

Rearrange according to the Reynolds average rule

$$
\frac{\overline{\partial(\bar{u}+u')}}{\partial t} = \frac{\partial \bar{u}}{\partial t} + \cancel{\frac{\partial \overline{u'}}{\partial t}} = \frac{\partial \bar{u}}{\partial t}
$$

$$
\frac{\overline{\partial(\bar{u}+u')^2}}{\partial x} = \frac{\partial}{\partial x}\overline{(\bar{u}^2 + \cancel{2\bar{u}u'} + u'^2)} = \frac{\partial \bar{u}^2}{\partial x} + \frac{\partial \overline{u'^2}}{\partial x}
$$

$$
\frac{\overline{\partial(\bar{u}+u')(\bar{v}+v')}}{\partial y} = \frac{\partial}{\partial y}\overline{(\bar{u}\,\bar{v} + \cancel{\bar{u}v'} + \cancel{u'\bar{v}} + u'v')} = \frac{\partial \bar{u}\,\bar{v}}{\partial y} + \frac{\partial \overline{u'v'}}{\partial y}
$$

$$
\frac{\overline{\partial(\bar{u}+u')(\bar{w}+w')}}{\partial z} = \frac{\partial}{\partial z}\overline{(\bar{u}\,\bar{w} + \cancel{\bar{u}w'} + \cancel{u'\bar{w}} + u'w')} = \frac{\partial \bar{u}\,\bar{w}}{\partial z} + \frac{\partial \overline{u'w'}}{\partial z}
$$

$$
\therefore \rho\left(\frac{\partial \bar{u}}{\partial t} + \frac{\partial \bar{u}^2}{\partial x} + \frac{\partial \bar{u}\,\bar{v}}{\partial y} + \frac{\partial \bar{u}\,\bar{w}}{\partial z}\right) = \rho g_x - \frac{\partial \overline{p_x}}{\partial x} + \frac{\partial \bar{\tau}_{yx}}{\partial y} + \frac{\partial \bar{\tau}_{zx}}{\partial z}
$$

$$
- \rho\left(\frac{\partial \overline{u'^2}}{\partial x} + \frac{\partial \overline{u'v'}}{\partial y} + \frac{\partial \overline{u'w'}}{\partial z}\right)
$$

Subtract Continuity Eq. of mean motion ($\bar{u}\dfrac{\partial \bar{u}}{\partial x}+\bar{u}\dfrac{\partial \bar{v}}{\partial y}+\bar{u}\dfrac{\partial \bar{w}}{\partial z}=0$)

$$\rho\left(\frac{\partial \bar{u}}{\partial t}+\frac{\partial \bar{u}^2}{\partial x}+\frac{\partial \bar{u}\,\bar{v}}{\partial y}+\frac{\partial \bar{u}\,\bar{w}}{\partial z}\right)-\left(\bar{u}\frac{\partial \bar{u}}{\partial x}+\bar{u}\frac{\partial \bar{v}}{\partial y}+\bar{u}\frac{\partial \bar{w}}{\partial z}\right)$$

$$=\rho g_x-\frac{\partial \bar{p}_x}{\partial x}+\frac{\partial \bar{\tau}_{yx}}{\partial y}+\frac{\partial \bar{\tau}_{zx}}{\partial z}-\rho\left(\frac{\partial \overline{u'^2}}{\partial x}+\frac{\partial \overline{u'v'}}{\partial y}+\frac{\partial \overline{u'w'}}{\partial z}\right)$$

$$\therefore \rho\left(\frac{\partial \bar{u}}{\partial t}+\bar{u}\frac{\partial \bar{u}}{\partial x}+\bar{v}\frac{\partial \bar{u}}{\partial y}+\bar{w}\frac{\partial \bar{u}}{\partial z}\right)$$

$$=\rho g_x-\frac{\partial \bar{p}_x}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}+\frac{\partial \bar{\tau}_{zx}}{\partial z}-\rho\left(\frac{\partial \overline{u'^2}}{\partial x}+\frac{\partial \overline{u'v'}}{\partial y}+\frac{\partial \overline{u'w'}}{\partial z}\right) \tag{8.18}$$

•turbulence acceleration terms

•mean transport of fluctuating momentum by turbulent velocity fluctuations

y-direction:

$$\rho\left(\frac{\partial \bar{v}}{\partial t}+\bar{u}\frac{\partial \bar{v}}{\partial x}+\bar{v}\frac{\partial \bar{v}}{\partial y}+\bar{w}\frac{\partial \bar{v}}{\partial z}\right)=\rho g_y+\frac{\partial \bar{\tau}_{xy}}{\partial x}-\frac{\partial \bar{p}_y}{\partial y}+\frac{\partial \bar{\tau}_{zy}}{\partial z}$$

$$-\rho\left(\frac{\partial \overline{v'u'}}{\partial x}+\frac{\partial \overline{v'^2}}{\partial y}+\frac{\partial \overline{v'w'}}{\partial z}\right)$$

z-direction:

$$\rho\left(\frac{\partial \bar{w}}{\partial t}+\bar{u}\frac{\partial \bar{w}}{\partial x}+\bar{v}\frac{\partial \bar{w}}{\partial y}+\bar{w}\frac{\partial \bar{w}}{\partial z}\right)=\rho g_z+\frac{\partial \bar{\tau}_{xz}}{\partial x}+\frac{\partial \bar{\tau}_{yz}}{\partial y}-\frac{\partial \bar{p}_z}{\partial z}$$

$$-\rho\left(\frac{\partial \overline{w'u'}}{\partial x}+\frac{\partial \overline{w'v'}}{\partial y}+\frac{\partial \overline{w'^2}}{\partial z}\right)$$

