노트 제목 2010-04-06

- For (x, r, θ) coord.

$$\frac{\partial \langle U\rangle}{\partial x} + \frac{1}{r}\frac{\partial (r\langle V\rangle)}{\partial r} = 0$$

(*) $$\langle U\rangle \frac{\partial \langle U\rangle}{\partial x} + \langle V\rangle \frac{\partial \langle U\rangle}{\partial r} = \frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \langle U\rangle}{\partial r}\right) - \frac{1}{r}\frac{\partial}{\partial r}\left(r\langle uv\rangle\right)$$

$$\frac{1}{\rho}\langle p\rangle = \frac{p_0}{\rho} - \langle v^2\rangle + \int_r^{\infty} \frac{\langle v^2\rangle - \langle w^2\rangle}{r'}\,dr'$$

- self similarity

$$\bar{f}(\xi) = \frac{\langle U(x,r)\rangle}{U_0(x)}, \quad \bar{g}(\xi) = \frac{\langle uv\rangle}{U_0(x)^2}$$

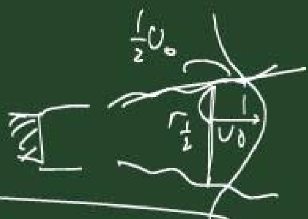
$$\xi = r/r_{\frac{1}{2}}(x)$$

→ (*) and neglect the viscous term ($\because \nu_T/\nu \gg 1$)

$$\Rightarrow \underbrace{[\xi \bar{f}^2]}_{\text{ft. of } \xi}\left\{\underbrace{\frac{r_{\frac{1}{2}}}{U_0}\frac{dU_0}{dx}}_{\text{ft. of } x}\right\} - \underbrace{\left[\bar{f}'\int_0^{\xi} \xi \bar{f}\,d\xi\right]}_{\text{ft. of } \xi}\left\{\underbrace{\frac{r_{\frac{1}{2}}}{U_0}\frac{dU_0}{dx} + 2\frac{dr_{\frac{1}{2}}}{dx}}_{\text{ft. of } x}\right\} = -\underbrace{[\xi \bar{g}]}_{\text{ft. of } \xi}$$

should be indep. of x

$$\frac{r_{\frac{1}{2}}}{U_0}\frac{dU_0}{dx} = C$$



$$\frac{dr_{\frac{1}{2}}}{dx} = S \rightarrow r_{\frac{1}{2}} = Sx + S_0$$

$\therefore$ $\boxed{\text{the jet spreading rate is linear in } x}$

$$\frac{r_{\frac{1}{2}}}{U_0}\frac{dU_0}{dx} = C \rightarrow U_0 \sim x^n \qquad n : \text{unspecified}.$$

Now, let us go back to mtm eq.

neglect the viscous term and multiply by r.

$$(*) \rightarrow \frac{\partial}{\partial x}\left(r\langle U\rangle^2\right) + \frac{\partial}{\partial r}\left(r\langle U\rangle\langle V\rangle + r\langle uv\rangle\right) = 0$$

$$\int \rightarrow \underline{\frac{d}{dx}\int_0^\infty r\langle U\rangle^2 dr} = -\left[r\langle U\rangle\langle V\rangle + r\langle uv\rangle\right]_0^\infty = 0$$

U_J

$\boxed{\rho U_J^2 A}$

mtm flux $\quad \dot{M}(x) = \underline{\int_0^\infty 2\pi r \rho \langle U\rangle^2 dr}$

$$\frac{d\dot{M}}{dx} = 0 \rightarrow \dot{M}(x) \text{ is indep of } x.$$

$$\boxed{\therefore \text{ momentum flux is conserved.}}$$

Using the self similarity,

$$\dot{M} = 2\pi\rho\left(r_{\frac{1}{2}}U_0\right)^2 \underbrace{\int_0^\infty \xi \bar{f}(\xi)^2 d\xi}_{\text{const}} = \text{const}.$$

