3. Linear Spaces and Linear Operators

- ✓ Linear Spaces
- ✓ Basis and Representation
- ✓ Linear Operators
- ✓ Similarity Transform
- ✓ Functions of Square Matrix
- ✓ Lyapunov Equation
- ✓ Useful Formulas
- ✓ Matrix Properties

Linear Spaces

Definition: Field $\mathcal{F}$ is a set of scalars and over $\mathcal{F}$, addition, multiplication are defined such that they satisfy

1) $\alpha+\beta\in\mathcal{F},\ \alpha\beta\in\mathcal{F},\ \forall\alpha,\beta\in\mathcal{F}$
2) Commutative:
   $\alpha+\beta=\beta+\alpha,\ \alpha\beta=\beta\alpha,\ \forall\alpha,\beta\in\mathcal{F}$
3) Associative:
   $(\alpha+\beta)+\gamma=\alpha+(\beta+\gamma),\ (\alpha\beta)\gamma=\alpha(\beta\gamma),\ \forall\alpha,\beta,\gamma\in\mathcal{F}$
4) Distributive: $\alpha(\beta+\gamma)=\alpha\beta+\alpha\gamma),\ \forall\alpha,\beta,\gamma\in\mathcal{F}$
5) $\exists$ Identity i.e., $0\in\mathcal{F},\ 1\in\mathcal{F}$ such that
   $\alpha+0=\alpha,\ 1\cdot\alpha=\alpha,\ \forall\alpha\in\mathcal{F}$
6) $\exists$ Additive Inverse $\beta\in\mathcal{F}$ such that $\alpha+\beta=0,\ \forall\alpha\in\mathcal{F}$
7) $\exists$ Multiplicative Inverse $\gamma\in\mathcal{F}$ such that $\alpha\gamma=1,\ \forall\alpha\in\mathcal{F}$

Linear Spaces

Example

- Binary Field {0,1} with operations of
  addition: 0+0=1+1=0, 1+0=1;
  multiplication: 0*1=0*0=0, 1*1=1.

- Positive Real is not Field because of no additive inverse

Linear Spaces

Definition: Linear Space over a Field $\mathcal{F}$: $(X, \mathcal{F})$

$(X, \mathcal{F})$ consists of a set X of vectors, a Field $\mathcal{F}$, two operations of vector addition and scalar multiplication satisfying

1) Vector addition: $\mathbf{x}_1 + \mathbf{x}_2 \in X$, $\forall \mathbf{x}_1, \mathbf{x}_2 \in X$
2) Commutative: $\mathbf{x}_1 + \mathbf{x}_2 = \mathbf{x}_2 + \mathbf{x}_1$, $\forall \mathbf{x}_1, \mathbf{x}_2 \in X$
3) Associative: $(\mathbf{x}_1 + \mathbf{x}_2) + \mathbf{x}_3 = \mathbf{x}_1 + (\mathbf{x}_2 + \mathbf{x}_3)$, $\forall \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \in X$
4) $\exists\ \mathbf{0} \in X$ such that $\mathbf{x} + \mathbf{0} = \mathbf{x}$, $\forall \mathbf{x} \in X$
5) $\exists\ \bar{\mathbf{x}} \in X$ such that $\mathbf{x} + \bar{\mathbf{x}} = \mathbf{0}$, $\forall \mathbf{x} \in X$
6) Scalar multiplication: $\alpha \mathbf{x} \in X$, $\forall \mathbf{x} \in X, \forall \alpha \in \mathcal{F}$
7) $\alpha(\beta \mathbf{x}) = \alpha\beta \mathbf{x}$, $\forall \mathbf{x} \in X, \forall \alpha, \beta \in \mathcal{F}$
8) $\alpha(\mathbf{x}_1 + \mathbf{x}_2) = \alpha \mathbf{x}_1 + \alpha \mathbf{x}_2$, $\forall \mathbf{x}_1, \mathbf{x}_2 \in X, \forall \alpha \in \mathcal{F}$
9) $(\alpha + \beta)\mathbf{x} = \alpha \mathbf{x} + \beta \mathbf{x}$, $\forall \mathbf{x} \in X, \forall \alpha, \beta \in \mathcal{F}$
10) $\exists\ 1 \in \mathcal{F}$ such that $1 \cdot \mathbf{x} = \mathbf{x}$, $\forall \mathbf{x} \in X$

Linear Spaces

Example

- **(R, R), (C, C), (C, R) :** Linear Space
- **(R, C):** Not Linear Space because it does not satisfy (6)
- Define $\mathbf{R_n[s]}$ be real coefficient polynomial of **s** with order less than **n,**

  $\mathbf{(R_n[s], R), (R[s], R[s])}$: Linear Space

  $\mathbf{(R_n[s], R[s])}$: Not Linear Space
- $\mathbf{(R^n, R)}$: Linear Space, usually we use $\mathbf{R^n}$

Linear Spaces

Example

X : Sol. Set of homogeneous differential eq.

$$X = \{x \mid \ddot{x} + 2\dot{x} + 3x = 0\}$$
$$\Rightarrow x = \alpha e^{-\lambda_1 t} + \beta e^{-\lambda_2 t} \in X$$
$$\Rightarrow Linear \quad Space$$

X : Sol. Set of Nonhomogeneous differential eq.

$$X = \{x \mid \ddot{x} + 2\dot{x} + 3x = C\}$$
$$\Rightarrow x = \alpha e^{-\lambda_1 t} + \beta e^{-\lambda_2 t} + v(t) \in X, v(t): \text{equal to all sol.s}$$
$$\Rightarrow x_1 + x_2 \notin X$$
$$\Rightarrow Not \quad Linear \quad Space$$

Linear Spaces

Definition: Subspace

If $(X, \mathcal{F})$ is Linear Space, $(Y, \mathcal{F})$ is Linear Space, and $Y \subset X$

then $(Y, \mathcal{F})$ is Subspace of $(X, \mathcal{F})$.

Example

(R^n, R) is Subspace of (C^n, R)

(R^2, R) is Subspace of (R^3, R)

Note)

If $Y \subset X$, 2), 3), 7) - 10) are satisfied, then

only if for LS Y satisfy 1) & 4) - 6),

$(Y, \mathcal{F})$ is Subspace of $(X, \mathcal{F})$.

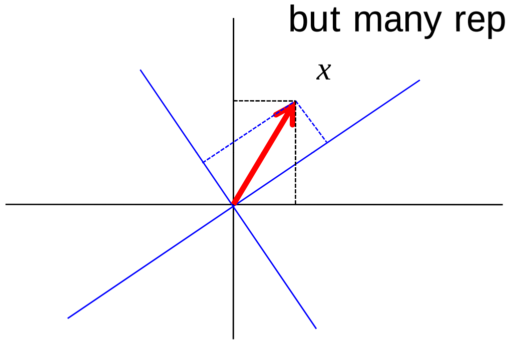
$\Rightarrow$ Only check !!

$$\alpha_1 y_1 + \alpha_2 y_2 \in Y, \quad \forall y_1, y_2 \in Y, \quad \forall \alpha_1, \alpha_2 \in \mathcal{F}$$

Basis and Representation

Linear (Vector) Space

Vector has unique direction and magnitude
but many representations



$$\overline{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\hat{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Basis: Coordinate system for representation

Basis consists of a set of Linearly Independent Vectors

Basis and Representation

Definition: Linearly Independent

A set of $x_1, x_2, \ldots, x_n$ in $(X, \mathcal{F})$ is linearly independent if and only if $\sum_{i=1}^{n} \alpha_i x_i = 0$ implies $\alpha_1 = \alpha_2 = \ldots = \alpha_n = 0$, otherwise, linearly dependent.

