8. State Feedback and State Estimators

- ✓ State Feedback Controller Design
- ✓ Regulation and Tracking
- ✓ State Estimator Design
- ✓ Feedback from Estimated States
- ✓ State Feedback-Multivariable Case
- ✓ State Estimator-Multivariable Case
- ✓ Feedback from Estimated States-Multivariable Case

State Feedback Controller Design

State Feedback

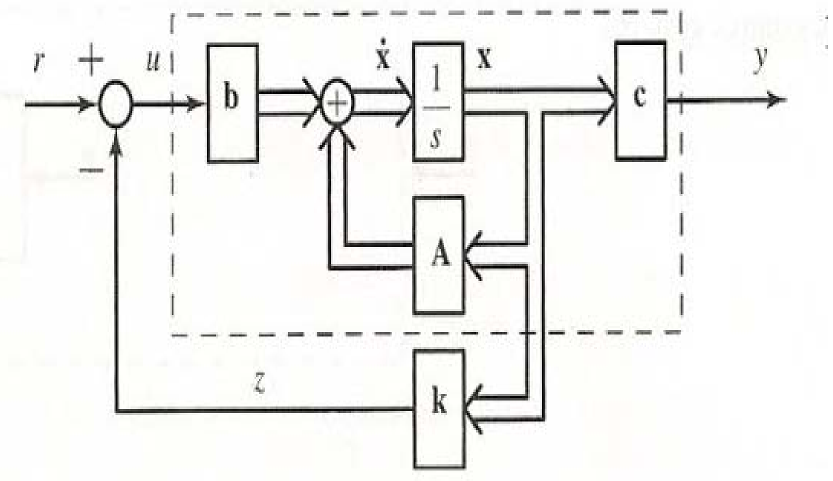
$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u$$

$$y = \mathbf{c}\mathbf{x}$$

$$u = r - \mathbf{k}\mathbf{x} = r - \begin{bmatrix} k_1 & k_2 & \cdots & k_n \end{bmatrix}\mathbf{x} = r - \sum_{i=1}^{n} k_i x_i$$

$$\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{b}\mathbf{k})\mathbf{x} + \mathbf{b}r$$

$$y = \mathbf{c}\mathbf{x}$$



State Feedback Controller Design

Theorem 8.1

The pair $(\mathbf{A}-\mathbf{bk},\mathbf{b})$, for any $1\times n$ real constant vector $\mathbf{k}$, is controllable if and only if $(\mathbf{A},\mathbf{b})$ is controllable.

Proof : We show the theorem for $n=4$.

$$C=\begin{bmatrix}\mathbf{b} & \mathbf{Ab} & \mathbf{A}^2\mathbf{b} & \mathbf{A}^3\mathbf{b}\end{bmatrix}$$

and

$$C_f=\begin{bmatrix}\mathbf{b} & (\mathbf{A}-\mathbf{bk})\mathbf{b} & (\mathbf{A}-\mathbf{bk})^2\mathbf{b} & (\mathbf{A}-\mathbf{bk})^3\mathbf{b}\end{bmatrix}$$

$$C_f=C\begin{bmatrix}1 & -\mathbf{kb} & -\mathbf{k}(\mathbf{A}-\mathbf{bk})\mathbf{b} & -\mathbf{k}(\mathbf{A}-\mathbf{bk})^2\mathbf{b}\\ 0 & 1 & -\mathbf{kb} & -\mathbf{k}(\mathbf{A}-\mathbf{bk})\mathbf{b}\\ 0 & 0 & 1 & -\mathbf{kb}\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$\to \rho C_f=\rho C \to$ Controllability is invariant.

