



10 Fourier Series

10.1 Periodic Functions

- **Definition :** A function $f(x)$ is called periodic, if it is defined for all real x and if there is some positive number p such that

$$f(x + p) = f(x) \quad \text{for all } x \quad (1)$$

- The number p is called a period of $f(x)$

Example: $\sin(x + 2\pi) = \sin x$, $\cos(x + 2\pi) = \cos x$

(note) $f = c = \text{constant}$ for every x

- Examples of non-periodic functions : x , x^2 , x^3 , e^x , $\cosh x$, ...

- From (1)

$$f(x + 2p) = f[(x + p) + p] = f(x + p) = f(x)$$

- For any integer n

$$f(x + np) = f(x) \quad \text{for all } x$$

- Furthermore, if $f(x)$ and $g(x)$ have period p , then the function

$$\begin{aligned} h(x) &= af(x) + bg(x) \\ h(x + p) &= af(x + p) + bg(x + p) = af(x) + bg(x) = h(x) \\ \therefore h(x + p) &= h(x) \end{aligned}$$

a, b : constant, period p

- Fundamental period: a smallest period $p (> 0)$ of $f(x)$

$$\cos x \quad \text{and} \quad \sin x \rightarrow 2\pi$$

$$\cos 2x \quad \text{and} \quad \sin 2x \rightarrow \pi$$

$$f = \text{constant} \rightarrow \text{no fundamental period}$$

10.2 Trigonometric Series

- Trigonometric functions with a period of 2π

$$1, \quad \cos x, \quad \sin x, \quad \cos 2x, \quad \sin 2x, \quad \dots \quad \cos nx, \quad \sin nx, \quad \dots$$

- Trigonometric Series

$$a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \dots$$

where $a_0, a_1, b_1, a_2, a_3, b_2, b_3 \dots$ are real constants.

- Trigonometric system

$$a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad (2)$$

If the series (2) converges, its sum will be a function of period 2π : Fourier series of f

10.3 Fourier Series

Euler Formulas for the Fourier Coefficients

- $f(x)$ is a periodic function of period 2π and is integrable over a period. $f(x)$ can be represented by a trigonometric series,

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx). \quad (3)$$

→ Determination of the constant term a_0

$$\begin{aligned} \int_{-\pi}^{\pi} f(x) dx &= \int_{-\pi}^{\pi} [a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)] dx \\ \int_{-\pi}^{\pi} f(x) dx &= \int_{-\pi}^{\pi} a_0 dx + \sum_{n=1}^{\infty} (\int_{-\pi}^{\pi} a_n \cos nx dx + \int_{-\pi}^{\pi} b_n \sin nx dx) \\ \because a_n \int_{-\pi}^{\pi} \cos nx dx &= b_n \int_{-\pi}^{\pi} \sin nx dx = 0 \\ a_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx \end{aligned}$$

→ Determination of the coefficients a_n of the cosine terms by multiplying (3) by $\cos mx$ where m is any fixed positive integer,

$$\int_{-\pi}^{\pi} f(x) \cos mx dx = \int_{-\pi}^{\pi} [a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)] \cos mx dx$$

- R.H.S of the above equation

$$a_0 \int_{-\pi}^{\pi} \cos mx dx + \sum_{n=1}^{\infty} [a_n \int_{-\pi}^{\pi} \cos nx \cdot \cos mx dx + b_n \int_{-\pi}^{\pi} \sin nx \cdot \cos mx dx]$$

- 1st integral:

$$a_0 \int_{-\pi}^{\pi} \cos mx dx = 0$$

- 2nd and 3rd integrals:

$$\begin{aligned} \int_{-\pi}^{\pi} \cos nx \cdot \cos mx dx &= \frac{1}{2} \int_{-\pi}^{\pi} \cos(n+m)x dx + \frac{1}{2} \int_{-\pi}^{\pi} \cos(n-m)x dx \\ \int_{-\pi}^{\pi} \sin nx \cdot \cos mx dx &= \frac{1}{2} \int_{-\pi}^{\pi} \sin(n+m)x dx + \frac{1}{2} \int_{-\pi}^{\pi} \sin(n-m)x dx \end{aligned}$$