Definition: Dimension of Linear Space

Maximum number of linearly independent vectors in LS $(X, \mathcal{F})$

Example

- (R^n, R): n-dimensional vector space
- Functional linear space: the set of all real valued functions

  $(f(t), R)$, $f(t) = \sum_{i=0}^{\infty} \alpha_i t^i$

  Basis: $1, t, t^2, \ldots$ Dimension: infinite

Basis and Representation

Definition: Basis

A set of linearly independent vectors (LIVs) of LS $(X, \mathcal{F})$ is basis if every vectors in X can be expressed as a unique linear combination of these LIVs.

Theorem

In n-dim. LS, any set of n LIVs can be basis.

Note

Let $e_1, e_2, \ldots, e_n \in X$ be basis,
for $x \in X$,

$$x = \sum_{i=1}^{n} e_i \beta_i \text{ (Linear Combination)}$$
$$= [\, e_1, e_2, \ldots, e_n]\beta = E\beta,$$

where $\beta = [\, \beta_1, \beta_2, \ldots, \beta_n]^T \in \mathcal{F}^n$, $E = [\, e_1, e_2, \ldots, e_n]$

Basis and Representation

Definition: Representation

β is called representation of x with respect to the basis $\{e_1, e_2, \ldots, e_n\}$.

Example

In $(R_4[s], R)$,
for $x = 3s^3 + 2s^2 - 2s + 10$,

if basis is $\{s^3, s^2, s, 1\}$,

$$x = [s^3, s^2, s, 1]\begin{bmatrix} 3 \\ 2 \\ -2 \\ 10 \end{bmatrix}$$

if basis is $\{s^3 - s^2, s^2 - s, s - 1, 1\}$,

$$x = [s^3 - s^2, s^2 - s, s - 1, 1]\begin{bmatrix} 3 \\ 5 \\ 3 \\ 13 \end{bmatrix}$$

Basis and Representation

Change of Basis: Various forms of state variable description

$$x = [\, e_1, e_2, \ldots, e_n]\beta = [\, \bar{e}_1, \bar{e}_2, \ldots, \bar{e}_n]\bar{\beta} \qquad (*)$$

$$e_i = [\, \bar{e}_1, \bar{e}_2, \ldots, \bar{e}_n]\begin{bmatrix} p_{1i} \\ p_{2i} \\ \cdots \\ p_{ni} \end{bmatrix} = \bar{E}p_i \quad , \text{ i=1,2, ..., n}$$

$$[\, e_1, e_2, \ldots, e_n] = [\bar{E}p_1 \; \bar{E}p_2 \; \ldots \; \bar{E}p_n] = \bar{E}[\, p_1 \; p_2 \; \ldots \; p_n] = [\, \bar{e}_1, \bar{e}_2, \ldots, \bar{e}_n]P$$

From (*)

$$x = [\, \bar{e}_1, \bar{e}_2, \ldots, \bar{e}_n]P\beta = [\, \bar{e}_1, \bar{e}_2, \ldots, \bar{e}_n]\bar{\beta}$$

$$\Rightarrow \bar{\beta} = P\beta$$

$i -$ th column of P = representation of e_i w.r.t. $\{\bar{e}_i\}$ new basis

Similarly,

$$\Rightarrow \beta = P^{-1}\bar{\beta} = Q\bar{\beta}$$

$i -$ th column of Q = representation of $\bar{e}_i$ w.r.t. $\{e_i\}$

Basis and Representation

Norms of Vectors

Any real valued function of x, $\|x\|$, is defined as a norm if it has the following properties:

$$1.\ \|x\| \geq 0 \ \ \forall x \ \ \& \ \|x\| = 0 \ \text{ iff } \ x = 0$$

$$2.\ \|\alpha x\| = |\alpha| \, \|x\| \ \ \forall \alpha \in R$$

$$3.\ \|x_1 + x_2\| \leq \|x_1\| + \|x_2\| \ \ \forall x_1, x_2$$

$$\text{(Trangular inequality)}$$

$$\|x\|_1 := \sum_{i=1}^{n} |x_i|$$

$$\|x\|_2 := \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2} = \sqrt{x^T x}$$

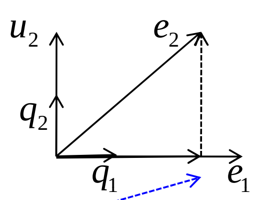
$$\|x\|_\infty := \max_i |x_i|$$

Basis and Representation

Orthonormalization

Othorgonal : $x_i^T x_j \begin{cases} = 0 & \text{if } i \neq j \\ \neq 0 & \text{if } i = j \end{cases}$

Othonormal : $x_i^T x_j \begin{cases} = 0 & \text{if } i \neq j \\ = 1 & \text{if } i = j \end{cases}$



Schmidt Orthonormalization procedure

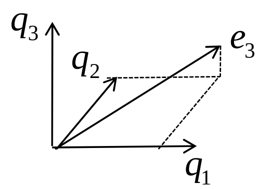
LI vectors $e_1, e_2, \ldots, e_m,$

$u_1 = e_1 \qquad q_1 := u_1 / \|u_1\|$

$u_2 = e_2 - (q_1^T e_2) q_1 \qquad q_2 := u_2 / \|u_2\|$

...

$u_m = e_m - \sum_{k=1}^{m-1} (q_k^T e_m) q_k \qquad q_m := u_m / \|u_m\|$



Linear Operators

Linear Operators, Linear Mappings, Linear Transformations

$$L:(X,F)\rightarrow(Y,F)$$

Definition: A function L is Linear Operator if and only if

$$L(\alpha_1 x_1+\alpha_2 x_2)=\alpha_1 L(x_1)+\alpha_2 L(x_2)\quad \forall x_1,x_2\in X,\ \forall \alpha_1,\alpha_2\in F$$

Example: Convolution integral

$$y(t)=\int_0^t g(t-\tau)u(\tau)d\tau$$

Funeral Rites During 2002 World Cup in Seoul
월드컵 때 장례식장

Linear Operators

Matrix Representation of Linear Operators

$\{x_i\}$: Basis of X
$\{u_i\}$: Basis of Y

$$
\begin{array}{lccc}
\textbf{Operator} & & y = Lx & \\
& \{u_i\} \downarrow & & \downarrow\downarrow \{x_i\} \\
\textbf{Represen.} & & \beta = A\alpha &
\end{array}
$$