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State Feedback Controller Design

Example 8.1

$$\dot{\mathbf{x}} = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u$$

$$y = \begin{bmatrix} 1 & 2 \end{bmatrix}\mathbf{x}$$

$$C = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix},\ O = \begin{bmatrix} 1 & 2 \\ 7 & 4 \end{bmatrix} : controllable\ \ \& \ observable$$

$$u = r - \begin{bmatrix} 3 & 1 \end{bmatrix}\mathbf{x}$$

$$\dot{\mathbf{x}} = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}r$$

$$y = \begin{bmatrix} 1 & 2 \end{bmatrix}\mathbf{x}$$

$$C_f = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix},\ O_f = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} : controllable\ \ \& \ not\ observable$$

State Feedback Controller Design

Example 8.2

$$\dot{\mathbf{x}} = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$\Delta(s) = (s-1)^2 - 9 = s^2 - 2s - 8 = (s-4)(s+2) : unstable$$

$$u = r - \begin{bmatrix} k_1 & k_2 \end{bmatrix} \mathbf{x}$$

$$\dot{\mathbf{x}} = \left(\begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} - \begin{bmatrix} k_1 & k_2 \\ 0 & 0 \end{bmatrix} \right) \mathbf{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} r = \begin{bmatrix} 1-k_1 & 3-k_2 \\ 3 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} r$$

$$\Delta_f(s) = (s-1+k_1)(s-1) - 3(3-k_2) = s^2 + (k_1-2)s + (3k_2 - k_1 - 8)$$

If we want to place the eigenvalues at $-1 \pm j2$,

$$\Delta_f(s) = (s-1-j2)(s-1+j2) = s^2 + 2s + 5$$

$$\rightarrow k_1 = 4, k_2 = 17/3.$$

$$\rightarrow Stabilizaed.$$

State Feedback Controller Design

Theorem 8.2

Consider the state equation in (8.1) with $n = 4$ and the characteristic polynomial

$$\Delta(s) = \det(sI - A) = s^4 + \alpha_1 s^3 + \alpha_2 s^2 + \alpha_3 s + \alpha_4$$

If (8.1) is controllable, then it can be transformed by the transformation $\bar{\mathbf{x}} = \mathbf{P}\mathbf{x}$ with

$$\mathbf{Q} := \mathbf{P}^{-1} = \begin{bmatrix} \mathbf{b} & \mathbf{A}\mathbf{b} & \mathbf{A}^2\mathbf{b} & \mathbf{A}^3\mathbf{b} \end{bmatrix} \begin{bmatrix} 1 & \alpha_1 & \alpha_2 & \alpha_3 \\ 0 & 1 & \alpha_1 & \alpha_2 \\ 0 & 0 & 1 & \alpha_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

State Feedback Controller Design

into the controllable canonical form

$$\dot{\mathbf{x}} = \bar{\mathbf{A}}\bar{\mathbf{x}} + \bar{\mathbf{b}}u = \begin{bmatrix} -\alpha_1 & -\alpha_2 & -\alpha_3 & -\alpha_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \bar{\mathbf{x}} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \bar{\mathbf{c}}\bar{\mathbf{x}} = \begin{bmatrix} \beta_1 & \beta_2 & \beta_3 & \beta_4 \end{bmatrix} \bar{\mathbf{x}}$$

Furthermore, the transfer function of (8.1) with $n = 4$ equals

$$g(s) = \frac{\beta_1 s^3 + \beta_2 s^2 + \beta_3 s + \beta_4}{s^4 + \alpha_1 s^3 + \alpha_2 s^2 + \alpha_3 s + \alpha_4}$$

State Feedback Controller Design

Proof :

C & $\bar{C}$ is nonsingular since the eq. is controllable and

$\bar{C} = \mathbf{P}C.$

$\mathbf{P} = \bar{C}C^{-1}$ or $\mathbf{Q} := \mathbf{P}^{-1} = C\bar{C}^{-1}$

$$\bar{C} = \begin{bmatrix} 1 & -\alpha_1 & \alpha_1{}^2 - \alpha_2 & -\alpha_3{}^3 + 2\alpha_1\alpha_2 - \alpha_3 \\ 0 & 1 & -\alpha_1 & \alpha_1{}^2 - \alpha_2 \\ 0 & 0 & 1 & -\alpha_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\bar{C}^{-1} = \begin{bmatrix} 1 & \alpha_1 & \alpha_2 & \alpha_3 \\ 0 & 1 & \alpha_1 & \alpha_2 \\ 0 & 0 & 1 & \alpha_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$ is assumed. Proved by multiplying.

