4. State-space Solutions and Realizations

- ✓ Solution of LTI State Equations
- ✓ Solution of Discrete-time LTI Equations
- ✓ Equivalent State Equation
- ✓ Realizations
- ✓ Solution of LTV State Equations
- ✓ Solution of Discrete-time LTV Equations
- ✓ Equivalent Time-varying Equations
- ✓ Time-varying Realizations

Solutions of LTI State Equations

Solutions of LTI State Equations

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

$\times\ e^{-At}$

$$e^{-At}\dot{x}(t) - e^{-At}Ax(t) = e^{-At}Bu(t)$$
$$\frac{d}{dt}(e^{-At}x(t)) = e^{-At}Bu(t)$$

By integration

$$e^{-A\tau}x(\tau)\Big|_{\tau=0}^{t} = \int_0^t e^{-A\tau}Bu(\tau)d\tau$$
$$e^{-At}x(t) - x(0) = \int_0^t e^{-A\tau}Bu(\tau)d\tau$$
$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

Solutions of LTI State Equations

Taking derivative of $x(t)$

$$\dot{x}(t) = Ae^{At}x(0) + A\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau + Bu(t)$$
$$= Ax(t) + Bu(t)$$

$$y(t) = Ce^{At}x(0) + C\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau + Du(t)$$
$$= Cx(t) + Du(t)$$

Solutions of LTI State Equations

Calculation of e^{At}

1. $e^{At} = h(A)$
2. $A = Q\hat{A}Q^{-1}, \quad e^{At} = Qe^{\hat{A}t}Q^{-1}$
3. Infinite Power Series
4. $e^{At} = L^{-1}(s\mathbf{I} - A)^{-1}$

Calculation of $(s\mathbf{I} - A)^{-1}$

1. Direct Cal. of $(s\mathbf{I} - A)^{-1}$
2. $f(A) = h(A)$
3. $(s\mathbf{I} - A)^{-1} = Q(s\mathbf{I} - \hat{A})^{-1}Q^{-1}$
4. Infinite Power Series
5. Problem 3.26 (Leverrier algorithm)

Solutions of LTI State Equations

Example

$$\dot{x}(t) = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$x(t) = e^{At} x(0) + \int_0^t e^{A(t-\tau)} Bu(\tau) d\tau$$

$$(sI - A)^{-1} = \begin{bmatrix} s & 1 \\ -1 & s+2 \end{bmatrix}^{-1} = \frac{1}{s^2 + 2s + 1} \begin{bmatrix} s+2 & -1 \\ 1 & s \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{s+2}{(s+1)^2} & \dfrac{-1}{(s+1)^2} \\ \dfrac{1}{(s+1)^2} & \dfrac{s}{(s+1)^2} \end{bmatrix}$$

Solutions of LTI State Equations

Example(cont.)

$$e^{At} = L^{-1}[(sI - A)^{-1}] =$$

$$= L^{-1}\begin{bmatrix} \dfrac{s+2}{(s+1)^2} & \dfrac{-1}{(s+1)^2} \\ \dfrac{1}{(s+1)^2} & \dfrac{s}{(s+1)^2} \end{bmatrix} = \begin{bmatrix} (1+t)e^{-t} & -te^{-t} \\ te^{-t} & (1-t)e^{-t} \end{bmatrix}$$

$$x(t) = \begin{bmatrix} (1+t)e^{-t} & -te^{-t} \\ te^{-t} & (1-t)e^{-t} \end{bmatrix} x(0) + \begin{bmatrix} \int_0^t (t-\tau)e^{-(t-\tau)}u(\tau)d\tau \\ \int_0^t [1-(t-\tau)]e^{-(t-\tau)}u(\tau)d\tau \end{bmatrix}$$

HW4-1

Problem 4.1 p.117 in Text

Solutions of LTI State Equations

Note)

$$e^{At} = Qe^{\hat{A}t}Q^{-1} \leftarrow \hat{A}: \text{ Jordan form}$$

$$e^{\hat{A}t} = \begin{bmatrix} e^{\lambda_1 t} & te^{\lambda_1 t} & t^2 e^{\lambda_1 t}/2! & 0 & 0 \\ 0 & e^{\lambda_1 t} & te^{\lambda_1 t} & \dots & \\ & 0 & e^{\lambda_1 t} & 0 & \\ & \dots & 0 & e^{\lambda_1 t} & 0 \\ 0 & & & 0 & e^{\lambda_2 t} \end{bmatrix}$$

Solution of Discrete Linear Equations

Discretization

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$
$$\dot{x}(t) = \lim_{T \to 0} \frac{x(t+T) - x(t)}{T}$$

Approximated Eq.

$$x(t+T) = x(t) + Ax(t)T + Bu(t)T$$
$$t = kT,\ x(t) = x(kT) := x[k]$$
$$x[k+1] = (\mathrm{I} + TA)x[k] + TBu[k]$$
$$y[k] = Cx[k] + Du[k]$$
$$\Rightarrow \text{ least accurate results}$$

Solution of Discrete Linear Equations

Different Method

$$x[k] = x(kT) = e^{AkT}x(0) + \int_0^{kT} e^{A(kT-\tau)}Bu(\tau)d\tau$$

$$x[k+1] = e^{A(k+1)T}x(0) + \int_0^{(k+1)T} e^{A((k+1)T-\tau)}Bu(\tau)d\tau$$

$$= e^{AT}\left[e^{AkT}x(0) + \int_0^{kT} e^{A(kT-\tau)}Bu(\tau)d\tau\right]$$

$$+ \int_{kT}^{(k+1)T} e^{A(kT+T-\tau)}Bu(\tau)d\tau$$

$$(u(t) = u(kT) := u[k] \quad kT \le t < (k+1)T$$

$$x[k+1] = e^{AT}x[k] + \int_0^{T} e^{A\alpha}d\alpha Bu[k]$$

$$\mathrm{x}[k+1] = A_d x[k] + B_d u[k]$$

Perception and Intelligence Laboratory
School of Electrical Engineering at SNU

Solution of Discrete Linear Equations

$$A_d = e^{AT}, \;\; B_d = (\int_0^T e^{A\tau} d\tau)B, \; C_d = C, \; D_d = D$$

$$(\int_0^T e^{A\tau} d\tau) = \int_0^T (I + A\tau + A^2 \frac{\tau^2}{2!} + \cdots) d\tau$$

