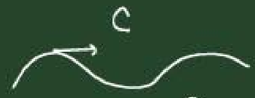
노트 제목 2010-05-11

8. **Standing waves** : waves that remain stationary
- the surface moves vertically only.

by superimposing two identical traveling waves which are moving in opposite directions.

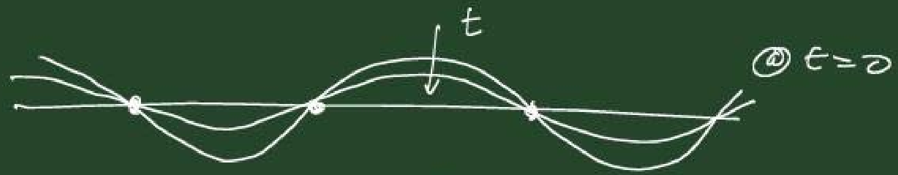
Two traveling waves

$$\eta_1(x,t) = \frac{1}{2}\varepsilon \sin\frac{2\pi}{\lambda}(x-ct)$$



$$\eta_2(x,t) = \frac{1}{2}\varepsilon \sin\frac{2\pi}{\lambda}(x+ct)$$

$$\eta(x,t) = \eta_1(x,t) + \eta_2(x,t)$$
$$= \varepsilon \sin\frac{2\pi}{\lambda}x \cos\frac{2\pi c}{\lambda}t$$



At any time, a sine ft in x,
for any x, oscillates vertically in t.

<u>Such a wave, in which the entire surface oscillates in time, is called a standing wave.</u>

complex potential

$\rightarrow F(z,t) = F_1(z,t) + F_2(z,t)$

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$$= -\frac{c\varepsilon}{\sinh\left(\frac{2\pi h}{\lambda}\right)} \sin\frac{2\pi}{\lambda}(z+ih)\sin\frac{2\pi c t}{\lambda}.$$

9. Particle paths for standing waves

same procedures as for traveling waves

$$\frac{dz_1^*}{dt} = \frac{dF}{dz} = -\frac{\frac{2\pi}{\lambda}c\varepsilon}{\sinh\frac{2\pi h}{\lambda}}\cos\frac{2\pi}{\lambda}(z+ih)\sin\frac{2\pi c t}{\lambda}$$

(x_1, y_1)

$$\rightarrow z_1^* = \frac{\varepsilon}{\sinh\frac{2\pi h}{\lambda}}\cos\frac{2\pi}{\lambda}(z+ih)\cos\frac{2\pi c}{\lambda}t = r_1 e^{-i\theta_1}$$

$z_1^* = (x_1 - iy_1)$

$$\rightarrow r_1 = \frac{\varepsilon}{\sinh\frac{2\pi h}{\lambda}}\cos\frac{2\pi c t}{\lambda}\sqrt{\cos^2\frac{2\pi x}{\lambda}\cosh^2\frac{2\pi(y+h)}{\lambda} + \sin^2\frac{2\pi x}{\lambda}\sinh^2\frac{2\pi(y+h)}{\lambda}}$$

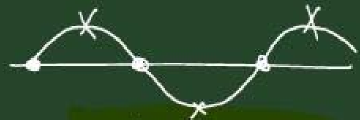
$$\theta_1 = \tan^{-1}\left(\tan\frac{2\pi x}{\lambda}\tanh\frac{2\pi}{\lambda}(y+h)\right)$$

where r_1 oscillates in time

θ_1 is const for given x & y.

$\rightarrow$ particle trajectories will be straight lines.

$$\eta = \varepsilon \sin\frac{2\pi x}{\lambda}\cos\frac{2\pi ct}{\lambda}$$



$$\frac{e^{\theta}+e^{-\theta}}{2}$$

For $x = \frac{n\lambda}{2}$, $r_1 = \varepsilon\cos\frac{2\pi ct}{\lambda}\cdot\frac{\cosh\frac{2\pi}{\lambda}(y+h)}{\sinh\frac{2\pi h}{\lambda}}$