State Feedback Controller Design

Theorem 8.3

If the n-dimensional state equation in (8.1) is controllable, then by the state feedback $u = r - \mathbf{kx}$, where $\mathbf{k}$ is a $1 \times n$ real constant vector, the eigenvalues of $\mathbf{A} - \mathbf{bk}$ can arbitrarily be assigned provided that complex conjugate eigenvalues are assigned in pairs.

State Feedback Controller Design

Proof :

$$u = r - \mathbf{kx} = r - \mathbf{kP}^{-1}\bar{\mathbf{x}} =: r - \bar{\mathbf{k}}\bar{\mathbf{x}}$$

$$\Delta_f(s) = s^4 + \bar{\alpha}_1 s^3 + \bar{\alpha}_2 s^2 + \bar{\alpha}_3 s + \bar{\alpha}_4$$

$$\bar{\mathbf{k}} = \begin{bmatrix} \bar{\alpha}_1 - \alpha_1 & \bar{\alpha}_2 - \alpha_2 & \bar{\alpha}_3 - \alpha_3 & \bar{\alpha}_4 - \alpha_4 \end{bmatrix}$$

$$\dot{\bar{\mathbf{x}}} = (\bar{\mathbf{A}} - \bar{\mathbf{b}}\bar{\mathbf{k}})\bar{\mathbf{x}} + \bar{\mathbf{b}}r = \begin{bmatrix} -\bar{\alpha}_1 & -\bar{\alpha}_2 & -\bar{\alpha}_3 & -\bar{\alpha}_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \bar{\mathbf{x}} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} r$$

$$y = \begin{bmatrix} \beta_1 & \beta_2 & \beta_3 & \beta_4 \end{bmatrix} \bar{\mathbf{x}}$$

$$\mathbf{k} = \bar{\mathbf{k}}\mathbf{P} = \bar{\mathbf{k}}\bar{C}C^{-1}$$

State Feedback Controller Design

Alternative derivation of $\bar{\mathbf{k}}$:

$$\Delta_f(s) = \det(s\mathbf{I} - \mathbf{A} + \mathbf{bk}) = \det\left((s\mathbf{I} - \mathbf{A})\left[\mathbf{I} + (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{bk}\right]\right)$$

$$= \det(s\mathbf{I} - \mathbf{A})\det\left[\mathbf{I} + (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{bk}\right]$$

$$\Delta_f(s) = \Delta(s)\left[1 + \mathbf{k}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}\right]$$

$$\Delta_f(s) - \Delta(s) = \Delta(s)\mathbf{k}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b} = \Delta(s)\bar{\mathbf{k}}(s\mathbf{I} - \bar{\mathbf{A}})^{-1}\bar{\mathbf{b}}$$

State Feedback Controller Design

$$\bar{\mathbf{c}}\left(s\mathbf{I}-\bar{\mathbf{A}}\right)^{-1}\bar{\mathbf{b}}=\frac{\beta_1 s^3+\beta_2 s^2+\beta_3 s+\beta_4}{\Delta(s)}$$

$$\bar{\mathbf{k}}\left(s\mathbf{I}-\bar{\mathbf{A}}\right)^{-1}\bar{\mathbf{b}}=\frac{\bar{k}_1 s^3+\bar{k}_2 s^2+\bar{k}_3 s+\bar{k}_4}{\Delta(s)}$$

Then

$$\Delta_f(s)-\Delta(s)=\bar{k}_1 s^3+\bar{k}_2 s^2+\bar{k}_3 s+\bar{k}_4$$

$$\rightarrow \bar{\mathbf{k}}=\left[\bar{\alpha}_1-\alpha_1 \quad \bar{\alpha}_2-\alpha_2 \quad \bar{\alpha}_3-\alpha_3 \quad \bar{\alpha}_4-\alpha_4\right]$$

Feedback Transfer Function

$$g(s)=\mathbf{c}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{b}=\frac{\beta_1 s^3+\beta_2 s^2+\beta_3 s+\beta_4}{s^4+\alpha_1 s^3+\alpha_2 s^2+\alpha_3 s+\alpha_4}$$

$$g_f(s)=\mathbf{c}(s\mathbf{I}-\mathbf{A}+\mathbf{bk})^{-1}\mathbf{b}=\frac{\beta_1 s^3+\beta_2 s^2+\beta_3 s+\beta_4}{s^4+\bar{\alpha}_1 s^3+\bar{\alpha}_2 s^2+\bar{\alpha}_3 s+\bar{\alpha}_4}$$