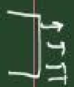
$$\therefore r_{\frac{1}{2}}U_0 = \text{const} \rightarrow U_0 \sim x^{-1}$$

$$\rightarrow \boxed{U_0 \text{ decays as } x^{-1}}$$

$2\pi r dr$

$\underline{\int \rho\langle U\rangle dA}$

mass flux $\quad \dot{m}(x) = \int_0^\infty 2\pi r \rho \langle U\rangle dr$



$$= 2\pi\rho\, r_{\frac{1}{2}}\left(r_{\frac{1}{2}}U_0\right)\int_0^\infty \xi \bar{f}(\xi) d\xi \sim x$$

entrainment

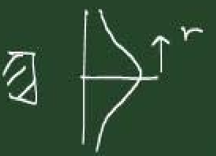
kinetic energy flux $\quad \int \rho\underline{\langle U\rangle}\cdot\underline{\tfrac{1}{2}\langle U\rangle^2}\, d\underline{A}$

$$\dot{E}(x) = \int_0^\infty 2\pi r \rho \frac{1}{2}\langle U\rangle^3 dr$$

$$= \frac{\pi\rho}{r_{\frac{1}{2}}}\left(r_{\frac{1}{2}}U_0\right)^3\int_0^\infty \xi \bar{f}^3 d\xi \sim x^{-1}$$

- Uniform turbulent viscosity

$$\langle uv \rangle = -\nu_T \frac{\partial \langle U \rangle}{\partial r}$$



$\frac{\partial \langle U \rangle}{\partial r} < 0$

$\langle uv \rangle > 0$

$\nu_T(x, r)$

With the self similarity $\hat{\nu}_T(\eta) = \dfrac{\nu_T(x,r)}{r_{\frac{1}{2}}(x)\, U_0(x)}$

exp. result → $\hat{\nu}_T(\eta)$ is within 15% of the value 0.028.

→ assume $\hat{\nu}_T$ is constant.

Then, the boundary layer mtm eq. becomes

$$\Rightarrow \langle U \rangle \frac{\partial \langle U \rangle}{\partial x} + \langle V \rangle \frac{\partial \langle U \rangle}{\partial r} = \frac{\nu_T}{r} \frac{\partial}{\partial r}\left(r \frac{\partial \langle U \rangle}{\partial r} \right)$$

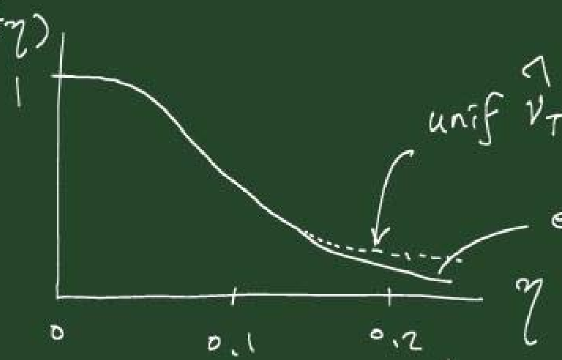
Schlichting (1933) obtained the solution
by defining stream ft. and using the self similarity.
PDE → ODE → solve.

$$\rightarrow f(\eta) = \frac{\langle U \rangle}{U_0} = \frac{1}{(1 + a\eta^2)^2}$$

derivation: pp. 120–122.

$$\eta = \frac{r}{x - x_0}, \quad a = \frac{\sqrt{2}-1}{S^2}, \quad \hat{\nu}_T = \frac{S}{8(\sqrt{2}-1)}$$

$\hat{\nu}_T = 0.028$



unif $\hat{\nu}_T$ w/ $S = 0.094$
not bad.

exp. → ν_T is zero at the edge of jet.

Turbulent Reynolds number

$$Re_T = \frac{U_0(x)\, r_{\frac{1}{2}}(x)}{\nu_T} = \frac{1}{\hat{\nu}_T} \simeq 35.$$

const.

5.3 Round jet : kinetic energy

kinetic energy : $E(\underline{x}, t) = \frac{1}{2}\underline{U}(\underline{x}, t)\cdot\underline{U}(\underline{x}, t)$

Mean of E : $\langle E(\underline{x}, t)\rangle = \bar{E}(\underline{x}, t) + k(\underline{x}, t)$

kinetic energy of mean flow $\bar{E} = \frac{1}{2}\langle\underline{U}\rangle\cdot\langle\underline{U}\rangle$

turbulent kinetic energy $k = \frac{1}{2}\langle u_i u_i\rangle$

- $\frac{DE}{Dt} + \nabla\cdot\underline{T} = -2\nu S_{ij}S_{ij} \leftarrow U_i\left(\rho\frac{\partial U_i}{\partial t} = \cdots\right)$