$\theta_1 = 0$ or π

$\Rightarrow$ horizontal motion

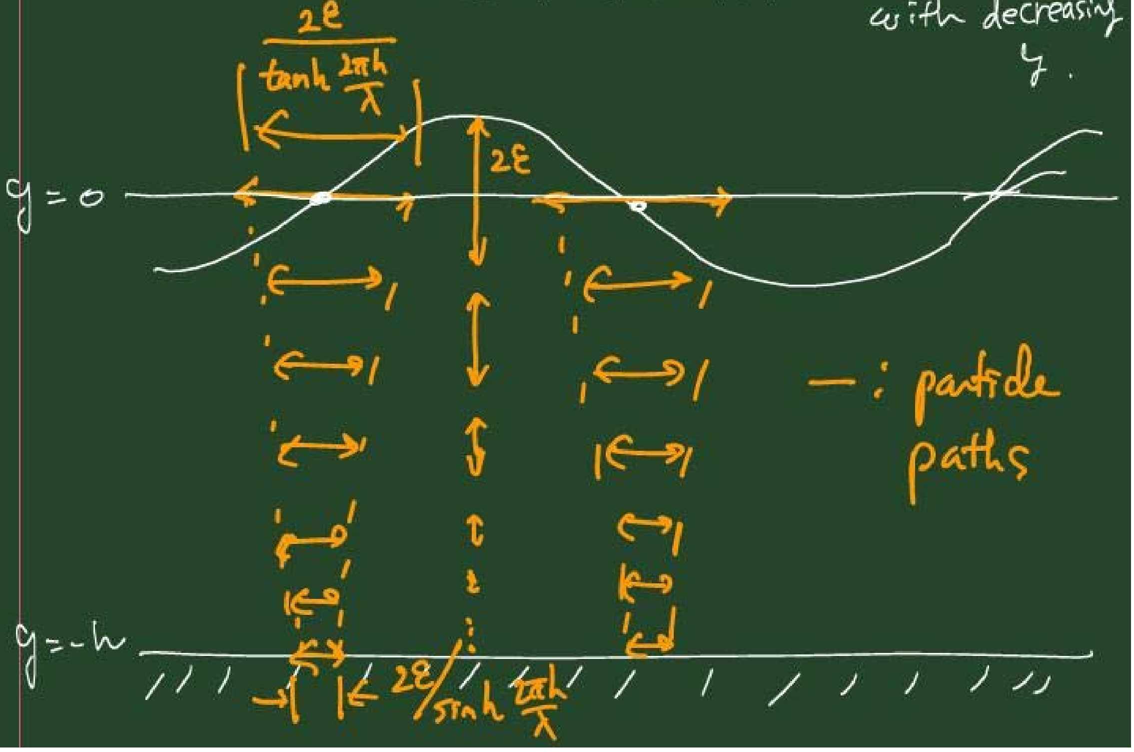
decreases as y decreases

For $x = \frac{(2n+1)\lambda}{4}$, $r_1 = \varepsilon\cos\frac{2\pi ct}{\lambda}\,\frac{\sinh\frac{2\pi}{\lambda}(y+h)}{\sinh\frac{2\pi h}{\lambda}}$

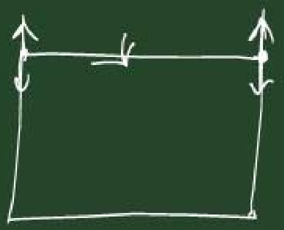
$\updownarrow$ $\theta_1 = \frac{\pi}{2}$ or $\frac{3\pi}{2}$ $\downarrow$

$\rightarrow$ vertical motion

decreases with decreasing y.



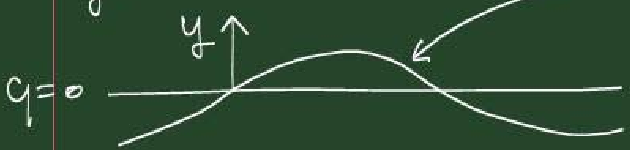
* waves in rectangular vessels



particle paths should be vertical
→ standing-wave types

10. Propagation of waves at an interface

$y>0$ ρ_2, $U_2 \longrightarrow$



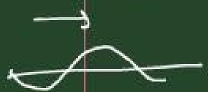
$y = \eta(x,t)$
$= \varepsilon\, e^{i\frac{2\pi}{\lambda}(x-\sigma t)}$

σ : complex number

$y<0$ ρ_1, $U_1 \longrightarrow$

when σ is real, the wave is traveling in the x direction w/ velocity σ.

when σ is imaginary,
the wave is decaying if $\sigma_i < 0$
or growing if $\sigma_i > 0$.
$\Downarrow$
represents an unstable interface.