State Feedback Controller Design

Example 8.3

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 5 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ -2 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \mathbf{x}$$

$\rightarrow$ *Controllable*

$\rightarrow$ its eigenvalues can be assigned arbitrarily.

$$\Delta(s) = s^2(s^2 - 5) = s^4 + 0 \cdot s^3 - 5s^2 + 0 \cdot s + 0$$

State Feedback Controller Design

$$\mathbf{P}^{-1} = C\overline{C}^{-1} = \begin{bmatrix} 0 & 1 & 0 & 2 \\ 1 & 0 & 2 & 0 \\ 0 & -2 & 0 & -10 \\ -2 & 0 & -10 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & -5 & 0 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 & -3 \\ 1 & 0 & -3 & 0 \\ 0 & -2 & 0 & 0 \\ -2 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{P} = \begin{bmatrix} 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & 0 & -\frac{1}{6} \\ -\frac{1}{3} & 0 & -\frac{1}{6} & 0 \end{bmatrix}$$

State Feedback Controller Design

Let the desired eigenvalues be $-1.5 \pm 0.5j$ and $-1 \pm j$

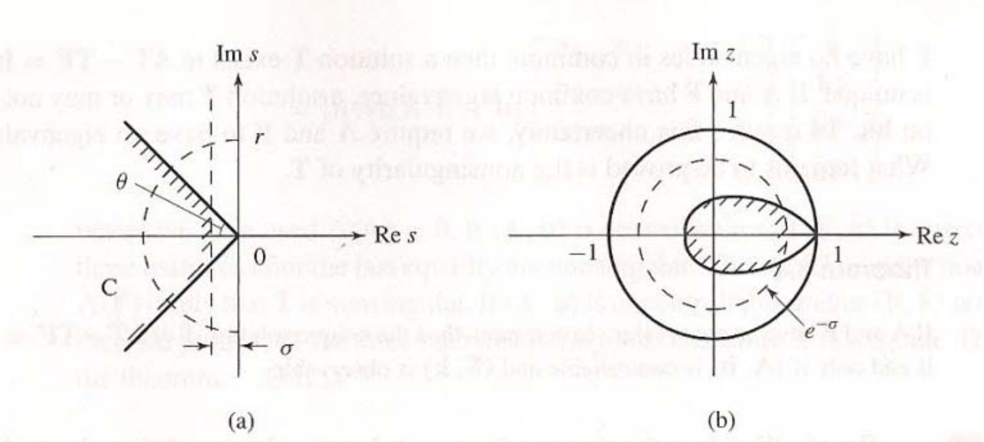
$$\Delta_f(s) = (s+1.5-0.5j)(s+1.5+0.5j)(s+1-j)(s+1+j)$$
$$= s^4 + 5s^3 + 10.5s^2 + 11s + 5$$

$$\bar{\mathbf{k}} = [5-0 \quad 10.5+5 \quad 11-0 \quad 5-0] = [5 \quad 15.5 \quad 11 \quad 5]$$

$$\mathbf{k} = \bar{\mathbf{k}}\mathbf{P} = \left[-\tfrac{5}{3} \quad -\tfrac{11}{3} \quad -\tfrac{103}{12} \quad -\tfrac{13}{3}\right]$$

State Feedback Controller Design

How to determine the desired eigenvalues?



Find $\mathbf{k}$ to minimize the objective function

$$J = \int_0^\infty \left[\mathbf{x}'(t)\mathbf{Q}\mathbf{x}(t) + \mathbf{u}(t)\mathbf{R}\mathbf{u}(t) \right] dt$$

$\rightarrow$ *Optimal Control Theory*

State Feedback Controller Design

Solving the Lyapunov Equation

Procedure 8.1

Consider controllable $(\mathbf{A},\mathbf{b})$. Find $\mathbf{k}$ such that $(\mathbf{A}-\mathbf{bk})$ has any set of desired eigenvalues that contains no eigenvalues of $\mathbf{A}$.