- When $n \neq m$ and the integration is performed over the common period of 2π ,

$$\int_{-\pi}^{\pi} \cos(n+m)x dx = \int_{-\pi}^{\pi} \cos(n-m)x dx = 0$$

$$\int_{-\pi}^{\pi} \sin(n+m)x dx = \int_{-\pi}^{\pi} \sin(n-m)x dx = 0$$

- When $n = m$, all the integrals are zero except

$$\int_{-\pi}^{\pi} \cos(n-m)x dx = \int_{-\pi}^{\pi} dx = 2\pi$$

- From (1)

$$\therefore a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos mx dx \quad m = 1, 2, 3, \dots$$

→ Determination of the coefficients b_n of the sine terms by multiplying (3) by $\sin mx$ where m is any fixed positive integer,

$$\int_{-\pi}^{\pi} f(x) \sin mx dx = \int_{-\pi}^{\pi} [a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)] \sin mx dx$$

- R.H.S of the above equation

$$a_0 \int_{-\pi}^{\pi} \sin mx dx + \sum_{n=1}^{\infty} [a_n \int_{-\pi}^{\pi} \cos nx \cdot \sin mx dx + b_n \int_{-\pi}^{\pi} \sin nx \cdot \sin mx dx]$$

- 1st integral:

$$a_0 \int_{-\pi}^{\pi} \sin mx dx = 0$$

- 2nd and 3rd integrals:

$$\begin{aligned} \int_{-\pi}^{\pi} \cos nx \cdot \sin mx dx &= \frac{1}{2} \int_{-\pi}^{\pi} \sin(m+n)x dx - \frac{1}{2} \int_{-\pi}^{\pi} \sin(n-m)x dx \\ \int_{-\pi}^{\pi} \sin nx \cdot \sin mx dx &= \frac{1}{2} \int_{-\pi}^{\pi} \cos(m-n)x dx - \frac{1}{2} \int_{-\pi}^{\pi} \cos(n+m)x dx \end{aligned}$$

- When $n \neq m$ and the integration is performed over the common period of 2π , the above integrals are zero.

- When $n = m$, the above integrals are zero with same reason except

$$\frac{1}{2} \int_{-\pi}^{\pi} \cos(m-n)x dx = \frac{1}{2} \int_{-\pi}^{\pi} dx = \pi$$

$$\therefore b_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin mx dx \quad m = 1, 2, 3, \dots$$

Summary of These Calculations: Fourier Coefficients, Fourier Series, Euler formula

$$\begin{aligned}
 (a) \quad a_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx \\
 (b) \quad a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cdot \cos nx dx \quad n = 1, 2, \dots \\
 (c) \quad b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cdot \sin nx dx \quad n = 1, 2, \dots
 \end{aligned} \tag{4}$$

Example 1. Rectangular wave

$$\begin{aligned}
 f(x) &= -k && \text{if } -\pi < x < 0 \\
 f(x) &= k && \text{if } 0 < x < \pi \\
 f(x + 2\pi) &= f(x)
 \end{aligned}$$

Solution.

$$\therefore a_0 = 0$$

$$\begin{aligned}
 a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cdot \cos nx dx = \frac{1}{\pi} \left[\int_{-\pi}^0 (-k) \cos nx dx + \int_0^{\pi} k \cos nx dx \right] \\
 &= \frac{1}{\pi} \left[-k \frac{\sin nx}{n} \Big|_{-\pi}^0 + k \frac{\sin nx}{n} \Big|_0^{\pi} \right] = 0
 \end{aligned}$$

$$\begin{aligned}
 b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cdot \sin nx dx = \frac{1}{\pi} \left[\int_{-\pi}^0 (-k) \sin nx dx + \int_0^{\pi} k \sin nx dx \right] \\
 &= \frac{1}{\pi} \left[k \frac{\cos nx}{n} \Big|_{-\pi}^0 - k \frac{\cos nx}{n} \Big|_0^{\pi} \right] = \frac{k}{n\pi} [\cos 0 - \cos(-n\pi) - \cos n\pi + \cos 0] \\
 &= \frac{2k}{n\pi} (1 - \cos n\pi)
 \end{aligned}$$

- Now, $\cos \pi = -1$, $\cos 2\pi = 1$, $\cos 3\pi = -1$, etc :

$$\cos n\pi = \begin{cases} -1 & \text{for odd } n \\ 1 & \text{for even } n \end{cases}$$

$$1 - \cos n\pi = \begin{cases} 2 & \text{for odd } n \\ 0 & \text{for even } n \end{cases}$$