Rearrange (8.18)

$$\rho\left(\frac{\partial \bar{u}}{\partial t}+\bar{u}\frac{\partial \bar{u}}{\partial x}+\bar{v}\frac{\partial \bar{u}}{\partial y}+\bar{w}\frac{\partial \bar{u}}{\partial z}\right)$$

$$=\rho g_x+\frac{\partial}{\partial x}\left(\underline{-\bar{p}_x-\rho\overline{u'^2}}\right)+\frac{\partial}{\partial y}(\underline{\bar{\tau}_{yx}-\rho\overline{u'v'}})+\frac{\partial}{\partial z}(\underline{\bar{\tau}_{zx}-\rho\overline{u'w'}})$$

Sum of apparent stress of <u>the mean motion</u> and <u>additional apparent stress due to turbulent fluctuations</u>

Introduce Newtonian stress relations: Eqs. 5.29 & 5.30

$$\sigma_x=-p+2\mu\frac{\partial u}{\partial x}-\frac{2}{3}\mu\nabla\cdot\vec{q}$$

$$\sigma_y=-p+2\mu\frac{\partial v}{\partial y}-\frac{2}{3}\mu\nabla\cdot\vec{q}$$

$$\sigma_z=-p+2\mu\frac{\partial w}{\partial z}-\frac{2}{3}\mu\nabla\cdot\vec{q}$$

$$\tau_{yx}=\tau_{xy}=\mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)$$

$$\tau_{yz}=\tau_{zy}=\mu\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)$$

$$\tau_{zx}=\tau_{xz}=\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)$$

Substitute velocity decomposition, Eqs (8.17) into Eqs. (5.29) & (5.30) and average over time

for incompressible fluid ($\nabla\cdot\vec{q}=0$)

1) x-direction:

$$\bar{\sigma}_x = -\bar{p}_x = -\overline{(\bar{p} + \cancel{p'})} + 2\mu \frac{\partial \overline{(\bar{u} + \cancel{u'})}}{\partial x} = -\bar{p} + 2\mu \frac{\partial \bar{u}}{\partial x}$$

$$\bar{\tau}_{yx} = \mu \left\{ \frac{\partial \overline{(\bar{v} + \cancel{v'})}}{\partial x} + \frac{\partial \overline{(\bar{u} + \cancel{u'})}}{\partial y} \right\} = \mu \left(\frac{\partial \bar{v}}{\partial x} + \frac{\partial \bar{u}}{\partial y} \right)$$

$$\bar{\tau}_{zx} = \mu \left\{ \frac{\partial \overline{(\bar{u} + \cancel{u'})}}{\partial z} + \frac{\partial \overline{(\bar{w} + \cancel{w'})}}{\partial x} \right\} = \mu \left(\frac{\partial \bar{u}}{\partial z} + \frac{\partial \bar{w}}{\partial x} \right) \qquad \text{(8.20 a)}$$

(2) y-direction:

$$-\bar{p}_y = -\bar{p} + 2\mu \frac{\partial \bar{v}}{\partial y}$$

$$\bar{\tau}_{xy} = \mu \left(\frac{\partial \bar{v}}{\partial x} + \frac{\partial \bar{u}}{\partial y} \right)$$

$$\bar{\tau}_{zy} = \mu \left(\frac{\partial \bar{w}}{\partial y} + \frac{\partial \bar{v}}{\partial z} \right) \qquad \text{(8.20 b)}$$

(3) z-direction:

$$-\bar{p}_z = -\bar{p} + 2\mu \frac{\partial \bar{w}}{\partial z}$$

$$\bar{\tau}_{xz} = \mu \left(\frac{\partial \bar{u}}{\partial z} + \frac{\partial \bar{w}}{\partial x} \right)$$

$$\bar{\tau}_{yz} = \mu \left(\frac{\partial \bar{w}}{\partial y} + \frac{\partial \bar{v}}{\partial z} \right) \qquad \text{(8.20 c)}$$

Substitute Eq. (8.20) into Eq. (8.18)

$$\rho\left(\frac{\partial\overline{u}}{\partial t}+\overline{u}\frac{\partial\overline{u}}{\partial x}+\overline{v}\frac{\partial\overline{u}}{\partial y}+\overline{w}\frac{\partial\overline{u}}{\partial z}\right)$$

$$=\rho g_x-\frac{\partial}{\partial x}\left(\overline{p}-2\mu\frac{\partial\overline{u}}{\partial x}\right)+\frac{\partial}{\partial y}\left(\mu\frac{\partial\overline{v}}{\partial x}+\mu\frac{\partial\overline{u}}{\partial y}\right)+\frac{\partial}{\partial z}\left(\mu\frac{\partial\overline{u}}{\partial z}+\mu\frac{\partial\overline{w}}{\partial x}\right)$$

$$-\rho\left(\frac{\partial\overline{u'^2}}{\partial x}+\frac{\partial\overline{u'v'}}{\partial y}+\frac{\partial\overline{u'w'}}{\partial z}\right)$$