1. Select an $n\times n$ matrix $\mathbf{F}$ that has the set of desired eigenvalues.
2. Select an arbitrary $1\times n$ vector $\bar{\mathbf{k}}$ such that $(\mathbf{F},\bar{\mathbf{k}})$ is observable.
3. Solve the unique $\mathbf{T}$ in the Lyapunov equation $\mathbf{AT}-\mathbf{TF}=\mathbf{b}\bar{\mathbf{k}}$.
4. Compute the feedback gain $\mathbf{k}=\bar{\mathbf{k}}\mathbf{T}^{-1}$.

Note:

$(\mathbf{A}-\mathbf{bk})\mathbf{T}=\mathbf{TF}$ or $\mathbf{A}-\mathbf{bk}=\mathbf{TFT}^{-1}\rightarrow\mathbf{A}-\mathbf{bk}$ is similar to $\mathbf{F}$.

State Feedback Controller Design

Theorem 8.4

If $\mathbf{A}$ and $\mathbf{F}$ have no eigenvalues in common,

then the unique solution $\mathbf{T}$ of $\mathbf{AT} - \mathbf{TF} = \mathbf{b}\bar{\mathbf{k}}$ is nonsingular

if and only if $(\mathbf{A}, \mathbf{b})$ is controllable and $(\mathbf{F}, \bar{\mathbf{k}})$ is observable.

Proof :

$$\Delta(s) = s^4 + \alpha_1 s^3 + \alpha_2 s^2 + \alpha_3 s + \alpha_4$$

$$\Delta(\mathbf{A}) = \mathbf{A}^4 + \alpha_1 \mathbf{A}^3 + \alpha_2 \mathbf{A}^2 + \alpha_3 \mathbf{A} + \alpha_4 \mathbf{I} = 0$$

$$\Delta(\mathbf{F}) = \mathbf{F}^4 + \alpha_1 \mathbf{F}^3 + \alpha_2 \mathbf{F}^2 + \alpha_3 \mathbf{F} + \alpha_4 \mathbf{I}$$

Since A and F have no common eigenvalues, $\Delta(\bar{\lambda}_i) \neq 0$.

If $\bar{\lambda}_i$ is eigenvalue of $\mathbf{F}$, $\Delta(\bar{\lambda}_i)$ is eigenvalue of $\Delta(\mathbf{F})$(Problem 3.19)

$\det \Delta(\mathbf{F}) = \prod_i \Delta(\bar{\lambda}_i) \neq 0 \rightarrow \Delta(\mathbf{F})$ is nonsingular.

State Feedback Controller Design

Substituting $\mathbf{AT} = \mathbf{TF} + \mathbf{b}\bar{\mathbf{k}}$ into $\mathbf{A}^2\mathbf{T} - \mathbf{AF}^2$ yields

$$\mathbf{A}^2\mathbf{T} - \mathbf{TF}^2 = \mathbf{A}(\mathbf{TF} + \mathbf{b}\bar{\mathbf{k}}) - \mathbf{TF}^2 = \mathbf{Ab}\bar{\mathbf{k}} + (\mathbf{AT} - \mathbf{TF})\mathbf{F}$$
$$= \mathbf{Ab}\bar{\mathbf{k}} + \mathbf{b}\bar{\mathbf{k}}\mathbf{F}$$

$$\begin{array}{lr}
\mathbf{IT} - \mathbf{TI} = \mathbf{0} & \times\alpha_4 \\
\mathbf{AT} - \mathbf{TF} = \mathbf{b}\bar{\mathbf{k}} & \times\alpha_3 \\
\mathbf{A}^2\mathbf{T} - \mathbf{TF}^2 = \mathbf{Ab}\bar{\mathbf{k}} + \mathbf{b}\bar{\mathbf{k}}\mathbf{F} & \times\alpha_2 \\
\mathbf{A}^3\mathbf{T} - \mathbf{TF}^3 = \mathbf{A}^2\mathbf{b}\bar{\mathbf{k}} + \mathbf{Ab}\bar{\mathbf{k}}\mathbf{F} + \mathbf{b}\bar{\mathbf{k}}\mathbf{F}^2 & \times\alpha_1 \\
+\ \mathbf{A}^4\mathbf{T} - \mathbf{TF}^4 = \mathbf{A}^3\mathbf{b}\bar{\mathbf{k}} + \mathbf{A}^2\mathbf{b}\bar{\mathbf{k}}\mathbf{F} + \mathbf{Ab}\bar{\mathbf{k}}\mathbf{F}^2 + \mathbf{b}\bar{\mathbf{k}}\mathbf{F}^3 & \times 1 \\
\hline
\Delta(\mathbf{A})\mathbf{T} - \mathbf{T}\Delta(\mathbf{F}) = \Delta(\mathbf{A})\mathbf{T} \text{ since } \Delta(\mathbf{F}) = 0 &
\end{array}$$