- Hence the Fourier coefficients b_n of our function are

$$\begin{aligned}
 b_1 &= \frac{4k}{\pi}, \quad b_2 = 0, \quad b_3 = \frac{4k}{3\pi}, \quad b_5 = \frac{4k}{5\pi}, \dots \\
 f(x) &= \frac{4k}{\pi} \left(\sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \dots \right)
 \end{aligned} \tag{5}$$

- The partial sums are

$$S_1 = \frac{4k}{\pi} \sin x, \quad S_2 = \frac{4k}{\pi} (\sin x + \frac{1}{3} \sin 3x), \quad \text{etc.}$$

- At $x = \pi/2$, $f(\pi/2) = k$

$$\begin{aligned} f\left(\frac{\pi}{2}\right) &= k = \frac{4k}{\pi} \left(1 - \frac{1}{3} + \frac{1}{5} - + \dots\right) \\ \therefore 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + - \dots &= \frac{\pi}{4} \end{aligned}$$

Orthogonality of the Trigonometric System

- The trigonometric system, $1, \cos x, \sin x, \cos 2x, \sin 2x, \dots, \cos nx, \sin nx, \dots$ is orthogonal on the interval $-\pi \leq x \leq \pi$.

$$\begin{aligned} \int_{-\pi}^{\pi} \cos mx \cdot \cos nxdx &= 0 \quad (m \neq n) \\ \int_{-\pi}^{\pi} \sin mx \cdot \sin nxdx &= 0 \quad (m \neq n) \end{aligned}$$

and for any integers m and n (including $m = n$)

$$\begin{aligned} \int_{-\pi}^{\pi} \cos mx \cdot \sin nxdx &= 0 \\ \because \int_{-\pi}^{\pi} \cos mx \cdot \sin nxdx &= \frac{1}{2} \int_{-\pi}^{\pi} \sin(n+m)x dx + \frac{1}{2} \int_{-\pi}^{\pi} \sin(n-m)x dx = 0 \end{aligned}$$

for any integer m and n .

10.4 Convergence and Sum of Fourier Series

Theorem 1 [Representation by a Fourier serie]

If a periodic function $f(x)$ with period 2π is piecewise continuous in the interval $-\pi \leq x \leq \pi$ and has a left-hand derivative and right-hand derivative at each point of that interval, then the Fourier series (3) of $f(x)$ [with coefficients (4)] is convergent. Its sum is $f(x)$, except at a point x_0 at which $f(x)$ is discontinuous and the sum of the series is the average of the left-hand and right-hand limit of $f(x)$ having continuous first and second derivatives.

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx = \frac{f(x) \cdot \sin nx}{n\pi} \Big|_{-\pi}^{\pi} - \frac{1}{n\pi} \int_{-\pi}^{\pi} f'(x) \cdot \sin nx dx \\ &= \frac{f'(x) \cdot \cos nx}{n^2\pi} \Big|_{-\pi}^{\pi} - \frac{1}{n^2\pi} \int_{-\pi}^{\pi} f''(x) \cdot \cos nx dx \end{aligned}$$

- In the above equation,

$$\frac{f'(x) \cdot \cos nx}{n^2\pi} \Big|_{-\pi}^{\pi} = [f'(\pi) \frac{\cos n\pi}{n^2\pi} - f'(-\pi) \frac{\cos(-n\pi)}{n^2\pi}] = 0$$

$$\cos(-n\pi) = \cos n\pi, \quad f'(\pi) = f(-\pi)$$

\therefore continuity and periodicity of the function $f(x)$.

- Since f'' is continuous in the interval of integration, we have $|f''(x)| < M$ for an appropriate constant M . Furthermore, $|\cos nx| \leq 1$. It follows that

$$|a_n| = \frac{1}{n^2\pi} \left| \int_{-\pi}^{\pi} f''(x) \cos nx dx \right| < \frac{1}{n^2\pi} \int_{-\pi}^{\pi} M dx = \frac{2M}{n^2}$$

- Similarly, $|b_n| < 2M/n^2$ for all n ,

$$\therefore |f(x)| < |a_0| + 2M(1 + 1 + \frac{1}{2^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{3^2} + \dots)$$

- It converges!

Example 2. Convergence at a jump as indicated in Theorem 1

The square wave in Example 1 has a jump at $x = 0$. Its left-hand limit there is $-k$ and its right-hand limit is k . Hence the average of these limit is 0. The Fourier series (5) of the square wave does indeed converge to the value when $x = 0$ because then all its terms are 0. Similarly for other jumps. This is in agreement with Theorem 1.