$$= TI + \frac{T^2}{2!} A + \frac{T^3}{3!} A^2 + \cdots$$

If A is nonsingular

$$= A^{-1}(-I + I + TA + \frac{T^2}{2!} A^2 + \frac{T^3}{3!} A^3 + \cdots \,)$$

$$= A^{-1}(-I + e^{AT})$$

Solution of Discrete Linear Equations

Solution

$$x[1] = Ax[0] + Bu[0]$$

$$x[2] = Ax[1] + Bu[1] = A^2 x[0] + ABu[0] + Bu[1]$$

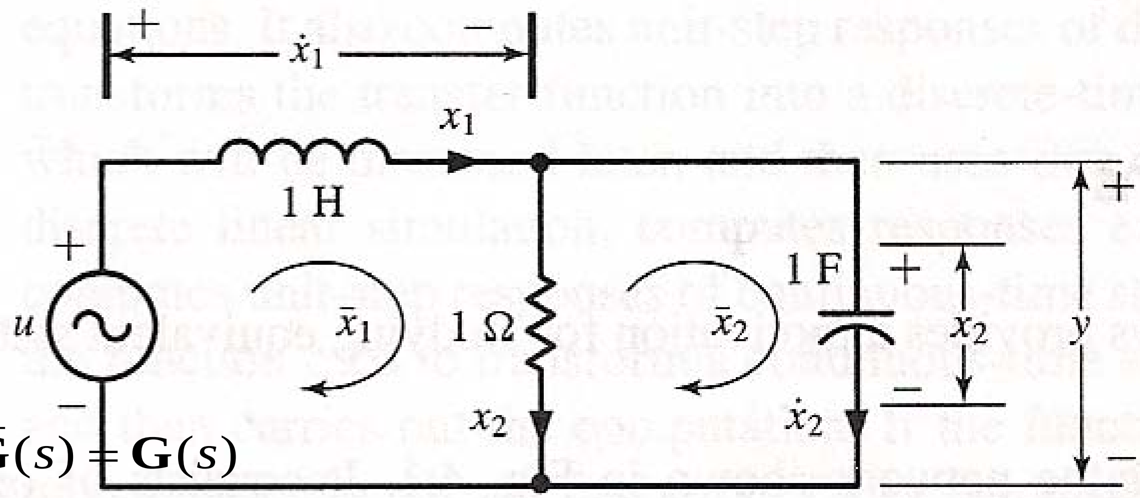
$$x[k] = A^k x[0] + \sum_{m=0}^{k-1} A^{k-1-m} Bu[m]$$

$$A^k = Q \cdot \hat{A}^k Q^{-1}$$

$$\hat{A}^k = \begin{bmatrix} \lambda_1^k & k\lambda_1^{k-1} & \frac{k(k-1)}{2!}\lambda_1^{k-2} & & 0 \\ 0 & \lambda_1^k & k\lambda_1^{k-1} & & \\ & 0 & \lambda_1^k & 0 & \\ 0 & & 0 & \cdots & 0 \\ 0 & 0 & & 0 & \lambda_2^k \end{bmatrix}$$

Equivalent State Equation

Example:



$\bar{\mathbf{G}}(s) = \mathbf{G}(s)$

state1: Inductor current $x_1(t)$, Capacitor voltage $x_2(t)$

state2: Loop current $\bar{x}_1(t)$, $\bar{x}_2(t)$

Equivalent State Equation

Example(cont.):

state1: Inductor current $x_1(t)$, Capacitor voltage $x_2(t)$

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)$$

$$y(t) = [0\ 1]x(t)$$

state2: Loop current $\overline{x}_1(t)$, $\overline{x}_2(t)$

$$\begin{bmatrix} \dot{\overline{x}}_1(t) \\ \dot{\overline{x}}_2(t) \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \overline{x}_1(t) \\ \overline{x}_2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = [1\ \ -1]\overline{x}(t)$$

The two equations describe the same circuit network, and They are said to be equivalent to each other.

Equivalent State Equation

Equivalence Transformation

Definition: Let P be nonsingular matrix and let

$$\bar{x} = Px$$

$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u(t)$$

$$y = \bar{C}\bar{x} + \bar{D}u(t)$$

where

$$\bar{A} = PAP^{-1},\ \bar{B} = PB,\ \bar{C} = CP^{-1},\ \bar{D} = D$$

is said to be equivalent to $\{A, B, C, D\}$

$\bar{x} = Px$ is called an equivalence transformation

Equivalent State Equation

$$x(t) = P^{-1}\overline{x}(t)$$

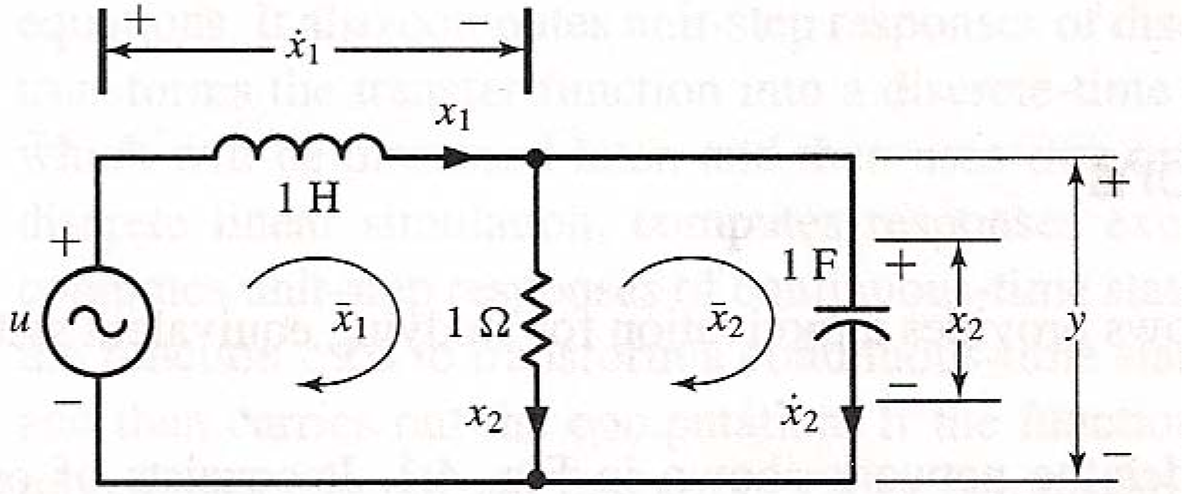
$$\begin{aligned}
\dot{x}(t) &= P^{-1}\dot{\overline{x}}(t) \\
&= P^{-1}\overline{A}\overline{x} + P^{-1}\overline{B}u(t) \\
&= P^{-1}\overline{A}Px + P^{-1}\overline{B}u(t) \\
&= Ax + Bu \\
&\Rightarrow \overline{A} = PAP^{-1}, \quad \overline{B} = PB
\end{aligned}$$

$$\begin{aligned}
\overline{\Delta}(\lambda) &= \det(\lambda\mathbf{I} - \overline{A}) = \det(\lambda PP^{-1} - PAP^{-1}) \\
&= \det(P(\lambda\mathbf{I} - A)P^{-1}) \\
&= \det P \det(\lambda\mathbf{I} - A) \det P^{-1} = \det(\lambda\mathbf{I} - A) \\
&= \Delta(\lambda)
\end{aligned}$$

$$\overline{\mathbf{G}}(s) = \mathbf{G}(s)$$

Equivalent State Equation

Example:



$$x_1(t) = \bar{x}_1(t)$$

$$x_2(t) = \bar{x}_1(t) - \bar{x}_2(t)$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix}$$