$$= \begin{bmatrix} \mathbf{b} & \mathbf{Ab} & \mathbf{A}^2\mathbf{b} & \mathbf{A}^3\mathbf{b} \end{bmatrix} \begin{bmatrix} \alpha_3 & \alpha_2 & \alpha_1 & 1 \\ \alpha_2 & \alpha_1 & 1 & 0 \\ \alpha_1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{\mathbf{k}} \\ \bar{\mathbf{k}}\mathbf{F} \\ \bar{\mathbf{k}}\mathbf{F}^2 \\ \bar{\mathbf{k}}\mathbf{F}^3 \end{bmatrix}$$

$\rightarrow \mathbf{T}$ is nonsingular.

State Feedback Controller Design

Selection of observable pair $\{\mathbf{F}, \bar{\mathbf{k}}\}$

$$\mathbf{F} = \begin{bmatrix} 0 & 0 & 0 & 0 & \alpha_5 \\ 1 & 0 & 0 & 0 & \alpha_4 \\ 0 & 1 & 0 & 0 & \alpha_3 \\ 0 & 0 & 1 & 0 & \alpha_2 \\ 0 & 0 & 0 & 1 & \alpha_1 \end{bmatrix}, \bar{\mathbf{k}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Desired Eigenvalues: $\lambda_1, \alpha_1 \pm j\beta_1, \alpha_2 \pm j\beta_2$

$$\mathbf{F} = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & \alpha_1 & \beta_1 & 0 & 0 \\ 0 & -\beta_1 & \alpha_1 & 0 & 0 \\ 0 & 0 & 0 & \alpha_2 & \beta_2 \\ 0 & 0 & 0 & -\beta_2 & \alpha_2 \end{bmatrix}, \begin{matrix} \bar{\mathbf{k}} = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \end{bmatrix} \\ \bar{\mathbf{k}} = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \end{bmatrix} \\ \bar{\mathbf{k}} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix} \end{matrix}$$

(Problem 6.16)

HW 8-1

1. Find the state feedback gain for the state equation

$$\dot{\mathbf{x}} = \begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u$$

so that the resulting system has eigenvalues -2 and $-1 \pm j$.

Use the method you think is the simplest by hand to carry out.

2. Consider a system with thransfer function

$$g(s) = \frac{(s-1)(s+2)}{(s+1)(s-2)(s+3)}.$$

Is it possible to change the transfer function to

$$g_f(s) = \frac{1}{s+3}$$

by the state feedback? Is the resulting system BIBO stable?

Is it asymptotically stable?

Regulation and Tracking

Regulation and Tracking

Regulation : to find out the state feedback gain
so that the output decay to zero.

Tracking : to find out the state feedback control $u(t)$
so that $y(t)$ approaches to $r(t) = a$

By stabilizing control $u = -\mathbf{kx}$ for $r(t) = 0$,

Zero input response

$$y(t) = ce^{(\mathbf{A}\text{-}\mathbf{bk})t}\mathbf{x}(0)$$

will decay to zero. The regulation can be easily achieved.

For tracking , we need a feedforward gain p as

$$u(t) = pr(t) - \mathbf{kx}.$$

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sy(s) = \lim_{s\to 0} sg_f(s)r(s)$$

$$= \lim_{s\to 0} sp\frac{\beta_1 s^3 + \beta_2 s^2 + \beta_3 s + \beta_4}{s^4 + \bar{\alpha}_1 s^3 + \bar{\alpha}_2 s^2 + \bar{\alpha}_3 s + \bar{\alpha}_4}\frac{a}{s} = \frac{p\beta_4}{\bar{\alpha}_4}a = a$$

$\rightarrow \quad p = \dfrac{\bar{\alpha}_4}{\beta_4}$, where β_4 should not be zero.