10.5 Functions of Any Period $p = 2L$

- If a function $f(x)$ of period $p = 2L$ has a Fourier series, the series is

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L})$$

- The Fourier coefficients of $f(x)$ is given by the Euler formulas

$$\begin{aligned} (a) \quad a_0 &= \frac{1}{2L} \int_{-L}^L f(x) dx \\ (b) \quad a_n &= \frac{1}{2L} \int_{-L}^L f(x) \cos \frac{n\pi x}{L} dx \quad n = 1, 2, \dots \\ (c) \quad b_n &= \frac{1}{2L} \int_{-L}^L f(x) \sin \frac{n\pi x}{L} dx \quad n = 1, 2, \dots \end{aligned} \tag{6}$$

Proof.

$$\begin{aligned} v &= \frac{\pi x}{L} \rightarrow x = \frac{Lv}{\pi} \\ x &= \pm L \rightarrow v = \pm \pi \end{aligned}$$

$$\implies f(x) = g(v)$$

$$g(v) = a_0 + \sum_{n=1}^{\infty} (a_n \cdot \cos nv + b_n \cdot \sin nv)$$

- Coefficients:

$$\begin{aligned}
 a_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} g(v) dv \\
 a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} g(v) \cos nv dv \\
 b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} g(v) \sin nv dv \\
 v &= \frac{\pi x}{L}, \quad dv = \frac{\pi}{L} dx
 \end{aligned} \tag{7}$$

- (7) gives (6)!

- Interval of integration : In (6), we may replace the interval of integration by any interval of length $p = 2L$, for example, by the interval $0 \leq x \leq 2L$

Example 3. Periodic square wave

$$\begin{aligned}
 f(x) &= 0 && \text{if } -2 < x < -1 \\
 f(x) &= k && \text{if } -1 < x < 1 \\
 f(x) &= 0 && \text{if } 1 < x < 2
 \end{aligned}$$

$$p = 2L = 4, \quad \therefore L = 2$$

Solution.

$$\begin{aligned}
 a_0 &= \frac{1}{4} \int_{-2}^2 f(x) dx = \frac{1}{4} \int_{-1}^1 k dx = \frac{k}{2} \\
 a_n &= \frac{1}{2} \int_{-2}^2 f(x) \cos \frac{n\pi x}{2} dx = \frac{1}{2} \int_{-1}^1 k \cos \frac{n\pi x}{2} dx \\
 &= \frac{2k}{2n\pi} \sin \frac{n\pi x}{2} \Big|_{-1}^1 = \frac{2k}{n\pi} \sin \frac{n\pi}{2}
 \end{aligned}$$

$$\begin{aligned}
 a_n &= 0 && \text{if } n \text{ is even} \\
 a_n &= \frac{2k}{n\pi} && \text{if } n = 1, 5, 9, \dots \\
 a_n &= \frac{-2k}{n\pi} && \text{if } n = 3, 7, 11, \dots
 \end{aligned}$$

$$\begin{aligned}
 b_n &= \frac{1}{2} \int_{-2}^2 f(x) \sin \frac{n\pi x}{2} dx = \frac{1}{2} \int_{-1}^1 k \cdot \sin \frac{n\pi x}{2} dx = \frac{-k}{n\pi} \cos \frac{n\pi x}{2} \Big|_{-1}^1 = 0 \\
 \therefore f(x) &= \frac{k}{2} + \frac{2k}{\pi} \left(\cos \frac{\pi}{2}x - \frac{1}{3} \cos \frac{3\pi}{2}x + \frac{1}{5} \cos \frac{5\pi}{2}x - \dots \right)
 \end{aligned}$$

Example 4. Half - wave rectifier

$$\begin{aligned}
 u(t) &= 0 && \text{if } -L < t < 0 \\
 u(t) &= E \cdot \sin \omega t && \text{if } 0 < t < L
 \end{aligned}$$

$$p = 2L = \frac{2\pi}{\omega}$$

Solution.

$$\begin{aligned} a_0 &= \frac{\omega}{2\pi} \int_0^{\pi/\omega} E \cdot \sin \omega t dt = -\frac{E}{2\pi} \cos \omega t \Big|_0^{\pi/\omega} \\ &= -\frac{E}{2\pi} (\cos \pi - \cos 0) = -\frac{E}{2\pi} (-1 - 1) = \frac{E}{\pi} \end{aligned}$$

$$\begin{aligned} a_n &= \frac{\omega}{\pi} \int_0^{\pi/\omega} E \cdot \sin \omega t \cos n\omega t dt \\ &= \frac{\omega E}{2\pi} \int_0^{\pi/\omega} [\sin(1+n)\omega t + \sin(1-n)\omega t] dt \end{aligned}$$