Equivalent State Equation

Zero-state equivalent

Z-s-e if $D + C(s\mathbf{I} - A)^{-1}B = \bar{D} + \bar{C}(s\mathbf{I} - \bar{A})^{-1}\bar{B}, i.e.$

$\mathbf{G}(s) = \bar{\mathbf{G}}(s)$, state dimension may be different

Zero-input equivalent

Z-i-e if for zero input, outputs are identical.

Theorem 4.1

$\{A, B, C, D\} \,\&\, \{\bar{A}, \bar{B}, \bar{C}, \bar{D}\}$ are

Zero-state-equivalent iff $D = \bar{D}$ &

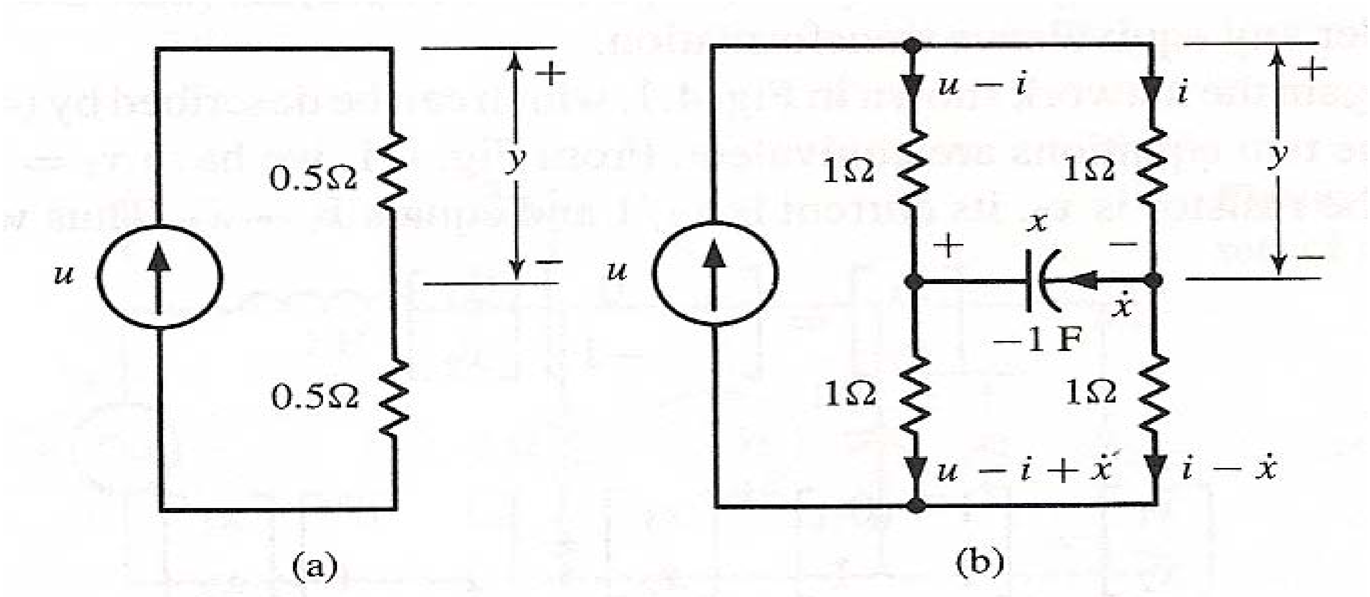
$CA^m B = \overline{CA}^m \bar{B}, \; m = 0, 1, 2, \cdots$

Pf.)

$$D + CBs^{-1} + CABs^{-2} + CA^2Bs^{-3} + \cdots$$
$$= \bar{D} + \bar{C}\bar{B}s^{-1} + \overline{CA}\bar{B}s^{-2} + \overline{CA}^2\bar{B}s^{-3} + \cdots$$

Equivalent State Equation

Example



$(a)\ \ y(t) = 0.5u(t),\ \ A = B = C = 0,\ D = 0.5$

$(b)\ \ \dot{x} = x(t),\ \ y = 0.5x(t) + 0.5u(t),\ \ \bar{A} = 1,\ \bar{B} = 0,\ \bar{C} = 0.5,\ \bar{D} = 0.5$

$\Rightarrow CA^m B = \bar{C}\bar{A}^m \bar{B} = 0 \Rightarrow$ zero state equivalent

Realizations

Realization

- $\mathbf{G}(s)$ is said to be realizable if there exists $\{A, B, C, D\}$ such that

  $$\mathbf{G}(s) = C(s\mathbf{I} - A)^{-1}B + D$$

- $\{A, B, C, D\}$ is called a realization of $\mathbf{G}(s)$

Theorem

$\mathbf{G}(s)$ is realizable *iff* $\mathbf{G}(s)$ is a proper rational matrix.

Realizations

Pf.)

$(\Rightarrow)$

$$\mathbf{G}(s) = C(s\mathbf{I} - A)^{-1}B + D$$

$$\mathbf{G}(\infty) = D, \mathbf{G}_{sp} = C(s\mathbf{I} - A)^{-1}B = \frac{1}{\det(s\mathbf{I} - A)} C[Adj(s\mathbf{I} - A)]B$$

$\rightarrow$ Every entry of $Adj(s\mathbf{I} - A)$ is the determinent of an

$(n-1)\times(n-1)$ submatrix of $(s\mathbf{I} - A)$,

thus it has at most degree $(n-1)$

$\rightarrow \mathbf{G}_{sp}$: strictly proper rational matrix

$\rightarrow C(s\mathbf{I} - A)^{-1}B + D$ is proper rational matrix

Realizations

Pf.cont)

$(\Leftarrow)$

Assume that $\mathbf{G}(s)$ is a $q \times p$ proper rational matrix.