$$=\rho g_x-\frac{\partial\overline{p}}{\partial x}+\mu\left(2\frac{\partial^2\overline{u}}{\partial x^2}+\frac{\partial^2\overline{v}}{\partial y\partial x}+\frac{\partial^2\overline{u}}{\partial y^2}+\frac{\partial^2\overline{u}}{\partial z^2}+\frac{\partial^2\overline{w}}{\partial z\partial x}\right)$$

$$-\rho\left(\frac{\partial\overline{u'^2}}{\partial x}+\frac{\partial\overline{u'v'}}{\partial y}+\frac{\partial\overline{u'w'}}{\partial z}\right)$$

$$=\rho g_x-\frac{\partial\overline{p}}{\partial x}+\underbrace{\mu\left(\frac{\partial^2\overline{u}}{\partial x^2}+\frac{\partial^2\overline{u}}{\partial y^2}+\frac{\partial^2\overline{u}}{\partial z^2}\right)}_{=\mu\nabla^2\overline{u}}+\underbrace{\mu\left(\frac{\partial^2\overline{u}}{\partial x^2}+\frac{\partial^2\overline{v}}{\partial y\partial x}+\frac{\partial^2\overline{w}}{\partial z\partial x}\right)}_{\text{(I)}}$$

$$-\rho\left(\frac{\partial\overline{u'^2}}{\partial x}+\frac{\partial\overline{u'v'}}{\partial y}+\frac{\partial\overline{u'w'}}{\partial z}\right)$$

By the way,

(I) $=\dfrac{\partial}{\partial x}\left(\dfrac{\partial\overline{u}}{\partial x}+\dfrac{\partial\overline{v}}{\partial y}+\dfrac{\partial\overline{w}}{\partial z}\right)=0$ ($\because$ Continuity Eq. for incompressible fluid)

Therefore, substituting this relation yields x-dir.:

$$\rho\left(\frac{\partial \bar{u}}{\partial t}+\bar{u}\frac{\partial \bar{u}}{\partial x}+\bar{v}\frac{\partial \bar{u}}{\partial y}+\bar{w}\frac{\partial \bar{u}}{\partial z}\right)$$
$$=\rho g_x-\frac{\partial \bar{p}}{\partial x}+\mu\nabla^2\bar{u}-\rho\left(\frac{\partial \overline{u'^2}}{\partial x}+\frac{\partial \overline{u'v'}}{\partial y}+\frac{\partial \overline{u'w'}}{\partial z}\right) \tag{8.22 a}$$

y-dir.:

$$\rho\left(\frac{\partial \bar{v}}{\partial t}+\bar{u}\frac{\partial \bar{v}}{\partial x}+\bar{v}\frac{\partial \bar{v}}{\partial y}+\bar{w}\frac{\partial \bar{v}}{\partial z}\right)$$
$$=\rho g_y-\frac{\partial \bar{p}}{\partial y}+\mu\nabla^2\bar{v}-\rho\left(\frac{\partial \overline{v'u'}}{\partial x}+\frac{\partial \overline{v'^2}}{\partial y}+\frac{\partial \overline{v'w'}}{\partial z}\right) \tag{8.22 b}$$

z-dir.:

$$\rho\left(\frac{\partial \bar{w}}{\partial t}+\bar{u}\frac{\partial \bar{w}}{\partial x}+\bar{v}\frac{\partial \bar{w}}{\partial y}+\bar{w}\frac{\partial \bar{w}}{\partial z}\right)$$
$$=\rho g_z-\frac{\partial \bar{p}}{\partial z}+\mu\nabla^2\bar{w}-\rho\left(\frac{\partial \overline{w'u'}}{\partial x}+\frac{\partial \overline{w'v'}}{\partial y}+\frac{\partial \overline{w'^2}}{\partial z}\right) \tag{8.22 c}$$

→ Reynolds Equations (temporal mean eq. of motion)

→ Navier-Stokes form for incompressible fluid

[Re] No. of Equations = 4

No. of Unknowns: 4 + 9 (turbulence fluctuating terms)

→ 9 products of $\overline{u'_i u'_j}$

→ one-point double correlation of velocity fluctuation

[Re]

1) Reynolds Equation of motion → solve for mean motion

$$\frac{\partial \bar{u}_i}{\partial t} + \bar{u}_j \frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial}{\partial x_j}\left(\overline{u'_i u'_j}\right) = \bar{X}_i - \frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_i} + \frac{\mu}{\rho}\frac{\partial^2 \bar{u}_i}{\partial x_j \partial x_j}$$

time rate of change of momentum

rate of convection of the momentum

rate of diffusion of momentum by turbulence

body force

force due to mean pressure

rate of molecular diffusion of momentum by viscosity

2) Navier-Stokes Eq. → apply to instantaneous motion

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = X_i - \frac{1}{\rho}\frac{\partial p}{\partial x_i} + \frac{\mu}{\rho}\frac{\partial^2 u_i}{\partial x_j \partial x_j}$$

8.5.2 Closure Model

Assumptions are needed to close the gap between No. of equations and No. unknowns.