Regulation and Tracking

Robust Tracking and Disturbance Rejection

Robust tracking:

When the parameter of the transfer function is perturbed, the feedforward gain p may not yield the exact tracking.

$\rightarrow$ nonrobust $\rightarrow$ robust design is required.

Disturbance rejection:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u + \mathbf{b}w$$

$$y = \mathbf{c}\mathbf{x}$$

To design the controller so that the output track the step response even with the presence of a disturbance $w(t)$.

Regulation and Tracking

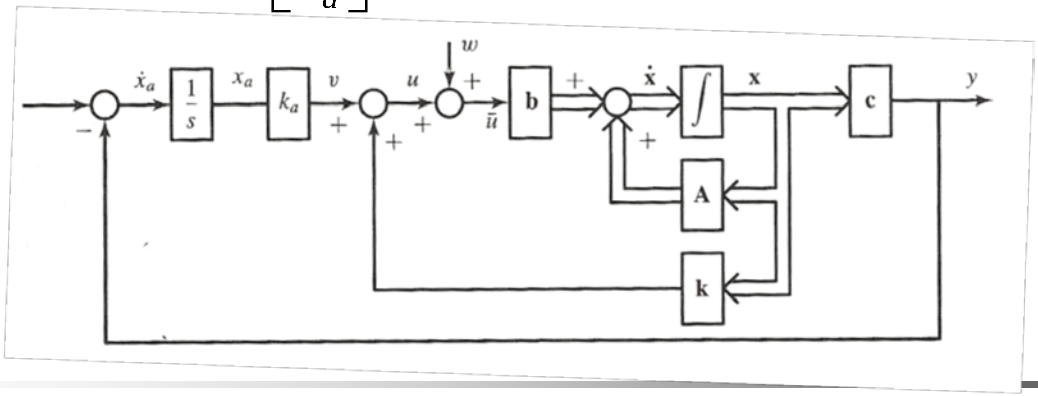
Robust Tracking and Disturbance Rejection

$$\dot{x}_a = r - y = r - \mathbf{c}\mathbf{x}$$

$$u = [\mathbf{k} \quad k_a]\begin{bmatrix}\mathbf{x} \\ x_a\end{bmatrix}$$

$$\begin{bmatrix}\dot{\mathbf{x}} \\ \dot{x}_a\end{bmatrix} = \begin{bmatrix}\mathbf{A}+\mathbf{b}\mathbf{k} & \mathbf{b}k_a \\ -\mathbf{c} & 0\end{bmatrix}\begin{bmatrix}\mathbf{x} \\ x_a\end{bmatrix} + \begin{bmatrix}\mathbf{0} \\ 1\end{bmatrix} r + \begin{bmatrix}\mathbf{b} \\ 0\end{bmatrix} w \quad \text{........ (8-29)}$$

$$y = [\mathbf{c} \quad 0]\begin{bmatrix}\mathbf{x} \\ x_a\end{bmatrix}$$



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Regulation and Tracking

Theorem 8.5

If $(\mathbf{A},\mathbf{b})$ is controllable and if $g(s)=\mathbf{c}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{b}$ has no zero at $s=0$, then all eigenvalues of the A-matrix in (8.29) can be assigned arbitrarily by selecting a feedback gain $[\mathbf{k} \quad k_a]$