Let $y_i = Lx_i$

$$
\begin{array}{ccc}
x_i & \xrightarrow{L} & y_i \\
\{x_i\}\downarrow & & \downarrow\{u_i\} \\
e_i & \xrightarrow{A} & a_i
\end{array}
\quad \Rightarrow \quad
\begin{array}{ccc}
x & \xrightarrow{L} & y \\
\{x_i\}\downarrow & & \downarrow\{u_i\} \\
\alpha & \xrightarrow{A} & \beta
\end{array}
$$

$$A = [a_{1,} a_2, \ldots, a_n]$$

$$a_i = rep.\ of\ y_i (= Lx_i)\ w.r.t.\ \{u_i\}$$

Linear Operators

Matrix Representation of Linear Operators

$$y_i = [u_{1,} u_2, \ldots, u_m] a_i$$

$$
\begin{aligned}
L[x_{1,} x_2, \ldots, x_m] &= [y_{1,} y_2, \ldots, y_m] \\
&= [u_{1,} u_2, \ldots, u_m][a_{1,} a_2, \ldots, a_m] \\
&= [u_{1,} u_2, \ldots, u_m] A
\end{aligned}
$$

From $y = Lx$, L: Unique

$$
\begin{aligned}
[u_{1,} u_2, \ldots, u_m]\beta &= L[x_{1,} x_2, \ldots, x_m]\alpha \\
&= [u_{1,} u_2, \ldots, u_m] A\alpha
\end{aligned}
$$

Hence $\beta = A\alpha$, A: Many depending on $\{x_i\}, \{u_i\}$

Linear Operators

Basis Changes

Operator $\quad x \xrightarrow{L} y \ (= Lx)$

Rep1: basis $[e_1 \ ... \ e_n]$ $\quad \alpha \xrightarrow{A} \beta \ (= A\alpha)$

$$P \downarrow \uparrow Q \qquad P \downarrow \uparrow Q$$

Rep2: basis $[\bar{e}_1 \ ... \ \bar{e}_n]$ $\quad \bar{\alpha} \xrightarrow{\bar{A}} \bar{\beta} \ (= \bar{A}\bar{\alpha})$

a_i : rep. of Le_i w.r.t. $\{e_i\}$

$\bar{a}_i$: rep. of $L\bar{e}_i$ w.r.t. $\{\bar{e}_i\}$

p_i : rep. of e_i w.r.t. $\{\bar{e}_i\}$

q_i : rep. of $\bar{e}_i$ w.r.t. $\{e_i\}$

Linear Operators

Similarity Transform

$$\bar{\alpha} = P\alpha, \quad \bar{\beta} = P\beta = PA\alpha = PAP^{-1}\bar{\alpha}$$

$$\bar{\beta} = \bar{A}\bar{\alpha}$$

$$\Rightarrow \qquad \bar{A} = PAP^{-1} = Q^{-1}AQ$$

Note: $\bar{A}$ & A are similar if there exists a nonsingular P

Linear Operators

Example

$$A=\begin{bmatrix} 3 & 2 & -1 \\ -2 & 1 & 0 \\ 4 & 3 & 1 \end{bmatrix} \qquad b=\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Let new basis be

$\{b,\ Ab,\ A^2b\}$: Linearly independent

Q=$[b\ Ab\ A^2b]$

$$\bar{A}=Q^{-1}AQ=\begin{bmatrix} 0 & 0 & 17 \\ 1 & 0 & -15 \\ 0 & 1 & 5 \end{bmatrix}$$

$$AQ=Q\bar{A}$$

$$[Aq_1\ \ldots\ Aq_n]=[q_1\ \ldots\ q_n]\bar{A}$$

$\bar{a}_i$: rep. of Aq_i w.r.t. $\{q_i\}$

Homework

HW1: Problem 3.1 in Text

Should submit the report within one week after finishing the lecture of this chapter

HW2

Given $A=\begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad b=\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \qquad \bar{b}=\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

What are the representations of A with respect to the basis $\{b,\ Ab,\ A^2b,\ A^3b\}$ and the basis $\{\bar{b},\ A\bar{b},\ A^2\bar{b},\ A^3\bar{b}\}$, respectively?

Drive using the definition of representation.

Linear Operators

Linear Algebraic Equations

$$Ax = y \qquad A: (\mathcal{F}^n, \mathcal{F}) \rightarrow (\mathcal{F}^m, \mathcal{F})$$

Definition: Range Space

$R(A)$={all y for which there is at least one x such that y=Ax}

Theorem:

$R(A)$ is Subspace of $(\mathcal{F}^m, \mathcal{F})$

Linear Operators

Example

$$y = \begin{bmatrix} y_1 \\ y_2 \\ \cdots \\ y_m \end{bmatrix} = Ax = [a_1 \ \ldots \ a_n] \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix} = \sum_i^n x_i a_i$$

$\Rightarrow$ y is spaned from $\{a_i\}$

$\Rightarrow$ Dim. of $R(A)$ is equal to number of LI vectors in $\{a_i\}$

$\Rightarrow$ Dim. of $R(A) = Rank$ of $A \leq m$

Linear Operators

Definition: Null Space

$N(A)$={all x for which $Ax=0$}, dim. of $N(A)$ =n- dim. of $R(A)$

Example

$$
\begin{aligned}
&\text{If}\ \ \text{dim. of } R(A) = n\\
&Ax = 0 \Rightarrow x = 0, \quad N(A) = \{x \mid Ax = 0\} = \{0\}:\ \text{not vector space}\\
&\text{Hence dim. of } N(A) = 0\\
&\text{If}\ \ \text{dim. of } R(A) = k \prec n,\ \exists x \neq 0 \text{ such that } Ax = 0
\end{aligned}
$$

$$
\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} =
\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
\qquad N(A) = \{\begin{bmatrix} 0 \\ 0 \\ x_3 \\ x_4 \end{bmatrix}\}
$$

$$
\begin{aligned}
&\text{dim. of } N(A) = n - k\\
&\Rightarrow \text{dim. of } R(A) + \text{dim. of } N(A) = n
\end{aligned}
$$

Linear Operators

Theorem

Let A: $\mathcal{F}^n \to \mathcal{F}^m$ cf.) $(\mathcal{F}^m, \mathcal{F})$

1. for given y, there exists x such that

$$Ax = y \quad \text{iff} \ \ \rho(A) = \rho([A\ y])$$

2. for all $y \in \mathcal{F}^m$, there exists x such that

$$Ax = y \quad \text{iff} \ \ \rho(A) = m \quad (\text{indefiniteness, } m < n)$$

Linear Operators

Theorem

Let x_p be a solution of $Ax = y$

$k = n - \rho(A) : nullity$

then

$x = x_p + \alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_k v_k$

is a solution of $Ax = y$,

where $\{v_i\}$ is a basis of $N(A)$

Pf.

$Ax_p = y$

$Ax = Ax_p + \sum \alpha_i A v_i = y$

Linear Operators

Theorem

$Ax = y, \ A:$ square

1. If A is nonsingular, $x = A^{-1}y, \ Ax = 0 \Rightarrow x = 0$

2. Iff A is singular, $Ax = 0$ has nonzero sol's.

Number of LI sol's is nullity of A.