- $a_1 = 0$ if $n = 1$.
- If $n = 2, 3, \dots$,

$$\begin{aligned} a_n &= \frac{\omega E}{2\pi} \left[-\frac{\cos(1+n)\omega t}{(1+n)\omega} - \frac{\cos(1-n)\omega t}{(1-n)\omega} \right]_0^{\pi/\omega} \\ &= \frac{E}{2\pi} \left[-\frac{\cos(1+n)\pi + 1}{1+n} - \frac{\cos(1-n)\pi + 1}{1-n} \right]_0^{\pi/\omega} \end{aligned}$$

- If n is odd $\cos(n+1)\pi = \cos(1-n)\pi = 1$,

$$\therefore a_n = 0$$

$$\begin{aligned} a_n &= \frac{E}{2\pi} \left(\frac{2}{1+n} + \frac{2}{1-n} \right) = -\frac{2E}{(n-1)(n+1)\pi} \quad (n = 2, 4, 6, \dots) \\ b_n &= \frac{\omega}{\pi} \int_0^{\pi/\omega} E \cdot \sin \omega t \cdot \sin n\omega t dt \\ &= \frac{\omega E}{2\pi} \int_0^{\pi/\omega} [-\cos(1+n)\omega t + \cos(1-n)\omega t] dt \end{aligned}$$

- If $n = 1$,

$$b_1 = \frac{\omega E}{2\pi} \int_0^{\pi/\omega} (-\cos 2\omega t + 1) dt = \frac{E}{2}$$

- If $n = 2, 3, \dots$,

$$b_n = \frac{\omega E}{2\pi} \left[-\frac{\sin(1+n)\omega t}{(1+n)\omega} + \frac{\sin(1-n)\omega t}{(1-n)\omega} \right]_0^{\pi/\omega} = 0$$

$$\therefore u(t) = \frac{E}{\pi} + \frac{E}{2} \sin \omega t - \frac{2E}{\pi} \left(\frac{1}{1 \cdot 3} \cos 2\omega t + \frac{1}{3 \cdot 5} \cos \omega t + \dots \right)$$

10.6 Fourier cosine/sine series

Even and Odd Functions

- A function $y = g(x)$ is even if

$$g(-x) = g(x) \quad \text{for all } x$$

- A function $h(x)$ is odd if

$$h(-x) = -h(x) \quad \text{for all } x$$

- $\cos nx$: even, $\sin nx$: odd

Three Key Facts for the Present Discussion

- 1. If $g(x)$ is an even function, then

$$\int_{-L}^L g(x)dx = 2 \int_0^L g(x)dx.$$

- 2. If $h(x)$ is an odd function, then

$$\int_{-L}^L h(x)dx = 0.$$

- 3. The product of an even and an odd function is odd.

Proof)

$$\begin{aligned} q &= gh \\ q(-x) &= g(-x) \cdot h(-x) = g(x)(-h(x)) = -g(x)h(x) = -q(x) \end{aligned}$$

Theorem 2 [Fourier cosine series, Fourier sine series]

- The Fourier series of an even function of period $2L$ is a "Fourier cosine series".

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cdot \cos \frac{n\pi x}{L} \quad (f : \text{even})$$

- Coefficients of the cosine series:

$$\begin{aligned} a_0 &= \frac{1}{L} \int_0^L f(x)dx \\ a_n &= \frac{2}{L} \int_0^L f(x) \cdot \cos \frac{n\pi x}{L} dx, \quad n = 1, 2, \dots \end{aligned}$$

- The Fourier series of an odd function of period $2L$ is a "Fourier sine series".

$$f(x) = \sum_{n=1}^{\infty} b_n \cdot \sin \frac{n\pi x}{L} \quad (f : \text{odd})$$

- Coefficients of the sine series:

$$b_n = \frac{2}{L} \int_0^L f(x) \cdot \sin \frac{n\pi x}{L} dx$$

- Derivation of the coefficients of the even series

$$a_0 = \frac{1}{2L} \int_{-L}^L f(x) dx = \frac{1}{2L} \left[\int_{-L}^0 f(x) dx + \int_0^L f(x) dx \right]$$