$\mathbf{G}(s) = \mathbf{G}(\infty) + \mathbf{G}_{sp}(s)$. Let

$d(s) = s^r + \alpha_1 s^{r-1} + \cdots + \alpha_r$

be the least common denominator of all entries of $\mathbf{G}_{sp}(s)$.

$$\mathbf{G}_{sp}(s) = \frac{1}{d(s)}\left[N_1 s^{r-1} + N_2 s^{r-2} + \qquad + N_r\right],$$

where N_i are $q \times p$ constant matrices. We claim that

$$\dot{x} = \begin{bmatrix} -\alpha_1 \mathbf{I}_p & \cdots - \alpha_r \mathbf{I}_p \\ \mathbf{I}_p & 0 \\ 0 \quad \ddots & \mathbf{I}_p \end{bmatrix} x + \begin{bmatrix} \mathbf{I}_p \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} N_1 & N_2 & \cdots N_r \end{bmatrix} x + \mathbf{G}(\infty) u$$

is a realization of $\mathbf{G}(s)$.

Realizations

Pf.cont)

Define

$$Z := (s\mathbf{I} - A)^{-1}B = [Z_1^T \ Z_2^T \ \ldots Z_r^T]^T$$

$$C(s\mathbf{I} - A)^{-1}B = N_1 Z_1 + \cdots + N_r Z_r$$

$$(s\mathbf{I} - A)Z = B$$

$$sZ = AZ + B$$

$$sZ_2 = Z_1 \ , \ sZ_3 = Z_2 \cdots sZ_r = Z_{r-1}$$

$$Z_2 = \frac{1}{s} Z_1 \qquad \ldots \qquad Z_r = \frac{1}{s} Z_{r-1}$$

$$Z_r = \frac{1}{s^{r-1}} Z_1$$

$$sZ_1 = -\alpha_1 Z_1 - \alpha_2 Z_2 \cdots - \alpha_r Z_r + \mathbf{I}_p$$

$$= -(\alpha_1 + \frac{\alpha_2}{s} + \cdots \frac{\alpha_r}{s^{r-1}}) Z_1 + \mathbf{I}_p$$

$$A = \begin{bmatrix} -\alpha_1 \mathbf{I}_p & \cdots & -\alpha_r \mathbf{I}_p \\ \mathbf{I}_p & & 0 \\ 0 & \ddots & \mathbf{I}_p \end{bmatrix}$$

$$B = \begin{bmatrix} \mathbf{I}_p \\ 0 \\ 0 \end{bmatrix}$$

Realizations

Pf.cont)

$$(s^r + \alpha_1 s^{r-1} + \cdots \alpha_r) Z_1 = s^{r-1} \mathbf{I}_p$$

$$Z_1 = \frac{s^{r-1}}{d(s)} \mathbf{I}_p$$

$$\vdots$$

$$Z_r = \frac{1}{d(s)} \mathbf{I}_p$$

$$C(S\mathbf{I} - A)^{-1} B = \frac{1}{d(s)} [N_1 s^{r-1} + \qquad N_r] = G_{sp}(s)$$

Realizations

Eample

$$G(s)=\begin{bmatrix}\dfrac{4s-10}{2s+1} & \dfrac{3}{s+2}\\ \dfrac{1}{(2s+1)(s+2)} & \dfrac{s+1}{(s+2)^2}\end{bmatrix}=\begin{bmatrix}2 & 0\\ 0 & 0\end{bmatrix}+\begin{bmatrix}\dfrac{-12}{2s+1} & \dfrac{3}{s+2}\\ \dfrac{1}{(2s+1)(s+2)} & \dfrac{s+1}{(s+2)^2}\end{bmatrix}$$

$$d(s)=(s+0.5)(s+2)^2=s^3+4.5s^2+6s+2:\ \text{least common denominator}$$

$$G_{sp}(s)=\frac{1}{s^3+4.5s^2+6s+2}\begin{bmatrix}-6(s+2)^2 & 3(s+2)(s+0.5)\\ 0.5(s+2) & (s+1)(s+0.5)\end{bmatrix}$$

$$=\frac{1}{d(s)}\left(\begin{bmatrix}-6 & 3\\ 0 & 1\end{bmatrix}s^2+\begin{bmatrix}-24 & 7.5\\ 0.5 & 1.5\end{bmatrix}s+\begin{bmatrix}-24 & 3\\ 1 & 1.5\end{bmatrix}\right)$$

$$\dot{x}=\begin{bmatrix}-4.5\mathbf{I} & -6\mathbf{I} & -2\mathbf{I}\\ \mathbf{I} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{I} & \mathbf{0}\end{bmatrix}x+\begin{bmatrix}\mathbf{I}\\ \mathbf{0}\\ \mathbf{0}\end{bmatrix}u$$

$$y=\begin{bmatrix}-6 & 3 & -24 & 7.5 & -24 & 3\\ 0 & 1 & 0.5 & 1.5 & 1 & 1.5\end{bmatrix}x+\begin{bmatrix}2 & 0\\ 0 & 0\end{bmatrix}u$$

Realizations

Eample (zero state equivalent)

$$G(s)=\begin{bmatrix}\dfrac{4s-10}{2s+1}\\[2ex]\dfrac{1}{(2s+1)(s+2)}\end{bmatrix}=\begin{bmatrix}2\\0\end{bmatrix}+\frac{1}{s^2+2.5s+1}\left(\begin{bmatrix}-6\\0\end{bmatrix}s+\begin{bmatrix}-12\\0.5\end{bmatrix}\right)$$

$$\dot{x}_1=\begin{bmatrix}-2.5 & -1\\1 & 0\end{bmatrix}x_1+\begin{bmatrix}1\\0\end{bmatrix}u_{1,}\quad y_1=\begin{bmatrix}-6 & -12\\0 & 0.5\end{bmatrix}x_1+\begin{bmatrix}2\\0\end{bmatrix}u_{1,}$$

$$\dot{x}_2=\begin{bmatrix}-4 & -4\\1 & 0\end{bmatrix}x_2+\begin{bmatrix}1\\0\end{bmatrix}u_{2,}\quad y_2=\begin{bmatrix}3 & 6\\1 & 1\end{bmatrix}x_2+\begin{bmatrix}0\\0\end{bmatrix}u_{2,}$$

by superposition principle,

$$y=y_1+y_2$$

$$\dot{x}=\begin{bmatrix}-2.5 & -1 & \mathbf{0} & \\ 1 & 0 & & \\ & & -4 & -4\\ \mathbf{0} & & 1 & 0\end{bmatrix}x+\begin{bmatrix}1 & 0\\0 & 0\\0 & 1\\0 & 0\end{bmatrix}u,\quad y=\begin{bmatrix}-6 & -12 & 3 & 6\\0 & 0.5 & 1 & 1\end{bmatrix}x+\begin{bmatrix}2 & 0\\0 & 0\end{bmatrix}u$$