Similarity Transform

Characteristic Polynomial

Eigenvalue λ
Eigenvector x (*non zero*)
$\exists\ x$ such that $Ax = \lambda x$
$\Rightarrow (A-\lambda I)x = 0 \Rightarrow$ *nullity* of $(A-\lambda I) \geq 1$
$\Rightarrow (A-\lambda I)$ is singular $\Rightarrow \Delta(\lambda) = \det(A-\lambda I) = 0$
$\Delta(\lambda)$ is called characteristic polynomial of A

Example

Companion form

$$\begin{bmatrix} 0 & 0 & 0 & -a_4 \\ 1 & 0 & 0 & -a_3 \\ 0 & 1 & 0 & -a_2 \\ 0 & 0 & 1 & -a_1 \end{bmatrix} \quad \Delta(\lambda) = \lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4$$

Similarity Transform

Similarity Transform to Jordan form

case 1: distinct eigenvalues

Theorem

Let $\lambda_i,\ i = 1,\ldots,n,$ be distinct eigenvalues,

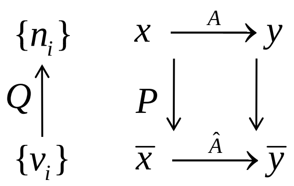
then eigenvectors $v_i,\ i = 1,\ldots,n,$ are linearly independent.

$\{v_i\}$ can be basis.

Similarity Transform

Similarity Transform to Jordan form

case 1: distinct eigenvalues



q_i : rep. of v_i w.r.t. $\{n_i\}$

$v_i = [n_1 \; \dots \; n_i \dots n_n] q_i = q_i$

$Q = [v_1 \; \dots \; v_i \dots v_n] \Rightarrow \hat{A} = Q^{-1} A Q = P A P^{-1}$

$\hat{a}_i$: rep. of $A v_i$ w.r.t. $\{v_i\}$

$$A v_i = \lambda_i v_i = [v_1 \; \dots \; v_i \dots v_n] \begin{bmatrix} 0 \\ \dots \\ \lambda_i \\ 0 \end{bmatrix} \Rightarrow \hat{A} = \begin{bmatrix} \lambda_1 & & & \\ & \dots & & \\ & & \lambda_i & \\ & & & \lambda_n \end{bmatrix}$$

P 56, 57, 58

Similarity Transform

Example

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix} \rightarrow \Delta(\lambda) = \det(\lambda \mathrm{I} - A) = (\lambda - 2)(\lambda + 1)\lambda$$

$$(A - 2\mathrm{I})\mathrm{q}_1 = \begin{bmatrix} -2 & 0 & 0 \\ 1 & -2 & 2 \\ 0 & 1 & -1 \end{bmatrix} \mathrm{q}_1 = 0 \rightarrow \mathrm{q}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$(A + \mathrm{I})\mathrm{q}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \mathrm{q}_2 = 0 \rightarrow \mathrm{q}_2 = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}, \; A\mathrm{q}_3 = 0 \rightarrow \mathrm{q}_3 = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

$$Q = \begin{bmatrix} 0 & 0 & 2 \\ 1 & -2 & 1 \\ 1 & 1 & -1 \end{bmatrix} \Rightarrow \hat{A} = Q^{-1} A Q = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Similarity Transform

Similarity Transform to Jordan form

case 2: not all distinct eigenvalues

Definition: Generalized eigenvector v of grade k iff

$$(A-\lambda I)^k v = 0 \text{ and } (A-\lambda I)^{k-1} v \neq 0$$

Example

$$A = \begin{bmatrix} \lambda & 1 & 1 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix} \Rightarrow (A-\lambda I) = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow (A-\lambda I)^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow (A-\lambda I)^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow v = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T \text{ is generalized eigenvector of grade 3.}$$

Similarity Transform

Derivation of Basis

$$\begin{aligned} v_k &:= v \\ v_{k-1} &:= (A-\lambda I)v = (A-\lambda I)v_k \\ v_{k-2} &:= (A-\lambda I)^2 v = (A-\lambda I)v_{k-1} \\ &\dots \\ v_1 &:= (A-\lambda I)^{k-1} v = (A-\lambda I)v_2 \\ \{v_i\} &:= \text{Chain of generalized eigenvectors} \end{aligned}$$

$$\hat{a}_i : \text{ rep.of } Av_i \text{ w.r.t. } \{v_i\}$$

$$\begin{aligned} (A-\lambda I)v_i &= v_{i-1} \\ Av_i &= \lambda v_i + v_{i-1} \end{aligned}$$

Jordan Block

$$= [v_1 \;\dots v_{i-1} \; v_i \;\dots v_n] \begin{bmatrix} 0 \\ 1 \\ \lambda \\ 0 \end{bmatrix} \rightarrow \hat{A} = \begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \lambda \end{bmatrix}$$

Similarity Transform

How to find generalized eigenvectors?

$$\det(sI - A) = (s-\lambda_1)(s-\lambda_2)(s-\lambda_3)^8 = 0$$
$$s = \lambda_1,\ \lambda_2,\ \lambda_3 (8 \text{ multiple roots})$$
$$(A-\lambda_1 I)x_1 = 0 \Rightarrow x_1 \text{ can be basis}$$
$$(A-\lambda_2 I)x_2 = 0 \Rightarrow x_2 \text{ can be basis}$$

8 generalized eigenvectors for λ_3

$$\rho(A-\lambda_3 I)^0 = 10,\ \nu_0 = 0$$
$$\rho(A-\lambda_3 I)^1 = 7,\ \nu_0 = 3,\ u_1\ w_1\ v_1$$
$$\rho(A-\lambda_3 I)^2 = 4,\ \nu_0 = 6,\ u_2\ w_2\ v_2$$
$$\rho(A-\lambda_3 I)^3 = 3,\ \nu_0 = 7,\ u_3$$
$$\rho(A-\lambda_3 I)^4 = 2,\ \nu_0 = 8,\ u_4$$
$$\rho(A-\lambda_3 I)^5 = 2,\ \nu_0 = 8$$

$\exists\ u \neq 0$ such that

$$(A-\lambda_3 I)^3 u \neq 0$$
$$(A-\lambda_3 I)^4 u = 0$$

There is 4 chains $\{u_1, u_2, u_3, u_4\}$

$\exists$ two w(or v) $\neq 0$ and such that

$$(A-\lambda_3 I)\ w(\text{or } v) \neq 0$$
$$(A-\lambda_3 I)^2 w(\text{or } v) = 0$$

There is 2 chains for each w(or v).
$\{w_1, w_2, v_1, v_2\}$

Similarity Transform

How to find generalized eigenvectors?

$$\det(sI - A) = (s-\lambda_1)(s-\lambda_2)(s-\lambda_3)^8 = 0$$
$$s = \lambda_1,\ \lambda_2,\ \lambda_3 (8 \text{ multiple roots})$$
$$(A-\lambda_1 I)x_1 = 0 \Rightarrow x_1 \text{ can be basis}$$
$$(A-\lambda_2 I)x_2 = 0 \Rightarrow x_2 \text{ can be basis}$$

8 generalized eigenvectors for λ_3

$$(A-\lambda_3 I)^4 u = 0,\ u_4 = u$$
$$(A-\lambda_3 I)^3 u_3 = 0,\ u_3 = (A-\lambda_3 I)u \neq 0$$
$$(A-\lambda_3 I)^2 u_2 = 0,\ u_2 = (A-\lambda_3 I)^2 u \neq 0$$
$$(A-\lambda_3 I)^1 u_1 = 0,\ u_1 = (A-\lambda_3 I)^3 u \neq 0$$