→ Turbulence modeling: Ch. 10

■ **Boussinesq**'s eddy viscosity model

$$-\overline{u'^2} = \varepsilon_x \frac{\partial \bar{u}}{\partial x}$$

$$-\overline{u'v'} = \varepsilon_y \frac{\partial \bar{u}}{\partial y} \qquad \boxed{-\overline{u'v'} \propto \frac{\partial \bar{u}}{\partial y}} \qquad \text{(A)}$$

$$-\overline{u'w'} = \varepsilon_z \frac{\partial \bar{u}}{\partial z}$$

Reynolds Equation in x-dir.:

$$\rho\left(\frac{\partial \bar{u}}{\partial t}+\bar{u}\frac{\partial \bar{u}}{\partial x}+\bar{v}\frac{\partial \bar{u}}{\partial y}+\bar{w}\frac{\partial \bar{u}}{\partial z}\right)$$

$$=\rho g_x-\frac{\partial \bar{p}}{\partial x}+\mu\left(\frac{\partial^2 \bar{u}}{\partial x^2}+\frac{\partial^2 \bar{u}}{\partial y^2}+\frac{\partial^2 \bar{u}}{\partial z^2}\right)-\rho\left(\frac{\partial \overline{u'^2}}{\partial x}+\frac{\partial \overline{u'v'}}{\partial y}+\frac{\partial \overline{u'w'}}{\partial z}\right)$$

$$=\rho g_x-\frac{\partial \bar{p}}{\partial x}+\rho\underline{\left[\frac{\partial}{\partial x}\left(\frac{\mu}{\rho}\frac{\partial \bar{u}}{\partial x}-\overline{u'^2}\right)+\frac{\partial}{\partial y}\left(\frac{\mu}{\rho}\frac{\partial \bar{u}}{\partial y}-\overline{u'v'}\right)+\frac{\partial}{\partial z}\left(\frac{\mu}{\rho}\frac{\partial \bar{u}}{\partial z}-\overline{u'w'}\right)\right]}$$

(B)

Substitute (A) and $\dfrac{\mu}{\rho}=\nu$ into (B)

$$\rho\left(\frac{\partial \bar{u}}{\partial t}+\bar{u}\frac{\partial \bar{u}}{\partial x}+\bar{v}\frac{\partial \bar{u}}{\partial y}+\bar{w}\frac{\partial \bar{u}}{\partial z}\right)$$

$$\boxed{\varepsilon_x=\varepsilon_y=\varepsilon_z=\varepsilon}$$

$$=\rho g_x-\frac{\partial \bar{p}}{\partial x}+\rho\left[\frac{\partial}{\partial x}\left\{(\nu+\varepsilon_x)\frac{\partial \bar{u}}{\partial x}\right\}+\frac{\partial}{\partial y}\left\{(\nu+\varepsilon_y)\frac{\partial \bar{u}}{\partial y}\right\}+\frac{\partial}{\partial z}\left\{(\nu+\varepsilon_z)\frac{\partial \bar{u}}{\partial z}\right\}\right]$$

$$=\rho g_x-\frac{\partial \bar{p}}{\partial x}+\rho(\nu+\varepsilon)\left(\frac{\partial^2 \bar{u}}{\partial x^2}+\frac{\partial^2 \bar{u}}{\partial y^2}+\frac{\partial^2 \bar{u}}{\partial z^2}\right)$$

$$=\rho g_x-\frac{\partial \bar{p}}{\partial x}+\rho(\nu+\varepsilon)\nabla^2\bar{u}$$

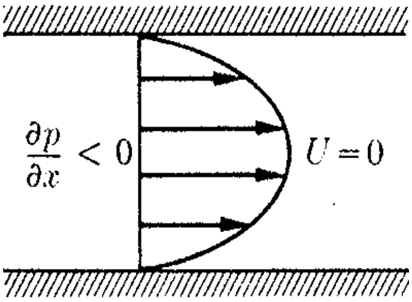
$$=\rho g_x-\frac{\partial \bar{p}}{\partial x}+(\mu+\eta)\nabla^2\bar{u}$$

where ν = kinematic molecular viscosity; ε = kinematic eddy viscosity; μ = dynamic molecular viscosity; η = dynamic eddy viscosity

8.5.3 Examples

(1) Turbulent flow between parallel plates

Apply Reynolds equations to steady uniform motion in the x-direction between parallel horizon walls



$$\frac{\partial (\;)}{\partial t} = 0 \quad \leftarrow \text{steady motion}$$

$$\frac{\partial (vel)}{\partial x} = 0 \leftarrow \text{uniform motion} \quad \left(\begin{array}{l} \frac{\partial u}{\partial x} = 0 \\ \frac{\partial u'}{\partial x} = 0 \end{array} \right.$$

$$\frac{\partial (\;)}{\partial z} = 0, \; w = 0 \leftarrow \text{2-D motion}$$

$$\overline{v} = \frac{1}{T}\int_0^T v\,dt = 0 \leftarrow \text{unidirectional mean flow}$$

$$v' \neq 0$$

Incorporate these assumptions into Eqs. (8.22)

$$x \;:\; \rho\left(\cancel{\frac{\partial \overline{u}}{\partial t}} + \overline{u}\cancel{\frac{\partial \overline{u}}{\partial x}} + \cancel{\overline{v}}\frac{\partial \overline{u}}{\partial y} + \cancel{\overline{w}}\frac{\partial \overline{u}}{\partial z} \right)$$

$$= \rho g_x - \frac{\partial \bar{p}}{\partial x} + \mu \nabla^2 \bar{u} - \rho \left(\cancel{\frac{\partial \overline{u'^2}}{\partial x}} + \frac{\partial \overline{u'v'}}{\partial y} + \cancel{\frac{\partial \overline{u'w'}}{\partial z}} \right)$$

$$\therefore 0 = \rho g_x - \frac{\partial \bar{p}}{\partial x} + \mu \frac{\partial^2 \bar{u}}{\partial y^2} - \rho \frac{\partial \overline{u'v'}}{\partial y}$$