$\frac{\partial E}{\partial t} + \frac{\partial}{\partial x_j}(EU_j)$ $\quad T_i = \frac{1}{\rho}U_i p - 2\nu U_j S_{ij}$: energy flux

$$\frac{d}{dt}\iiint_V E\, dV + \iint_A (\underline{U}E + \underline{T})\cdot\underline{n}\, dA = \underbrace{-\iiint_V 2\nu S_{ij}S_{ij}\, dV}_{<0}$$

dissipation ← sink of energy

- Mean kinetic energy

$$\frac{\bar{D}\langle E\rangle}{\bar{D}t} + \frac{\partial}{\partial x_j}\left(\langle U_j E\rangle + \langle T_j\rangle\right) = -\bar{\varepsilon} - \varepsilon$$

$\left(= \frac{\partial\langle E\rangle}{\partial t} + \langle\underline{U}\rangle\cdot\nabla\langle E\rangle\right)$ $\qquad \bar{S}_{ij} = \langle S_{ij}\rangle$

$\bar{\varepsilon} = 2\nu\bar{S}_{ij}\bar{S}_{ij}$: dissipation due to mean flow

$\varepsilon = 2\nu s_{ij}s_{ij}$: turbulent dissipation $\qquad s_{ij}$: fluctuating strain rate

$\bar{S}_{ij}$

$\varepsilon \gg \bar{\varepsilon} \sim O\left(\frac{1}{Re}\right)$

- Mean flow and turbulent kinetic energy

$$\frac{\bar{D}\bar{E}}{\bar{D}t} + \frac{\partial}{\partial x_j}\bar{T}_j = -\mathcal{P} - \bar{\varepsilon} \qquad \bar{E} = \frac{1}{2}\langle \underline{U}\rangle \cdot \langle \underline{U}\rangle$$

$$(\langle E\rangle = \bar{E} + k)$$

$$\frac{\bar{D}k}{\bar{D}t} + \frac{\partial}{\partial x_j}T_j' = \mathcal{P} - \varepsilon \qquad k = \frac{1}{2}\langle u_i u_i\rangle$$

$$\mathcal{P} = -\langle u_i u_j\rangle \frac{\partial\langle U_i\rangle}{\partial x_j} : \text{ production}$$

observations

generally positive
source in k-eq
sink in $\bar{E}$-eq.

① $\mathcal{P} = -\langle u_i u_j\rangle \dfrac{\partial\langle U_i\rangle}{\partial x_j} = -\langle u_i u_j\rangle(\bar{S}_{ij} + \bar{\Omega}_{ij})$

$= -\langle u_i u_j\rangle \bar{S}_{ij}$: only symmetric part of vel. grad. tensor affects production

② $\mathcal{P} = -\langle u_i u_j\rangle \dfrac{\partial\langle U_i\rangle}{\partial x_j}$ $\qquad \langle u_i u_j\rangle = a_{ij} + \frac{2}{3}k\delta_{ij}$

$= -a_{ij}\bar{S}_{ij}$: only anisotropic part of Reynolds stress tensor affects production.

③ $\mathcal{P} = -\langle u_i u_j\rangle \bar{S}_{ij} = 2\nu_T \bar{S}_{ij}\bar{S}_{ij} \geq 0$

④ In the boundary layer approx.

$\mathcal{P} = -\langle uv\rangle \dfrac{\partial\langle U\rangle}{\partial y}$ or $-\langle uv\rangle \dfrac{\partial\langle U\rangle}{\partial r}$

$= \nu_T \left(\dfrac{\partial\langle U\rangle}{\partial y}\right)^2$

Dissipation $\varepsilon = 2\nu s_{ij}s_{ij} \geq 0$

$= 2\nu s_{ij}\dfrac{\partial u_i}{\partial x_j}$: fluctuating vel. grad working against the fluctuating deviatoric stresses.

$$\hat{P} = \frac{P}{U_0^3/r_{\frac{1}{2}}} \doteq \left(\frac{-\langle uv \rangle}{U_0^2}\right) \cdot \left(\frac{r_{\frac{1}{2}}}{U_0} \cdot \frac{\partial \langle U \rangle}{\partial r}\right) \rightarrow \text{self-similar}$$

$$\hat{\varepsilon} = \frac{\varepsilon}{U_0^3/r_{\frac{1}{2}}} \longrightarrow \text{self-similar}$$

$\hat{P}$ & $\hat{\varepsilon}$ are self-similar and indep. of Re.