$\underline{u}_i = U_i\, \underline{e}_x + \nabla\phi_i$ $(i = 1 \text{ or } 2)$

ϕ_i : velocity potential for the perturbation to the uniform flow caused by the waves at the interface

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \underline{u}_i \cdot \nabla = \frac{\partial}{\partial t} + U_i \frac{\partial}{\partial x} + \nabla\phi_i \cdot \nabla$$

Kinematic b.c. $\frac{D}{Dt}(y-\eta) = 0$

$$\to -\frac{\partial \eta}{\partial t} - U_i \frac{\partial \eta}{\partial x} + \frac{\partial \phi_i}{\partial y} \underbrace{- \nabla\phi_i \cdot \nabla\eta}_{\text{small}} = 0$$

$$\to \frac{\partial \phi_i}{\partial y}(x,0,t) = \frac{\partial \eta}{\partial t}(x,t) + U_i \frac{\partial \eta}{\partial x}(x,t)$$

Dynamic b.c. (unsteady Bernoulli eq)

for const. press. surface.

p absorbed here.

$$\to \rho_i \frac{\partial \phi_i}{\partial t} + \frac{1}{2}\rho_i \underline{u}_i \cdot \underline{u}_i + \rho_i g \eta = \text{const.}$$

with F(t) absorbed into ϕ_i.

Governing eq. $\nabla^2 \phi_i = 0$ $(i = 1 \text{ or } 2)$

⋮ same procedure as before

$$\boxed{\sigma = \frac{\rho_2 U_2 + \rho_1 U_1}{\rho_2 + \rho_1} \pm \sqrt{\left(\frac{\rho_1 - \rho_2}{\rho_1 + \rho_2}\right)\frac{g\lambda}{2\pi} - \frac{\rho_1 \rho_2}{(\rho_2+\rho_1)^2}(U_2 - U_1)^2}}$$

↳ may be real, purely imaginary, or complex

Special cases

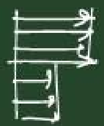
① $U_1 = U_2 = 0$, $\rho_2 = 0$ (e.g. air over water)

$\to \sigma = \pm\sqrt{\frac{g\lambda}{2\pi}}$: propagation speed for surface waves in deep l'qs.

real → interface is stable

② $\rho_2 = 0$.

$$\sigma = U_1 \pm \sqrt{\frac{g\lambda}{2\pi}} \quad \text{... same as ①}$$

③ $\rho_1 = \rho_2 \Longrightarrow$  shear layer

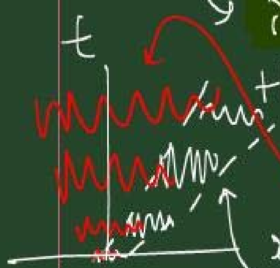
$$\sigma = \frac{U_2+U_1}{2} \pm \underbrace{i\,\frac{U_2-U_1}{2}}_{\text{imaginary}}$$

imaginary → interfacial wave grows exponentially in time

$\boxed{\text{Helmholtz instability}}$

or Rayleigh " ← interface at the shear layer is unstable

shear-layer instability



inviscid irrotational analysis shows that a shear layer has an absolute instability.

cf. convective instability

④ $U_1 = U_2 = 0$, $\rho_1 \neq \rho_2$

ρ_2

——— $y=0$

ρ_1

$$\sigma = \pm \sqrt{\frac{g\lambda}{2\pi}\left(\frac{\rho_1-\rho_2}{\rho_1+\rho_2}\right)}$$

for $\rho_1 > \rho_2$, σ real → stable

heavier fluid on top ← $\rho_1 < \rho_2$, σ imaginary → unstable → Taylor instability