$$= -\rho g \frac{\partial h}{\partial x} - \frac{\partial \bar{p}}{\partial x} + \mu \frac{\partial^2 \bar{u}}{\partial y^2} - \rho \frac{\partial \overline{u'v'}}{\partial y} \qquad \text{(A)}$$

$$y \ : \ \rho \left(\cancel{\frac{\partial \bar{v}}{\partial t}} + \bar{u} \cancel{\frac{\partial \bar{v}}{\partial x}} + \cancel{\bar{v}} \frac{\partial \bar{v}}{\partial y} + \cancel{\bar{w}} \frac{\partial \bar{v}}{\partial z} \right)$$

$$= \rho g_y - \frac{\partial \bar{p}}{\partial y} + \mu \nabla^2 \cancel{\bar{v}} - \rho \left(\cancel{\frac{\partial \overline{v'u'}}{\partial x}} + \frac{\partial \overline{v'^2}}{\partial y} + \cancel{\frac{\partial \overline{v'w'}}{\partial z}} \right)$$

$$\boxed{g_y = -g \frac{\partial h}{\partial y}}$$

$$\therefore 0 = \rho g_y - \frac{\partial \bar{p}}{\partial y} - \rho \frac{\partial \overline{v'^2}}{\partial y}$$

$$\frac{\partial}{\partial y}(\bar{p} + \gamma h) + \rho \frac{\partial \overline{v'^2}}{\partial y} = 0 \qquad (8.25)$$

Integrate (8.25)

$$\bar{p} + \gamma h + \rho \overline{v'^2} = const. \qquad (8.26)$$

→ In turbulent flow, static pressure distribution in planes perpendicular to flow direction <u>differs from the hydrostatic pressure by $\rho \overline{v'^2}$</u>

Rearrange (A)

$$\frac{\partial}{\partial x}(\overline{p}+\gamma h) = -\rho\frac{\partial \overline{u'v'}}{\partial y} + \mu\frac{\partial^2 \overline{u}}{\partial y^2} = \frac{\partial}{\partial y}\left(-\rho\overline{u'v'} + \cancel{\mu\frac{\partial \overline{u}}{\partial y}}\right) \tag{D}$$

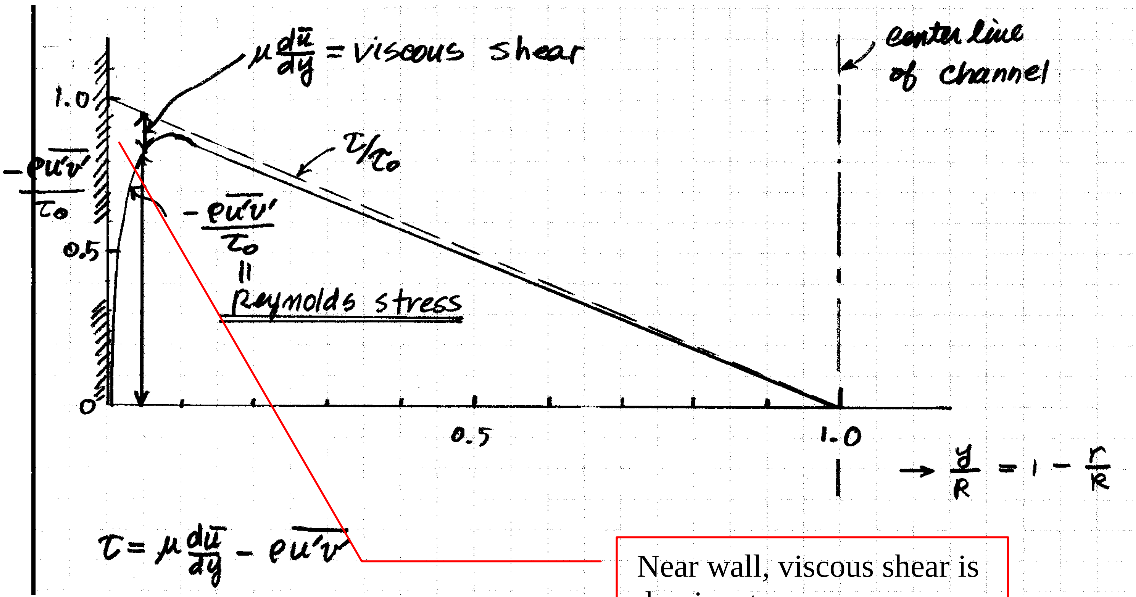
neglect since turbulence contribution to shear is dominant

Integrate (D) w.r.t. y (measured from centerline between the plate)

$$\frac{d}{dx}(\overline{p}+\gamma h)y = -\rho\overline{u'v'} = \tau$$

$$\tau_{tur} = -\rho\overline{u'v'} \;\propto\; y$$

→ τ distribution is linear with distance from the wall for both laminar and turbulent flows.



Near wall, viscous shear is dominant.