In similar way,
$\{w_1, w_2, v_1, v_2\}$ can be obtained.

$$\rho(A-\lambda_3 I)^0 = 10,\ \nu_0 = 0$$
$$\rho(A-\lambda_3 I)^1 = 7,\ \nu_0 = 3,\ u_1\ w_1\ v_1$$
$$\rho(A-\lambda_3 I)^2 = 4,\ \nu_0 = 6,\ u_2\ w_2\ v_2$$
$$\rho(A-\lambda_3 I)^3 = 3,\ \nu_0 = 7,\ u_3$$
$$\rho(A-\lambda_3 I)^4 = 2,\ \nu_0 = 8,\ u_4$$
$$\rho(A-\lambda_3 I)^5 = 2,\ \nu_0 = 8$$

Similarity Transform

How to find generalized eigenvectors?

$$Q = [x_1 \ x_2 \ w_1 \ w_2 \ v_1 \ v_2 \ u_1 \ u_2 \ u_3 \ u_4 \ ]$$

$$\hat{A} = Q^{-1}AQ$$

$$\hat{A} = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda_3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_3 \end{bmatrix}$$

Jordan Block

HW3

Transform the following matrix to Jordan form

$$A = \begin{bmatrix} 3 & -1 & 1 & 1 & 0 & 0 \\ 1 & 1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Motivations

Linear algebra for linear time invariant systems
Linear space and operator theory for linear time varying system
Stability for linear time invariant systems
General definition and Theorem on stability for general systems

Repetitive & tedious training is required for learning of language, mathematics, skill, mind control, sports, ...

Mathematics is useful for analysis, writing a paper, proof, ...

Overcoming of tedious training phase must give you freedom in the future.

Functions of Square Matrix

Square Matrix $A, \quad A^k := AA \cdots A$

Let $f(\lambda)$ be a ploynomial

$$f(\lambda) = \lambda^3 + 2\lambda^2 - 6$$
$$f(A) = A^3 + 2A^2 - 6I \Leftarrow \text{ Ploynomial of } A$$

$$A = Q^{-1}\bar{A}Q$$
$$A^k = Q^{-1}\bar{A}Q\ Q^{-1}\bar{A}Q ... = Q^{-1}\bar{A}^k Q$$
$$\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}^k = \begin{bmatrix} A_1^k & 0 \\ 0 & A_2^k \end{bmatrix}$$

Definition

Minimal polynomial of A is defined as monic polynomial $f(\lambda)$ of least degree such that $f(A) = 0$.

Functions of Square Matrix

Definition

Largest order of Jordan blocks for λ_i is index of λ_i in A

Theorem

Minimal polynomial of A is $f(\lambda)=\prod_{i=1}^{m}(\lambda-\lambda_i)^{\bar{n}_i}$
where $\bar{n}_i$ is index of λ_i in A.

Ex.) Charcteristic poly. $\Delta(\lambda)=(\lambda-3)^3(\lambda-1)$

$$\begin{bmatrix}3&0&0&0\\0&3&0&0\\0&0&3&0\\0&0&0&1\end{bmatrix}\quad\begin{bmatrix}3&1&0&0\\0&3&0&0\\0&0&3&0\\0&0&0&1\end{bmatrix}\quad\begin{bmatrix}3&1&0&0\\0&3&1&0\\0&0&3&0\\0&0&0&1\end{bmatrix}$$

$(\lambda-3)(\lambda-1)\quad(\lambda-3)^2(\lambda-1)\quad(\lambda-3)^3(\lambda-1):$ min. poly.

Functions of Square Matrix

Cayley-Hamilton Theorem

$$\Delta(\lambda)=\det(A-\lambda I):=\lambda^n+\alpha_1\lambda^{n-1}+\ldots+\alpha_n$$
$$\Rightarrow \Delta(A)=0$$

Remark:

$$\Delta(\lambda)=\varphi(\lambda)h(\lambda),\ \ \varphi(\lambda):\ \text{minimal polynomial}$$
$$\Rightarrow \Delta(A)=\varphi(A)h(A)=0\cdot h(A)=0$$

Functions of Square Matrix

Calculation of Function of Square Matrix $f(A)$

$$f(\lambda)=\Delta(\lambda)g(\lambda)+h(\lambda),\ \ order\ of\ h(\lambda)\ is\ n-1$$

$$f(A)=\Delta(A)g(A)+h(A)=h(A)$$

$$=\beta_{n-1}A^{n-1}+\beta_{n-2}A^{n-2}+\ldots+\beta_0 I$$

Example

Compute A^{100}, where $A=\begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}$

Minimal polynormal

$$\varphi(\lambda)=(\lambda-1)^2,\ \ \varphi(A)=(A-I)^2=0$$

$$f(\lambda)=\lambda^{100}=(\lambda-1)^2 g(\lambda)+\beta_1+\beta_2\lambda$$

$$f(1)=\beta_1+\beta_2=1$$

$$f'(1)=\beta_2=100\rightarrow\beta_1=-99$$

$$f(A)=A^{100}=\beta_1 I+\beta_2 A=\begin{bmatrix}1 & 200\\ 0 & 1\end{bmatrix}$$

Functions of Square Matrix

Example

Compute e^{At}, where $A=\begin{bmatrix}0 & 0 & -2\\ 0 & 1 & 0\\ 1 & 0 & 3\end{bmatrix}$

$$\Delta(\lambda)=(\lambda-1)^2(\lambda-2),$$

$$f(\lambda)=e^{\lambda t}=(\lambda-1)^2(\lambda-2)g(\lambda)+\beta_1+\beta_2\lambda+\beta_3\lambda^2$$

$$f(1)=\beta_1+\beta_2+\beta_3=e^t$$

$$f'(1)=\beta_2+2\beta_3=te^t$$

$$f(2)=\beta_1+2\beta_2+4\beta_3=e^{2t}$$

$$f(A)=e^{At}=\beta_1 I+\beta_2 A+\beta_3 A^2=\begin{bmatrix}2e^t-e^{2t} & 0 & 2e^t-2e^{2t}\\ 0 & e^t & 0\\ -e^t+e^{2t} & -te^t & -e^t+2e^{2t}\end{bmatrix}$$

Functions of Square Matrix

Theorem

For given $f(\lambda)$ and an $n\times n$ matrix A with characteristic polynomial

$$\Delta(\lambda)=\prod_{i=1}^{m}(\lambda-\lambda_i)^{n_i},$$

where $n=\sum_{i=1}^{m} n_i$.

$$f(\lambda)=\Delta(\lambda)g(\lambda)+h(\lambda)$$
$$f^{(l)}(\lambda_i)=h^{(l)}(\lambda_i),\ \ l=1,2,\ldots,n_i-1$$

where $f^{(l)}(\lambda_i)=\left.\dfrac{d^l f(\lambda)}{d\lambda^l}\right|_{\lambda=\lambda_i}$.

Then

$$f(A)=h(A)$$

and $h(\lambda)$ is said to equal to $f(\lambda)$ on the Spectrum of A.