Solution of LTV Equation

Solution of Linear Time Varying Equation (LTV)

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \qquad EF \neq FE$$

$$y(t) = C(t)x(t) + D(t)u(t) \qquad \mathrm{e}^{(F+E)t} \neq \mathrm{e}^{Ft} \cdot \mathrm{e}^{Et}$$

$$\dot{x} = a(t)x(t), \; x(0)$$

$$x(t) = e^{\int_0^t a(\tau)d\tau} x(0)$$

$$\frac{d}{dt} e^{\int_0^t a(\tau)d\tau} = a(t)e^{\int_0^t a(\tau)d\tau} = e^{\int_0^t a(\tau)d\tau} a(t)$$

Solution of LTV Equation

$$x(t) = e^{\int_0^t A(\tau)d\tau} x(0) \cdots (*) \quad \text{Solution?}$$

$$e^{\int_0^t A(\tau)d\tau} = \mathbf{I} + \int_0^t A(\tau)d\tau + \frac{1}{2}(\int_0^t A(\tau)d\tau)(\int_0^t A(\tau)d\tau) + \ldots$$

$$\frac{d}{dt} e^{\int_0^t A(\tau)d\tau} = A(t) + \frac{1}{2} A(t)\int_0^t A(\tau)d\tau$$

$$+ \frac{1}{2}\int_0^t A(\tau)d\tau A(t)$$

$$\neq A(t)e^{\int_0^t A(\tau)d\tau} \quad (\because \frac{1}{2} A(t)\int_0^t A(\tau)d\tau \neq \frac{1}{2}\int_0^t A(\tau)d\tau A(t))$$

$$\Rightarrow \dot{x}(t) \neq A(t)x(t)$$

In general, (*) is not solution of linear time varying systems.

Solution of LTV Equation

Fundamental Matrix

Theorem: the set of all solutions of $\dot{x}(t) = A(t)x(t)$ forms an n-dimensional Linear Space.

Pf.) See the second edition.

Define **Fundamental Matrix** composed of n-linearly independent solutions as

$$X(t) = [x_1(t), x_2(t), \cdots, x_n(t)]$$

which is a solution of

$$\dot{X}(t) = A(t)X(t).$$

If $X(0) = [x_1(0), x_2(0), \cdots, x_n(0)]$ is nonsingular,

$X(t)$ can be Fundamental Matrix.

Solution of LTV Equation

Example

$$\dot{x}(t) = \begin{bmatrix} 0 & 0 \\ t & 0 \end{bmatrix} x(t)$$

The solution of $\dot{x}_1(t) = 0$ for $t_0 = 0$ is $x_1(t) = x_1(0)$;

the solution of $\dot{x}_2(t) = tx_1(t) = tx_1(0)$ is

$$x_2(t) = \int_0^t \tau x_1(0) d\tau + x_2(0) = 0.5t^2 x_1(0) + x_2(0)$$

$$x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow x(t) = \begin{bmatrix} 1 \\ 0.5t^2 \end{bmatrix}; \quad x(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow x(t) = \begin{bmatrix} 1 \\ 0.5t^2 + 2 \end{bmatrix}$$

$$X(t) = \begin{bmatrix} 1 & 1 \\ 0.5t^2 & 0.5t^2 + 2 \end{bmatrix}$$

can be Fundamental Matrix.

Solution of LTV Equation

Definition: Let $X(t)$ be any fundamental matrix of

$$\dot{x}(t) = A(t)x(t). \text{ Then,}$$

$$\Phi(t,t_0) := X(t)X^{-1}(t_0)$$

is called the state transition matrix.

The transition matrix is a unique solution of

$$\frac{\partial}{\partial t}\Phi(t,t_0) = A(t)\Phi(t,t_0)$$

with initial condition $\Phi(t_0,t_0) = \mathbf{I}$

Note) $\Phi(t,t) = \mathbf{I},\ \Phi^{-1}(t,t_0) = \Phi(t_0,t)$

$$\Phi(t,t_0) = \Phi(t,t_1)\Phi(t_1,t_0)$$

Solution of LTV Equation

Example

$$X(t) = \begin{bmatrix} 1 & 1 \\ 0.5t^2 & 0.5t^2 + 2 \end{bmatrix}$$

$$X^{-1}(t) = \begin{bmatrix} 0.25t^2 + 1 & -0.5 \\ -0.25t^2 & 0.5 \end{bmatrix}$$

$$\Phi(t, t_0) = X(t)\, X^{-1}(t_0) = \begin{bmatrix} 1 & 0 \\ 0.5(t^2 - t_0{}^2) & 1 \end{bmatrix}$$

Solution of LTV Equation

Claim:

$$x(t) = \Phi(t,t_0)x_0 + \int_{t_0}^{t} \Phi(t,\tau)B(\tau)u(\tau)d\tau$$

Pf.)

$$x(t_0) = x_0$$

$$\begin{aligned}
\dot{x} &= \frac{\partial}{\partial t}\Phi(t,t_0)x_0 + \frac{\partial}{\partial t}\int_{t_0}^{t} \Phi(t,\tau)B(\tau)u(\tau)d\tau \\
&= A(t)\Phi(t,t_0)x_0 + \Phi(t,t)B(t)u(t) + \int_{t_0}^{t} A(t)\Phi(t,\tau)B(\tau)u(\tau)d\tau \\
&= A(t)[\Phi(t,t_0)x_0 + \int_{t_0}^{t} \Phi(t,\tau)B(\tau)u(\tau)d\tau] + B(t)u(t) \\
&= A(t)x(t) + B(t)u(t)
\end{aligned}$$