(2) Equations for a turbulent boundary layer

Apply Prandtl's 2-D boundary-layer equations

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{\mu}{\rho}\frac{\partial^2 u}{\partial y^2} \tag{8.7a}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

$$\rightarrow\ u\frac{\partial u}{\partial x}+u\frac{\partial v}{\partial y}=0 \tag{8.7b}$$

Add Continuity Eq. and Eq. (8.7a)

$$\frac{\partial u}{\partial t}+\underbrace{2u\frac{\partial u}{\partial x}}_{\downarrow\ \frac{\partial u^2}{\partial x}}+\underbrace{\left(v\frac{\partial u}{\partial y}+u\frac{\partial v}{\partial y}\right)}_{\downarrow\ \frac{\partial uv}{\partial y}}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{\mu}{\rho}\frac{\partial^2 u}{\partial y^2} \tag{A}$$

Substitute <u>velocity decomposition</u> into (A) and <u>average over time</u>

$$\overline{\frac{\partial(\bar{u}+u')}{\partial t}} = \frac{\partial\bar{u}}{dt}$$

$$\overline{\frac{\partial(\bar{u}+u')^2}{\partial x}} = \frac{\partial\bar{u}^2}{dx}+\frac{\partial\overline{u'^2}}{\partial x}$$

$$\overline{\frac{\partial(\bar{u}+u')(\bar{v}+v')}{\partial y}} = \frac{\partial\bar{u}\,\bar{v}}{\partial y}+\frac{\partial\overline{u'v'}}{\partial y}$$

$$-\frac{1}{\rho}\frac{\partial}{\partial x}\overline{(\bar{p}+p')}=-\frac{1}{\rho}\frac{\partial\bar{p}}{\partial x}$$

$$\frac{\mu}{\rho}\frac{\partial^2}{\partial y^2}(\overline{u}+u')=\frac{\mu}{\rho}\frac{\partial^2\overline{u}}{\partial y^2}$$

Thus, (A) becomes

$$\therefore \frac{\partial\overline{u}}{\partial t}+\frac{\partial\overline{u}^2}{\partial x}+\frac{\partial\overline{u}\,\overline{v}}{\partial y}=-\frac{1}{\rho}\frac{\partial\overline{p}}{\partial x}+\frac{\mu}{\rho}\frac{\partial^2\overline{u}}{\partial y^2}-\frac{\partial\overline{u'^2}}{\partial x}-\frac{\partial\overline{u'v'}}{\partial y} \qquad \text{(B)}$$

Subtract Continuity eq. from (B)

$$\frac{\partial\overline{u}}{\partial t}+\overline{u}\frac{\partial\overline{u}}{\partial x}+\overline{v}\frac{\partial\overline{u}}{\partial y}=-\frac{1}{\rho}\frac{\partial\overline{p}}{\partial x}+\frac{\mu}{\rho}\frac{\partial^2\overline{u}}{\partial y^2}-\frac{\partial\overline{u'^2}}{\partial x}-\frac{\partial\overline{u'v'}}{\partial y}$$

$$\boxed{\rho\left(\frac{\partial\overline{u}}{\partial t}+\overline{u}\frac{\partial\overline{u}}{\partial x}+\overline{v}\frac{\partial\overline{u}}{\partial y}\right)=-\frac{\partial\overline{p}}{\partial x}-\rho\frac{\partial\overline{u'^2}}{\partial x}+\mu\frac{\partial^2\overline{u}}{\partial y^2}-\rho\frac{\partial\overline{u'v'}}{\partial y}}$$

→ x-eq.

Adopt similar equation as Eq. (8.25) for y-eq.

$$0=-\frac{\partial}{\partial y}(\overline{p}+\rho\overline{v'^2})$$

Continuity eq.:

$$\frac{\partial\overline{u}}{\partial x}+\frac{\partial\overline{v}}{\partial y}=0$$

8.6 Mixing Length and Similarity Hypotheses in Shear flow

In order to close the turbulent problem, theoretical assumptions are needed for the calculation of turbulent flows (Schlichting, 1979).

→We need to have empirical hypotheses to establish a relationship between the Reynolds stresses produced by the mixing motion and the mean values of the velocity components

8.6.1 Boussinesq's eddy viscosity model

For laminar flow;

$$\tau_l = \mu \frac{d\overline{u}}{dy}$$

For turbulent flow, use analogy with laminar flow;

$$\boxed{\tau_t = -\rho \overline{u'v'} = \eta \frac{d\overline{u}}{dy}} \tag{8.30}$$

where η = apparent (virtual) eddy viscosity

→ turbulent mixing coefficient

~ not a property of the fluid

~ depends on $\overline{u}$; $\eta \propto \overline{u}$

8.6.2 Prandtl's mixing length theory

~ express the momentum shear stresses in terms of mean velocity

■ Assumptions

1) Average distance traversed by a fluctuating fluid element before it acquired the velocity of new region is related to an average (absolute) magnitude of the fluctuating velocity.

$$l \propto |v'|$$

$$\overline{|v'|} \propto l\left|\frac{d\bar{u}}{dy}\right| \tag{8.31a}$$

where $l = l(y)$ = mixing length

2) Two orthogonal fluctuating velocities are proportional to each other.

$$\overline{|u'|} \propto \overline{|v'|} \propto l\left|\frac{d\bar{u}}{dy}\right| \tag{8.31b}$$

Substituting (8.31) into (8.13) leads to

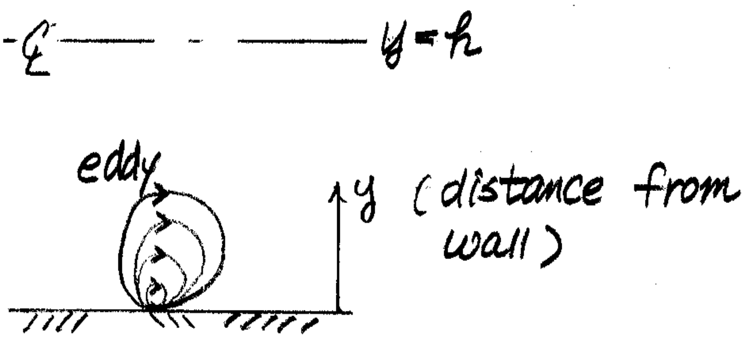
$$\boxed{\tau = -\rho\overline{u'v'} = \rho l^2\left|\frac{d\bar{u}}{dy}\right|\frac{d\bar{u}}{dy}} \tag{8.32}$$