Functions of Square Matrix

Matrix function based on Power Series

For given $f(\lambda)$ and an $n\times n$ matrix A,

$$e^{\lambda t}=1+\lambda t+\frac{1}{2!}\lambda^2t^2+\cdots=\sum_{k=0}^{\infty}\frac{1}{k!}\lambda^k t^k$$
$$e^{At}=\sum_{k=0}^{\infty}\frac{1}{k!}t^kA^k$$

Laplace Transform of e^{At}

$$\mathcal{L}\left(\frac{1}{k!}t^k\right)=s^{-(k+1)}$$
$$\mathcal{L}(e^{At})=\sum_{k=0}^{\infty}s^{-(k+1)}A^k=s^{-1}\sum_{k=0}^{\infty}(s^{-1}A)^k$$
$$\sum_{k=0}^{\infty}(s^{-1}\lambda)^k=\frac{1}{1-s^{-1}\lambda},\ \ \text{for } |s^{-1}\lambda|<1$$
$$\mathcal{L}(e^{At})=s^{-1}(I-s^{-1}A)^{-1}=(sI-A)^{-1}$$

Review

Linear Operator

$$L : R^n \to R^m$$

$$L(x) = y,\ \ x \in R^n,\ \ y \in R^m$$

Matrix Representation

$$Ax = y,\ \ x \in R^n,\ \ y \in R^m$$

Range Space

$$R(A) = \{y \mid Ax = y\}$$

$$y = \sum_{i=1}^{n} x_i \mathbf{a}_i$$

$$\to \rho R(A) = \#\ \text{of LI vectors in } \{\mathbf{a}_i\}$$

Null Space

$$N(A) = \{x \mid Ax = 0\}$$

$$\rho N(A) = n - \rho R(A)$$

$Ax = \lambda x$: eigenvector

can be basis of range space

$Ax = 0$: null vector

can be basis of null space

$\to$ Diagonal form

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Review

Simple Example

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ \ x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},\ \ y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$Ax = \begin{bmatrix} x_1 \\ 0 \end{bmatrix} = y = \begin{bmatrix} * \\ 0 \end{bmatrix} \to \rho\{y\} = \rho R(A) = 1$$

$$Ax = 0 \to \{\begin{bmatrix} 0 \\ * \\ * \end{bmatrix}\} = N(A) \to \rho N(A) = 2$$

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Lyapunov Equation

Lyapunov Equation

Problem to find $M \in R^{n\times m}$ satisfying the Lyapunov equation

$AM + MB = C,$

for given $A \in R^{n\times n}, \ B \in R^{m\times m}, \ C \in R^{n\times m}$.

Conversion to Linear Equation

For $A \in R^{3\times 3}, \ B \in R^{2\times 2}$

$$\underbrace{\begin{bmatrix} a_{11}+b_{11} & a_{12} & \cdot & & \\ a_{21} & \cdot & & & \\ & & \cdot & b_{21} & \\ & & & & a_{33}+b_{22} \end{bmatrix}}_{6\times 6} \underbrace{\begin{bmatrix} m_{11} \\ m_{21} \\ \cdots \\ m_{32} \end{bmatrix}}_{6\times 1} = \underbrace{\begin{bmatrix} c_{11} \\ c_{21} \\ \cdots \\ c_{32} \end{bmatrix}}_{6\times 1} \quad (mn = 6)$$

$\Rightarrow Qm = c$: Linear Equation

$\rightarrow$ if Q is nonsingular, the solution exists and unique

$\rightarrow$ if Q is singular, indeterminate or insoluble(insolvable)

Lyapunov Equation

Define Linear Mapping $L : R^{nm} \rightarrow R^{nm}$

$L(M) = AM + MB$

Let η be eigenvalue of linear mapping $L(\cdot)$

$L(M) = \eta M$

Let u and λ be right eigenvector and eigenvalue of A and

v and μ be left eigenvector and eigenvalue of B

$Au = \lambda u, \quad vB = \mu v$

$\Rightarrow L(uv) = Auv + uvB = \lambda uv + \mu uv = (\lambda + \mu)uv$

$\Rightarrow (\lambda + \mu)$ is eigenvalue of $L(\cdot)$

$\Rightarrow Q$ is nonsingular iff all $\eta_k = (\lambda_i + \mu_j)$ is nonzero

$\Rightarrow$ If some $\eta_k = (\lambda_i + \mu_j)$ is zero

case1: C is in range space of L, sol. exists and not unique

case2: otherwise, sol. does not exist.

HW4

Problem 3.31 in the Text.

Useful Formulas

Theorem

$$\rho(AB) \le \min(\rho(A),\ \rho(B)),\ \ A \in R^{m\times n},\ \ B \in R^{n\times p}$$

Bpf.)

$$AB = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ & \dots & \\ a_{m1} & & a_{mn} \end{bmatrix} \begin{bmatrix} \mathbf{b}_1 \\ \dots \\ \mathbf{b}_n \end{bmatrix} = \begin{bmatrix} \sum a_{1i}\mathbf{b}_i \\ \dots \\ \sum a_{mi}\mathbf{b}_i \end{bmatrix}$$

1. row of AB is spaned by $\{\mathbf{b}_j\}$
 $\Rightarrow$ rank of AB is not more than the number of LI vectors in $\{\mathbf{b}_j\}$

$$AB = \begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_n \end{bmatrix} \begin{bmatrix} b_{11} & \dots & b_{1p} \\ & \dots & \\ b_{n1} & & b_{np} \end{bmatrix} = \begin{bmatrix} \sum b_{i1}\mathbf{a}_i & \dots & \sum b_{in}\mathbf{a}_i \end{bmatrix}$$

2. column of AB is spaned by $\{\mathbf{a}_i\}$
 $\Rightarrow$ rank of AB is not more than the number of LI vectors in $\{\mathbf{a}_i\}$

Useful Formulas

Theorem

The rank of a matrix will not change after pre- or post-multiplying by a nonsingular matrix

$$\rho(A) = \rho(AC) = \rho(DA), A \in R^{m\times n}, C \in R^{n\times n}, D \in R^{m\times m}$$

Pf.)

$$P = AC$$
$$\rho(A) = \min(m,n), \rho(C) = n$$
$$\rightarrow \rho(A) \le \rho(C)$$
$$\rho(P) \le \min(\rho(A), \rho(C)) = \rho(A)$$
$$A = PC^{-1}$$
$$\rightarrow \rho(A) \le \rho(P)$$
$$\Rightarrow \rho(A) = \rho(P)$$

Useful Formulas

Theorem

$$\det(I_m + AB) = \det(I_n + BA), A \in R^{m\times n}, B \in R^{n\times m}$$

Pf.)

$$\text{Define } N = \begin{bmatrix} I_m & A \\ 0 & I_n \end{bmatrix}, Q = \begin{bmatrix} I_m & 0 \\ -B & I_n \end{bmatrix}, P = \begin{bmatrix} I_m & -A \\ B & I_n \end{bmatrix}$$
$$NP = \begin{bmatrix} I_m + AB & 0 \\ B & I_n \end{bmatrix}, QP = \begin{bmatrix} I_m & -A \\ 0 & I_n + BA \end{bmatrix}$$
$$\det N = \det I_m \det I_n = 1 = \det Q$$
$$\det NP = \det[I_m + AB] = \det N \det P = \det P$$
$$\det QP = \det[I_n + BA] = \det Q \det P = \det P$$

Matrix Properties

Fact: all eigenvalues of symetric real M are real.