- In the even function, $f(-x) = f(x)$

$$-x = x' \implies -dx = dx'$$

$$\int_{-L}^0 f(x) dx = - \int_L^0 (-x') dx' = \int_0^L f(x') dx'$$

$$\therefore a_0 = \frac{1}{L} \int_0^L f(x) dx$$

$$\begin{aligned} a_n &= \frac{1}{L} \int_{-L}^L f(x) \cos \frac{n\pi x}{L} dx \\ &= \frac{1}{L} \left[\int_{-L}^0 f(x) \cdot \cos \frac{n\pi x}{L} dx + \int_0^L f(x) \cdot \cos \frac{n\pi x}{L} dx \right] \end{aligned}$$

$$x = -x' \implies dx = -dx'$$

$$\begin{aligned} \int_{-L}^0 f(x) \cdot \cos \frac{n\pi x}{L} dx &= \int_L^0 f(-x') \cdot \cos \left(-\frac{n\pi x'}{L} \right) (-dx') \\ &= \int_0^L f(x') \cos \frac{n\pi x'}{L} dx' \end{aligned}$$

$$\therefore a_n = \frac{2}{L} \int_0^L f(x) \cdot \cos \frac{n\pi x}{L} dx$$

- Derivation of the coefficients of an odd function

$$\begin{aligned} b_n &= \frac{1}{L} \int_{-L}^L f(x) \sin \frac{n\pi x}{L} dx \\ &= \frac{1}{L} \left[\int_{-L}^0 f(x) \cdot \sin \frac{n\pi x}{L} dx + \int_0^L f(x) \cdot \sin \frac{n\pi x}{L} dx \right] \end{aligned}$$

- Since f is an odd function, $f(-x) = -f(x)$ and $\sin(-x) = -\sin x$

$$x = -x' \implies dx = -dx'$$

$$\begin{aligned}
\int_{-L}^0 f(x) \cdot \sin \frac{n\pi x}{L} dx &= \int_L^0 f(-x') \cdot \sin \left(-\frac{n\pi x'}{L} \right) (-dx') \\
&= \int_0^L f(x') \sin \frac{n\pi x'}{L} dx' \\
\therefore b_n &= \frac{2}{L} \int_0^L f(x) \cdot \sin \frac{n\pi x}{L} dx
\end{aligned}$$

- For an even function of a period of 2π ,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cdot \cos nx \quad (f : \text{even})$$

$$\begin{aligned}
a_0 &= \frac{1}{\pi} \int_0^\pi f(x) dx \\
a_n &= \frac{2}{\pi} \int_0^\pi f(x) \cdot \cos nx dx, \quad n = 1, 2, \dots
\end{aligned}$$

- Similarly, for an odd 2π -periodic function

$$\begin{aligned}
f(x) &= \sum_{n=1}^{\infty} b_n \cdot \sin nx \quad (f : \text{odd}) \\
b_n &= \frac{2}{\pi} \int_0^\pi f(x) \sin nx dx, \quad n = 1, 2, \dots
\end{aligned}$$

Theorem 3 [Sum of functions]

The Fourier coefficients of a sum $f_1 + f_2$ are the sums of the corresponding Fourier coefficients of f_1 and f_2 . The Fourier coefficients of cf are c times the corresponding Fourier coefficients of f .

Example 5. Rectangular pulse

$$f^*(x) = k + \frac{4k}{\pi} \left(\sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \dots \right)$$

Example 6. Sawtooth wave

$$\begin{aligned}
f(x) &= x + \pi && \text{if } -\pi < x < \pi \\
f(x + 2\pi) &= f(x)
\end{aligned}$$

$$\begin{aligned}
f &= f_1 + f_2 \quad \text{where } f_1 = x \quad \text{and } f_2 = \pi \\
\therefore f_2 &= \pi
\end{aligned}$$

- Since f_1 is odd, $a_n = 0$ for $n = 1, 2, \dots$.

$$\begin{aligned} b_n &= \frac{2}{\pi} \int_0^\pi f_1(x) \sin nx dx = \frac{2}{\pi} \int_0^\pi x \cdot \sin nx dx \\ &= \frac{2}{\pi} \left[-\frac{x \cdot \cos nx}{n} \Big|_0^\pi + \frac{1}{n} \int_0^\pi \cos nx dx \right] = -\frac{2}{n} \cos n\pi \end{aligned}$$

- Hence $b_1 = 2$, $b_2 = -2/2$, $b_3 = 2/3$, $b_4 = -2/4$, \dots

$$f(x) = \pi + 2(\sin x - \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x - \frac{1}{4} \sin 4x + \dots)$$