Therefore, combining (8.30) and (8.32), dynamic eddy viscosity can be expressed as

$$\boxed{\eta = \rho l^2\left|\frac{d\bar{u}}{dy}\right|} \tag{8.33}$$

→ Prandtl's formulation has a restricted usefulness because it is <u>not possible to predict mixing length function</u> for flows in general.

■ Near wall



$$l = \kappa y \qquad \text{(A)}$$

Where κ = von Karman constant

Substitute (A) into (8. 32)

$$\boxed{\tau = \rho \kappa^2 y^2 \left| \frac{d\bar{u}}{dy} \right| \frac{d\bar{u}}{dy}} \qquad (8.34)$$

[Re] Prandtl's mixing-length theory (Schlichting, 1979)

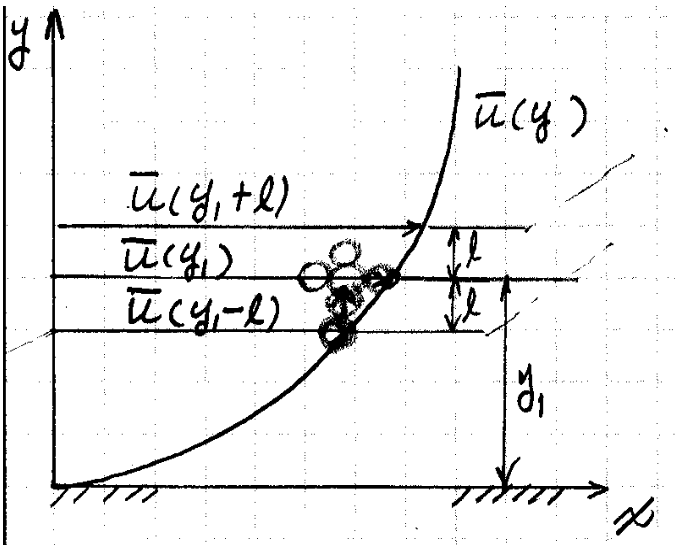
Consider simplest case of parallel flow in which the velocity varies only from streamline to streamline.

$$\rightarrow \begin{cases} \bar{u} = \bar{u}(y) \\ \bar{v} = \bar{w} = 0 \end{cases}$$

Shearing stress is given as

$$\tau'_{xy} = \tau_t = -\rho \overline{u'v'} = \eta \frac{d\bar{u}}{dy}$$

○ Simplified mechanism of the motion



1) Fluid particles move in lump both in longitudinal and in the transverse direction.

2) If a lump of fluid is displaced from a layer at $(y_1 - l)$ to a new layer y_1 , then, the difference in velocities is expressed as (use Taylor series and neglect high-order terms)

$$\Delta u_1 = \bar{u}(y_1) - \bar{u}(y_1 - l) \approx l \left(\frac{d\bar{u}}{dy} \right)_{y=y_1} \qquad ; v' > 0$$

where l = Prandtl's mixing length (mixture length)

For a lump of fluid which arrives at y_1 from the laminar at $y_1 + l$

$$\Delta u_2 = \bar{u}(y_1 + l) - \bar{u}(y_1) \approx l\left(\frac{d\bar{u}}{dy}\right)_{y=y_1} \qquad ; v' < 0$$

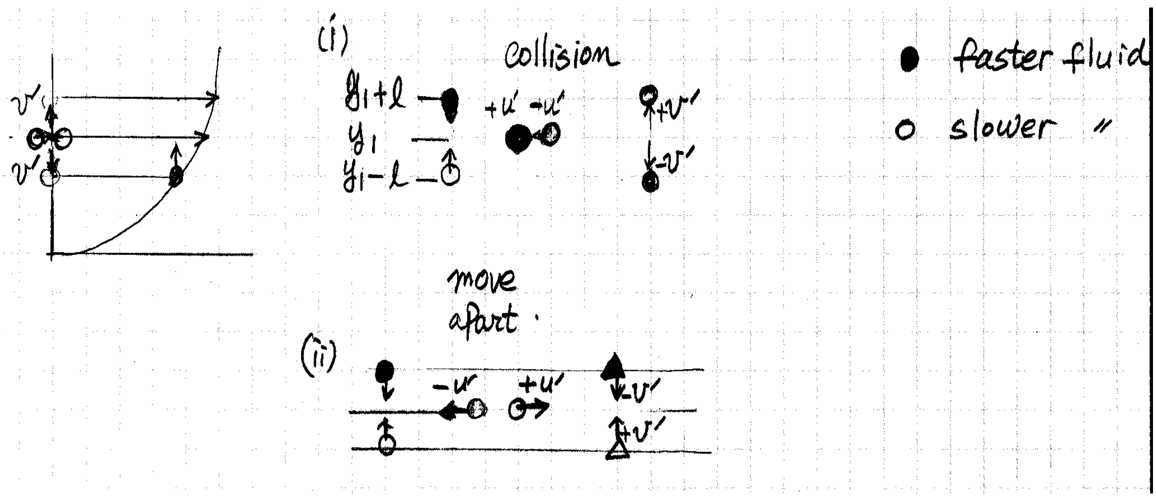
3) These velocity differences $(\Delta u_1, \Delta u_2)$ caused by the transverse motion can be regarded as the turbulent velocity fluctuation at y_1

$$\overline{|u'|} = \frac{1}{2}(|\Delta u_1| + |\Delta u_2|) = l\left|\left(\frac{d\bar{u}}{dy}\right)_{y_1}\right| \tag{2}$$

○ Physical interpretation of the mixing length l.