Solution of LTV Equation

Zero-input

$$y(t) = C(t)\Phi(t,t_0)x(t_0)$$

Zero-state

$$y(t) = \int_{t_0}^{t} C(t)\Phi(t,\tau)B(\tau)u(\tau)d\tau + D(t)u(t)$$

$$= \int_{t_0}^{t} [C(t)\Phi(t,\tau)B(\tau)u(\tau) + D(t)\delta(t-\tau)]u(\tau)d\tau$$

$$= \int_{t_0}^{t} G(t,\tau)u(\tau)d\tau$$

Impulse Response

$$G(t,\tau) = C(t)\Phi(t,\tau)B(\tau) + D(t)\delta(t-\tau)$$

$$= C(t)X(t)X^{-1}(\tau)B(\tau) + D(t)\delta(t-\tau)$$

Solution of LTV Equation

If $A(t)$ is commutative (diagonal or constant), i.e,

$$A(t)\left(\int_{t_0}^{t} A(\tau)d\tau\right) = \left(\int_{t_0}^{t} A(\tau)d\tau\right)A(t)$$

$$\Phi(t,t_0) = e^{\int_{t_0}^{t} A(\tau)d\tau} = \sum_{k=0}^{\infty} \frac{1}{k!}\left(\int_{t_0}^{t} A(\tau)d\tau\right)^k$$

If A is constant,

$$\Phi(t,t_0) = e^{A(t-t_0)},\ X(t) = e^{At}$$

Solution of Discrete-time LTV Equation

Discrete-Time Case

$$x[k+1] = A[k]x[k] + B[k]u[k]$$
$$y[k] = C[k]x[k] + D[k]u[k]$$

State transition matrix

$$\Phi[k+1, k_0] = A[k]\Phi[k, k_0], \;\; \Phi[k_0, k_0] = \mathbf{I}$$
$$\Phi[k, k_0] = A[k-1]A[k-2]\cdots A[k_0]$$
$$x[k_0+1] = A[k_0]x[k_0] + B[k_0]u[k_0]$$

Solution of Discrete-time LTV Equation

$$
\begin{aligned}
x[k_0+2] &= A[k_0+1]x[k_0+1] + B[k_0+1]u[k_0+1] \\
&= A[k_0+1]A[k_0]x[k_0] + A[k_0+1]B[k_0]u[k_0] \\
&\qquad + B[k_0+1]u[k_0+1] \\
&\vdots \\
x[k] &= A[k-1]\ldots A[k_0]x[k_0] + A[k-1]\ldots A[k_0+1]B[k_0]u[k_0]\ldots \\
&\quad + A[k-1]B[k-2]u[k-2] + B[k-1]u[k-1] \\
&= \Phi[k,k_0]x[k_0] + \sum_{m=k_0}^{k-1}\Phi[k,m+1]B[m]u[m]
\end{aligned}
$$

$$
y[k] = C[k]\Phi[k,k_0]x[k_0] + C[k]\sum_{m=k_0}^{k-1}\Phi[k,m+1]B[m]u[m] + D[k]u[k]
$$

$$
G[k,m] = C[k]\Phi[k,m+1]B[m] + D[m]\delta[k-m] : \text{ Impulse Response}
$$

Equivalent Time-varying Equations

Equivalent Time-varying Equations

Let $\bar{x} = P(t)x(t)$: $P(t)$ is called equivalence transformation

$$\{A(t), B(t), C(t), D(t)\} \leftrightarrow \{\bar{A}(t), \bar{B}(t), \bar{C}(t), \bar{D}(t)\}$$

equivalent

if it satisfies

$$\bar{A}(t) = [P(t)A(t) + \dot{P}(t)]P^{-1}(t)$$

$$\bar{B}(t) = P(t)B(t)$$

$$\bar{C}(t) = C(t)P(t)^{-1}$$

$$\bar{D}(t) = D(t)$$

under the assumption that

$P(t)$: nonsingular and $P(t)$ & $\dot{P}(t)$ are continueus for all t.

Equivalent Time-varying Equations

Verify :

$$\begin{aligned}\dot{\bar{x}} &= \dot{P}(t)x + P(t)\dot{x}(t) \\ &= \dot{P}(t)x + P(t)(A(t)x(t) + B(t)u(t)) \\ &= (\dot{P}(t) + P(t)A(t))P^{-1}(t)\bar{x}(t) + P(t)B(t)u(t) \\ &= \bar{A}(t)\bar{x}(t) + \bar{B}(t)u(t)\end{aligned}$$

Claim: $\bar{X}(t) = P(t)X(t)$ is fundamental matrix

Pf.)

$$\begin{aligned}\dot{\bar{X}} &= \dot{P}(t)X + P(t)\dot{X}(t) \\ &= \dot{P}(t)X + P(t)A(t)X(t) \\ &= (\dot{P}(t) + P(t)A(t))P^{-1}(t)\bar{X}(t) \\ &= \bar{A}(t)\bar{X}(t)\end{aligned}$$

Equivalent Time-varying Equations

Theorem

Let A_0 be an arbitrary constant. Then there exists an equivalence transformation For $\bar{A}(t) = A_0$.

Pf.)

$$X^{-1}X = \mathbf{I}$$

$$\dot{X}^{-1}X + X^{-1}\dot{X} = 0$$

$$\Rightarrow \dot{X}^{-1} = -X^{-1}\dot{X}X^{-1} = -X^{-1}A(t)$$

$$\bar{A}(t) = A_0$$

$$\bar{X}(t) = e^{A_0 t}$$

$$\bar{X}(t) = P(t)X(t) \rightarrow P(t) = \bar{X}(t)X(t)^{-1} = e^{A_0 t}X(t)^{-1}$$