Pf.)

Assume x be complex

$$(x^* M x)^* = x^* M^* x = x^* M x$$

This implies $x^* M x$ is real.

Let λ, v be eigenvalue and eigenvector of M

$$Mv = \lambda v$$

$$v^* M v = v^* \lambda v = \lambda v^* v$$

$\rightarrow \lambda$ should be real since $v^* v$ and $v^* M v$ are real.

Matrix Properties

Claim: every symmetric real matrix can be diagonalized by a similarity transform.

Pf.)

To show that there is no gneralized eigenvector of grade 2 or higher, suppose x be a generalized eigenvector of grade 2 or higher, i.e.,

$$(M - \lambda I)^2 x = 0 \cdots (1)$$

$$(M - \lambda I) x \neq 0 \quad \cdots (2)$$

From (2)

$$[(M - \lambda I)x]^* (M - \lambda I) x \neq 0$$

From (1)

$$[(M - \lambda I)x]^* (M - \lambda I) x = x^* (M - \lambda I)^2 x = 0$$

This contradicts.

Matrix Properties

Claim: Jordan form of symmetric real matrix M has no Jordan block of order of 2 or higher.

Note: A is called orthogonal (orthomormal) matrix if all columns are orthogonal(orthomormal).
If A is orthomormal ,

$$A^T A = I,\ \ A^T = A^{-1}:\ \text{called unitary matrix.}$$

Matrix Properties

Theorem

$$M = QDQ^{-1}, Q^T = Q^{-1}, D: \text{diagonal}, M: \text{symmetric real}$$

Pf.)

$$\text{Since } D^T = D, M^T = M$$

$$M = QDQ^{-1} = (QDQ^{-1})^T = Q^{-T} DQ^T$$

$$\Rightarrow Q^T = Q^{-1}$$

Positive Definiteness

M is positive definite , $M > 0$ if $x^T Mx > 0$ for every nonzero x

M is positive semidefinite , $M \geq 0$ if $x^T Mx \geq 0$ for every nonzero x

Matrix Properties

Theorem

M is positive definite (semidefinite) iff
any one of the following conditions holds
- every eigenvalue of M is positive (zero or positive),
- all leading principal minors of M are positive (all principal minors are zero or positive) (see [10])
- there exists nonsingular N (nonsingular or mxn matrix N with $m<n$) such that $M=N^TN$.

Note:

principal minors: det of 1x1, 2x2, 3x3 ... submatrix
leading principal minors include m_{11}

Matrix Properties

Theorem

1. mxn matrix H, $m \geq n$, has rank n iff
H^TH has rank n or det $H^TH \neq 0$
2. mxn matrix H, $m \leq n$, has rank m iff
HH^T has rank m or det $HH^T \neq 0$

Pf.)

(Necessity) $\rho(H^TH) = n \rightarrow \rho(H) = n$
by contraction, suppose
$\rho(H^TH) = n$, but $\rho(H) < n$
$\rightarrow \exists v \neq 0$ such that $Hv = 0$
$\rightarrow H^THv = 0$
$\rightarrow$ contradicts $\rho(H^TH) = n$

(Sufficiency) $\rho(H) = n \rightarrow \rho(H^TH) = n$
by contraction, suppose
$\rho(H) = n$, but $\rho(H^TH) < n$
$\rightarrow \exists v \neq 0$ such that $H^THv = 0$
$\rightarrow v^TH^THv = 0 = (Hv)^THv = \|Hv\|^2$
$\rightarrow Hv = 0$
$\rightarrow$ contradicts $\rho(H) = n$

Matrix Properties

Singular Value

$M = H^T H \geq 0$; eigenvalues $\lambda_i^2 \geq 0$

$$\lambda_1^2 \geq \lambda_2^2 \geq \cdots \geq \lambda_r^2 > 0 = \lambda_{r+1}^2 = \cdots = \lambda_n^2$$

Let $\bar{n} = \min(m, n)$

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0 = \lambda_{r+1} = \cdots = \lambda_{\bar{n}}$$

λ_i is called singular values of H

Matrix Properties

Example: Singular Value

$$H = \begin{bmatrix} -4 & -1 & 2 \\ 2 & 0.5 & -1 \end{bmatrix}$$

$$M = H^T H = \begin{bmatrix} 20 & 5 & -10 \\ 5 & 1.25 & -2.5 \\ -10 & -2.5 & 5 \end{bmatrix}$$

$$\det(\lambda I - M) = \lambda^3 - 26.25\lambda^2 = \lambda^2(\lambda - 26.25)$$

$\rightarrow$ singular values of H are $\sqrt{26.25} = 5.1235,\ 0$

Matrix Properties

Theorem: Singular Value Decomposition

Every $m \times n$ matrix H can be transformed into

$$H = RSQ^T$$

with $R^T R = RR^T = I_m$, $Q^T Q = QQ^T = I_n$, and

S is diagonal matrix with signular values

Q: orthonormalized eigenvectors of $H^T H$

R: orthonormalized eigenvectors of HH^T

Matrix Properties

Pf.)

$$\rho(H) = r = \rho(H^T H),\ \lambda_1^2 \geq \lambda_2^2 \cdots \lambda_r^2 > 0 = \lambda_{r+1} \cdots$$
$$Q = [q_1 \cdots q_r\ q_{r+1} \cdots q_n] = [Q_1\ Q_2]$$

q_i : orthonormalized eigenvectors of $H^T H$

note) $H^T H q_i = \lambda_i^2 q_i$, for $i = 1, \ldots, r$

$H^T H q_j = 0$, for $j = r+1, \ldots, n$ (Null space basis)

$$Q^T H^T H Q = \begin{bmatrix} \Lambda^2 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{matrix} Q_2^T H^T H Q_2 = 0 \\ Q_1^T H^T H Q_1 = \Lambda^2 \end{matrix}$$

$\Lambda^{-1} Q_1^T H^T H Q_1 \Lambda^{-1} = I \Rightarrow R_1^T R_1 = I$ by defining $R_1 = HQ_1\Lambda^{-1}$

Choose R_2 such that $R^T R = I, R = [R_1, R_2]$

$$R^T HQ = \begin{bmatrix} R_1^T \\ R_2^T \end{bmatrix} H[Q_1\ Q_2] = \begin{bmatrix} R_1^T HQ_1 & R_1^T HQ_2 \\ R_2^T HQ_1 & R_2^T HQ_2 \end{bmatrix}$$

$$R^T HQ = \begin{bmatrix} \Lambda & 0 \\ R_2^T R_1 \Lambda = 0 & 0 \end{bmatrix} := S \Rightarrow H = RSQ^T$$