= <u>distance in the transverse direction</u> which must be covered by an agglomeration of fluid particles travelling with its mean velocity in order to make the difference between it's velocity and the velocity in the new laminar equal to the <u>mean transverse fluctuation in turbulent flow</u>.

4) Transverse velocity fluctuation v' originates in two ways.



5) Transverse component v' is same order of magnitude as u'.

$$\overline{|v'|} = const \cdot \overline{|u'|} = const \cdot l \frac{d\bar{u}}{dy} \tag{3}$$

6) Fluid lumps which arrive at layer y_1 with a positive value of v' (upwards from layer) give rise mostly to a negative u'.

$$\therefore\ u'v' < 0$$

$$\overline{u'v'} = -c\overline{|u'|}\,\overline{|v'|} \tag{4}$$

where 0 < c < 1

Combine Eqs. 2-4

$$\overline{u'v'} = -constant \cdot -l^2 \left|\frac{d\bar{u}}{dy}\right| \frac{d\bar{u}}{dy}$$

Include constant into *l* (mixing length)

$$\overline{u'v'} = -l^2 \left|\frac{d\bar{u}}{dy}\right| \frac{d\bar{u}}{dy} \tag{5}$$

Therefore, shear stress is given as

$$\tau = -\rho\overline{u'v'} = \rho l^2 \left|\frac{d\bar{u}}{dy}\right| \frac{d\bar{u}}{dy} \tag{6}$$

→ Prandtl's mixing-length hypothesis

8.6.3 Von Karman's similarity hypothesis

○Assumptions

~Turbulent fluctuations are similar at all point of the field of flow (similarity rule).

→ Turbulent fluctuations differ from point to point only by time and length scale factors.

Velocity is characteristics of the turbulent fluctuating motion.

For 2-D mean flow in the x - direction, a necessary condition to secure compatibility between the similarity hypothesis and the vorticity transport equation is

$$l \sim \frac{d\bar{u} \,/\, dy}{d^2\bar{u} \,/\, dy^2}$$

$$l = \kappa \left| \frac{d\bar{u} \,/\, dy}{d^2\bar{u} \,/\, dy^2} \right| \qquad \text{(A)}$$

where κ = empirical dimensionless constant

Substituting (A) into (8.32) gives

$$\boxed{\tau = \rho\kappa^2 \frac{(d\bar{u} \,/\, dy)^4}{(d^2\bar{u} \,/\, dy^2)^2}} \qquad (8.35)$$

→ Von Karman's similarity rule

[Re] Prandtl's **velocity-distribution law**

For wall turbulence (immediate neighborhood of the wall)

$$\tau = \rho \kappa^2 y^2 \left(\frac{d\bar{u}}{dy} \right)^2 \tag{1}$$

$$\frac{d\bar{u}}{dy} = \frac{1}{\kappa y} \sqrt{\frac{\tau}{\rho}} = \frac{u_*}{\kappa y} \tag{2}$$

where $u_* = \sqrt{\tau / \rho}$ = shear velocity; κ = von Karman const ≈ 0.4

Integrate (2) w.r.t. *y*

$$\bar{u} = \frac{u_*}{\kappa} \ln y + C \tag{3}$$

→ **<u>Prandtl's velocity distribution law</u>**

Apply Prandtl's velocity distribution law to whole region

$$\bar{u} = \bar{u}_{\max} \quad at \quad y = h$$

$$\bar{u}_{\max} = \frac{u_*}{\kappa} \ln h + C \tag{4}$$

Subtract (3) from (4) to eliminate constant of integration

$$\frac{\bar{u}_{\max} - \bar{u}}{u_*} = \frac{1}{\kappa} \ln \frac{h}{y} \tag{5}$$

<u>→ Prandtl's universal velocity-defect law</u>

Homework Assignment # 5

Due: 1 week from today

8-1. The velocity data listed in Table were obtained at a point in a turbulent flow of sea water.

1) Compute the energy of turbulence per unit volume.

2) Determine the mean velocity in the x-direction $\bar{u}$ and verify that $\overline{u'}=0$.

3) Determine the magnitude of the three independent turbulent shear stresses in Eq. (8-21).

time, sec	u cm/s	u' cm/s	v' cm/s	w' cm/s
0.0	89.92	-4.57	1.52	0.91
0.1	95.10	0.61	0.00	-0.30
0.2	103.02	8.53	-3.66	-2.13
0.3	99.67	5.18	-1.22	-0.61
0.4	92.05	-2.44	-0.61	0.30
0.5	87.78	-6.71	2.44	0.91
0.6	92.96	-1.52	0.91	-0.61
0.7	90.83	-3.66	1.83	0.61
0.8	96.01	1.52	0.61	0.91
0.9	93.57	-0.91	0.30	-0.61
1.0	98.45	3.96	-1.52	-1.22