$$\bar{A}(t) = [P(t)A(t) + \dot{P}(t)]P^{-1}(t)$$

$$= [P(t)A(t) + A_0 e^{A_0 t}X(t)^{-1} - e^{A_0 t}X^{-1}A(t)]P^{-1}(t)$$

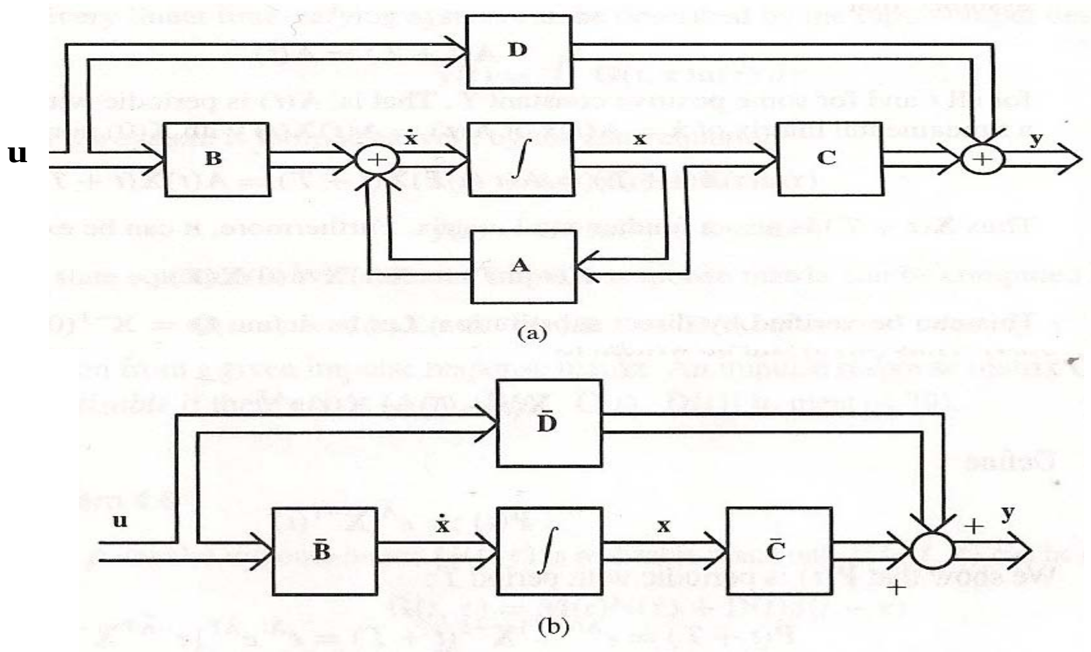
$$= A_0$$

Equivalent Time-varying Equations

If $A_0 = 0$

$$P(t) = X(t)^{-1}$$

$$\bar{A} = 0, \bar{B}(t) = X(t)^{-1} B(t), \bar{C}(t) = C(t) X(t), \bar{D}(t) = D$$



Equivalent Time-varying Equations

Definition

$P(t)$ is called a Lyapunov transformation if
$P(t)$ is nonsingular and
$P(t)$ & $P^{-1}(t)$ are continuous & bounded.

Then, $\bar{x} = P(t)x(t)$

$$\{A, B, C, D\} \leftrightarrow \{\bar{A}, \bar{B}, \bar{C}, \bar{D}\}$$

Lyapunov equivalent

Note) If $P(t)$ is Lyapunov transformation,
so is $P^{-1}(t)$.

Note) Lyapunov transformation preserves stability.

If LTI Case : equivalence transformation is always Lyapunov Tr.

If $P(t)$ should be Lyapunov Tr., it may not transformed into constant A_0,
however, if $A(t)$ is periodic, this is true.

Equivalent Time-varying Equations

Verify

Assume $A(t+T) = A(t)$ for all t .

Let $X(t)$ be fundamental matrix,

then $X(t+T)$ is also fundamental matrix

$(\because \dot{X}(t+T) = A(t+T)X(t+T) = A(t)X(t+T)).$

Furthermore it can be expressed as

$$X(t+T) = X(t)Q, \text{ where } Q \text{ is nonsingular matrix,}$$

$(\because \dot{X}(t+T) = \dot{X}(t)Q = A(t)X(t)Q = A(t)X(t+T)).$

$$\Rightarrow \quad X(t+T) = X(t)e^{\bar{A}T} \; (\Leftarrow \exists \cdot \bar{A} \ni Q = e^{\bar{A}T} \leftarrow \text{Problem 3.24})$$

Define $P(t) = e^{\bar{A}t}X^{-1}(t) (\Rightarrow \text{transformation to constant } \bar{A})$

$$P(t+T) = e^{\bar{A}(t+T)}X^{-1}(t+T) = e^{\bar{A}t}e^{\bar{A}T}e^{-\bar{A}T}X^{-1}(t) = P(t)$$

$\Rightarrow P(t)$: periodic $\Rightarrow$ bounded $\Rightarrow$ so is $\dot{P}(t)$

$\Rightarrow$ Lyapunov Transformation.

Equivalent Time-varying Equations

Theorem

Assume $A(t) = A(t+T)$ for all t, $X(t)$ be fundamental matrix.

Then $P(t) = e^{\bar{A}t} X^{-1}(t)$ is Lyapunov transformation that yields

Lyapunov equivalent equation of

$$\dot{\bar{x}}(t) = \bar{A}\bar{x}(t) + P(t)B(t)u(t)$$

$$y(t) = C(t)P^{-1}(t)\bar{x}(t) + D(t)u(t).$$

Note: The homogeneous part of the Theorem is called the **Theory of Floquet**.

Time-varying Realizations

Time varying realization

$$\mathbf{G}(t,\tau) \rightarrow \{A(t), B(t), C(t), D(t)\}$$

realization

$$\mathbf{G}(t,\tau) = C(t)X(t)X^{-1}(\tau)B(\tau) + D(t)\delta(t-\tau)$$

Theorem

$\mathbf{G}(t,\tau)$ is realizable iff it can be decomposed into

$$\mathbf{G}(t,\tau) = M(t)N(\tau) + D(t)\delta(t-\tau).$$

Time-varying Realizations

Pf.)

$$(Sufficiency)\ \ M(t) = C(t)X(t)$$

$$N(\tau) = X^{-1}(\tau)B(\tau)$$

$$\dot{x} = N(t)u(t)$$

$$y(t) = M(t)x(t) + D(t)u(t)$$

$$\dot{x} = 0x(t) \Rightarrow X(t) = \mathbf{I}$$

$$x(t) = \int_0^t \mathbf{I}\cdot\mathbf{I}^{-1}N(\tau)u(\tau)d\tau$$

$$y(t) = \int_0^t (M(t)\cdot\mathbf{I}\cdot\mathbf{I}^{-1}N(\tau) + D(t)\delta(t-\tau))u(\tau)d\tau$$

$$\mathbf{G}(t,\tau) = M(t)\cdot\mathbf{I}\cdot\mathbf{I}^{-1}N(\tau) + D(t)\delta(t-\tau)$$

Time-varying Realizations

Example

Consider $g(t) = te^{\lambda t}$

$$g(t,\tau) = g(t-\tau) = (t-\tau)e^{\lambda(t-\tau)}$$

$$= [e^{\lambda t}\ \ te^{\lambda t}]\begin{bmatrix} -\tau e^{-\lambda\tau} \\ e^{-\lambda\tau} \end{bmatrix}$$

Laplace transform of $g(t)$

$$L[g(t)] = \frac{1}{s^2 - 2\lambda s + \lambda^2}$$

Time varying eq.

$$\dot{x} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} -te^{-\lambda t} \\ e^{-\lambda t} \end{bmatrix} u$$

$$y = [e^{\lambda t}\ \ te^{\lambda t}]x.$$

Time invariant eq.

$$\dot{x} = \begin{bmatrix} 2\lambda & -\lambda^2 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$y = [0\ 1]x.$$

HW4-2

Problem 4.16, p. 119 in the Text

Summary

Solution of Linear Systems $\dot{x} = Ax + Bu$

$$x(t) = \Phi(t,t_0)x_0 + \int_{t_0}^{t} \Phi(t,\tau)B(\tau)u(\tau)d\tau \cdots (*) \leftarrow LTV + LTI$$

$$x(t) = e^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau \leftarrow LTI$$

$\Phi(t,t_0) = X(t)X^{-1}(t_0)$: state transition matrix

$X(t) = [x_1(t),\ x_2(t),\ \ldots, x_n(t)]$: fundamental matrix

$x_i(t)$: LI solutions of $\dot{x} = Ax + Bu$.