HW5

Find Singular Value Decomposition for the following matrix

$$H = \begin{bmatrix} -1 & 0 & 1 \\ 2 & -1 & 0 \end{bmatrix}.$$

Matrix Properties

Norm of Matrix (Induced Norm)

$$\|A\| = \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|} = \sup_{\|x\|=1} \|Ax\|$$

$$\|A\|_1 = \max_j (\sum_{i=1}^n |a_{ij}|), \text{ for } \|x\|_1 = 1, ex) x = [0 \ldots 1 \ldots 0]$$

$$\|A\|_2 = (\lambda_{\max}(A^* A))^{1/2}, \text{ for } \|x\|_2 = 1$$

$$\|A\|_\infty = \max_i (\sum_{j=1}^n |a_{ij}|), \text{ for } \|x\|_\infty = 1, ex) x = [-1 \ldots 1 \ldots -1]$$

$$\Leftarrow$$

$$\|A\|_2 = \sup_{\|x\|=1} (x^* A^* A x)^{1/2} = \sup_{\|x\|=1} (x^* A^* A \sum \alpha_i v_i)^{1/2}, x = \sum \alpha_i v_i$$

$$= \sup_{\|x\|=1} (x^* \sum \alpha_i \lambda_i v_i)^{1/2} \le \sup_{\|x\|=1} (x^* \lambda_{\max} \sum \alpha_i v_i)^{1/2} = (\lambda_{\max}(A^* A))^{1/2}$$

Matrix Properties

Examples

$$\|A\|_1 = \max_j (\sum_{i=1}^{n} |a_{ij}|), \text{ for } \|x\|_1 = 1, ex) x = [0 \ldots 1 \ldots 0]$$

$$A = \begin{bmatrix} 1 & -2 & 4 \\ -5 & 2 & 0 \\ 2 & 3 & 1 \end{bmatrix}$$

$$x^T = [\pm 1 \quad 0 \quad 0] \rightarrow \|Ax\|_1 = 8$$

$$x^T = [0 \quad \pm 1 \quad 0] \rightarrow \|Ax\|_1 = 7$$

$$x^T = [0 \quad 0 \quad \pm 1] \rightarrow \|Ax\|_1 = 5$$

$$\Rightarrow \|A\|_1 = 8$$

Matrix Properties

Examples

$$\|A\|_\infty = \max_i (\sum_{j=1}^{n} |a_{ij}|), \text{ for } \|x\|_\infty = 1, ex) x = [-1 \ldots 1 \ldots -1]$$

$$A = \begin{bmatrix} 1 & -2 & 4 \\ -5 & 2 & 0 \\ 2 & 3 & 1 \end{bmatrix}$$

$$x^T = [1 \quad -1 \quad 1] \rightarrow \|Ax\|_\infty = 7$$

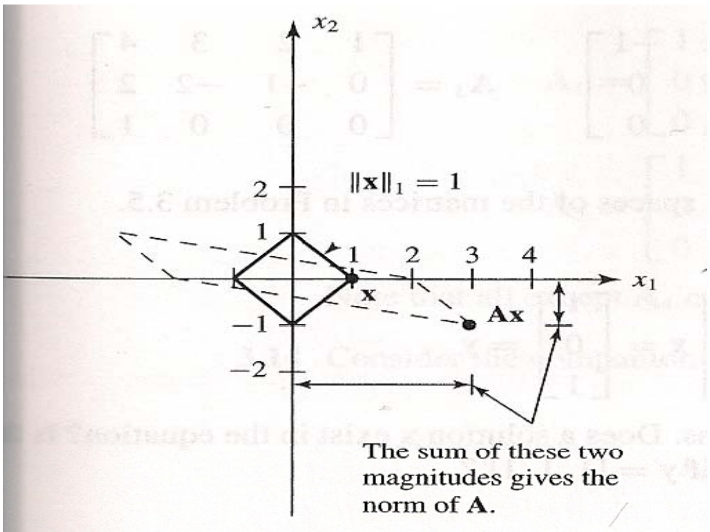
$$x^T = [-1 \quad 1 \quad 0] \rightarrow \|Ax\|_\infty = 7$$

$$x^T = [1 \quad 1 \quad 1] \rightarrow \|Ax\|_\infty = 6$$

$$\Rightarrow \|A\|_\infty = 7$$

Matrix Properties

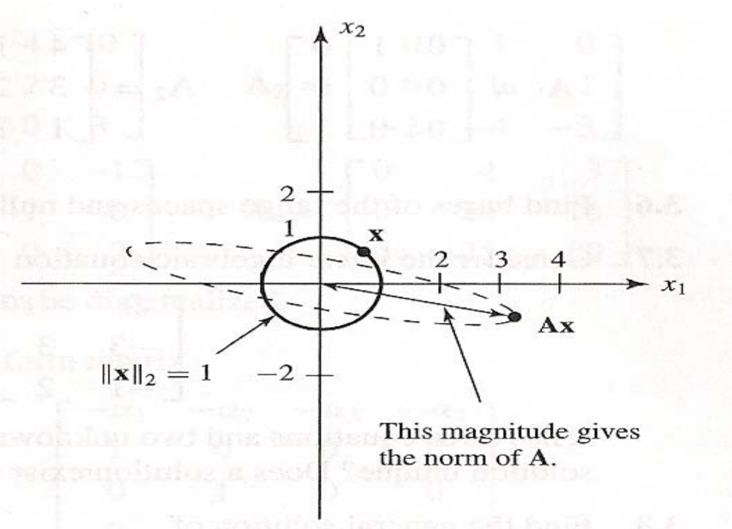
Norm of Matrix (Induced Norm)



(a)

Matrix Properties

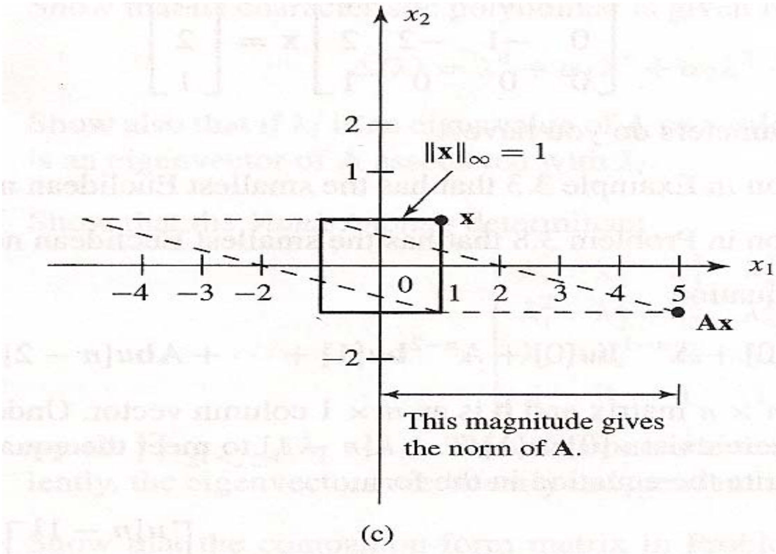
Norm of Matrix (Induced Norm)



(b)

Matrix Properties

Norm of Matrix (Induced Norm)



Summary

Field, Linear (Vector) Space
Basis, Linearly Independent Vectors,
Representation of Vectors and Linear Operators
Basis Change, Similarity Transform
Generalized Eigenvectors, Jordan Form
Function of Square Matrix
Range Space and Null Space in Linear Algebraic Equations
Lyapunov Equation
Singular Value Decomposition, Unitary Matrix
Matrix Norm
Useful Formula and Matrix Properties