Expected problem for exam:

Show that (*) is the solution of $\dot{x} = Ax + Bu$.

Find the solution of $\dot{x} = Ax + Bu$, where $A = \ldots$, $B = \ldots$

Summary

Solution of Discrete-time Linear Systems

$$x[k+1] = A[k]x[k] + B[k]u[k]$$

$$x[k] = \Phi[k,k_0]x[k_0] + \sum_{m=k_0}^{k-1} \Phi[k,m+1]B[m]u[m]$$

$$\Phi[k,k_0] = A[k-1]\ldots A[k_0]: \text{ state transition matrix}$$

$$= A^{k-k_0} \quad \text{for LTI.}$$

Summary

Equivalence

$$\dot{x} = Ax + Bu \qquad \bar{x} = Px \qquad \dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u$$

$$y = Cx + Du \qquad\longleftrightarrow\qquad y = \bar{C}\bar{x} + \bar{D}u$$

$$\bar{A} = [PA + \dot{P}]P^{-1},\ \dot{P} = 0 \text{ for LTI,}$$

$$\bar{B} = PB,\ \bar{C} = CP^{-1},\ \bar{D} = D$$

$$A(t) \longleftrightarrow \bar{A} = A_0 : \ \text{constant}$$

$$P(t) = e^{A_0 t} X^{-1}(t)$$

$$P(t)\ \&\ P^{-1}(t) : \text{nonsingular, continuous, bounded}$$

$$\rightarrow P(t)\ \&\ P^{-1}(t) : \text{Lyapunov transformation}$$

Zero State Equivalence

$$D + C(s\mathbf{I} - A)^{-1}B = \bar{D} + \bar{C}(s\mathbf{I} - \bar{A})^{-1}\bar{B}$$

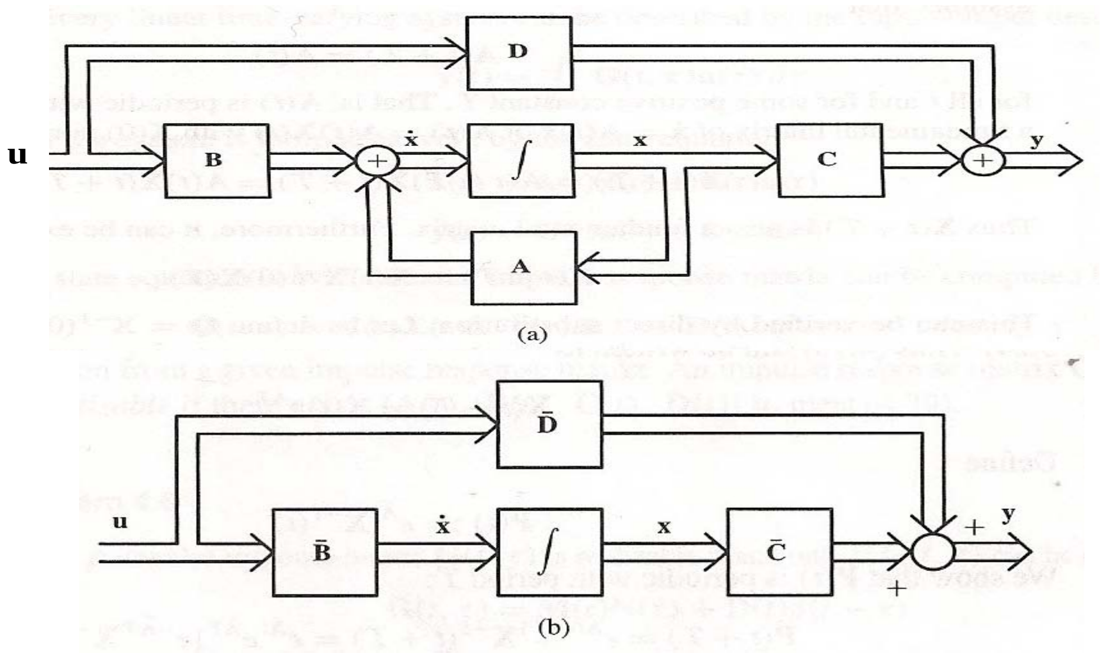
$$D = \bar{D}\ \&\ CA^m B = \overline{CA}^m \bar{B},\ m = 0, 1, 2, \cdots$$

Summary

If $A_0 = 0$

$$P(t) = X(t)^{-1}$$

$$\bar{A} = 0, \bar{B}(t) = X(t)^{-1} B(t), \bar{C}(t) = C(t) X(t), \bar{D}(t) = D$$



Summary

Realization

$\mathbf{G}(s)$ is a proper rational matrix.

$\Leftrightarrow$

$\exists$ a realization {A,B,C,D} such that $\mathbf{G}(s) = C'(s\mathbf{I} - A)^{-1}B + D.$

For example

$$\dot{x} = \begin{bmatrix} -\alpha_1\mathbf{I}_p & \cdots - \alpha_r\mathbf{I}_p \\ \mathbf{I}_p & 0 \\ 0 \quad \ddots & \mathbf{I}_p \end{bmatrix} x + \begin{bmatrix} \mathbf{I}_p \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} N_1 & N_2 & \cdots N_r \end{bmatrix} x + \mathbf{G}(\infty)u$$

is a realization of $\mathbf{G}(s)$. Here,

$\mathbf{G}(s) = \mathbf{G}(\infty) + \mathbf{G}_{sp}(s).$

$$\mathbf{G}_{sp}(s) = \frac{1}{d(s)}\left[N_1 s^{r-1} + N_2 s^{r-2} + \qquad + N_r \right]$$

$$d(s) = s^r + \alpha_1 s^{r-1} + \cdots + \alpha_r.$$