Proof :

$$\begin{bmatrix}\dot{\mathbf{x}}\\ \dot{x}_a\end{bmatrix}=\begin{bmatrix}\mathbf{A}+\mathbf{bk} & \mathbf{b}k_a\\ -\mathbf{c} & 0\end{bmatrix}\begin{bmatrix}\mathbf{x}\\ x_a\end{bmatrix}+\begin{bmatrix}\mathbf{0}\\ 1\end{bmatrix}r+\begin{bmatrix}\mathbf{b}\\ 0\end{bmatrix}w \quad \text{........ (8-29)}$$

can be expressed as

$$\begin{bmatrix}\dot{\mathbf{x}}\\ \dot{x}_a\end{bmatrix}=\begin{bmatrix}\mathbf{A} & 0\\ -\mathbf{c} & 0\end{bmatrix}\begin{bmatrix}\mathbf{x}\\ x_a\end{bmatrix}+\begin{bmatrix}\mathbf{b}\\ 0\end{bmatrix}u+\begin{bmatrix}\mathbf{0}\\ 1\end{bmatrix}r+\begin{bmatrix}\mathbf{b}\\ 0\end{bmatrix}w$$

$$\mathrm{u}=[\mathbf{k} \quad k_a]\begin{bmatrix}\mathbf{x}\\ x_a\end{bmatrix}.$$

Regulation and Tracking

Proof of Theorem 8.5

It is enough to show that

$$\begin{bmatrix} \mathbf{A} & \mathbf{0} \\ -\mathbf{c} & 0 \end{bmatrix} \qquad \begin{bmatrix} \mathbf{b} \\ 0 \end{bmatrix}$$

is controllable if only if $\beta_4 \neq 0$(the plant has no zero at $s = 0$).

We prove for $n = 4$.

$$\begin{bmatrix} \mathbf{b} & \mathbf{Ab} & \mathbf{A}^2\mathbf{b} & \mathbf{A}^3\mathbf{b} & \mathbf{A}^4\mathbf{b} \\ 0 & -\mathbf{cb} & -\mathbf{cAb} & -\mathbf{cA}^2\mathbf{b} & -\mathbf{cA}^3\mathbf{b} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -\alpha_1 & \alpha_1^2 - \alpha_2 & -\alpha_1(\alpha_1^2 - \alpha_2) + \alpha_2\alpha_1 - \alpha_3 & a_{15} \\ 0 & 1 & -\alpha_1 & \alpha_1^2 - \alpha_2 & a_{25} \\ 0 & 0 & 1 & -\alpha_1 & a_{35} \\ 0 & 0 & 0 & 1 & a_{45} \\ 0 & -\beta_1 & \beta_1\alpha_1 - \beta_2 & -\beta_1(\alpha_1^2 - \alpha_2) + \beta_2\alpha_1 - \beta_3 & a_{55} \end{bmatrix}$$

Regulation and Tracking

The rank of a matrix does not change by elementary operations.
Adding the second row multiplied β_1 to the last row, and adding the third row multiplied β_2 to the last row, and adding the fourth row multiplied β_3 to the last row, we obtain

$$\begin{bmatrix} 1 & -\alpha_1 & \alpha_1^2-\alpha_2 & -\alpha_1(\alpha_1^2-\alpha_2)+\alpha_2\alpha_1-\alpha_3 & a_{15} \\ 0 & 0 & -\alpha_1 & \alpha_1^2-\alpha_2 & a_{25} \\ 0 & 0 & 1 & -\alpha_1 & a_{35} \\ 0 & 0 & 0 & 1 & a_{45} \\ 0 & 0 & 0 & 0 & -\beta_4 \end{bmatrix}$$

which is nonsingular.

Regulation and Tracking

Characteristic polynomial of overall system

$$\Delta_f(s) = \det \begin{bmatrix} s\mathbf{I}-\mathbf{A}-\mathbf{bk} & -\mathbf{b}k_a \\ \mathbf{c} & s \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -\mathbf{c}(s\mathbf{I}-\mathbf{A}-\mathbf{bk})^{-1} & 1 \end{bmatrix} \begin{bmatrix} s\mathbf{I}-\mathbf{A}-\mathbf{bk} & -\mathbf{b}k_a \\ \mathbf{c} & s \end{bmatrix}$$

$$= \begin{bmatrix} s\mathbf{I}-\mathbf{A}-\mathbf{bk} & -\mathbf{b}k_a \\ 0 & s+\mathbf{c}(s\mathbf{I}-\mathbf{A}-\mathbf{bk})^{-1}\mathbf{b}k_a \end{bmatrix}$$

$$1\cdot\Delta_f(s) = \bar{D}(s)\left(s + \frac{\bar{N}(s)}{\bar{D}(s)} k_a \right)$$

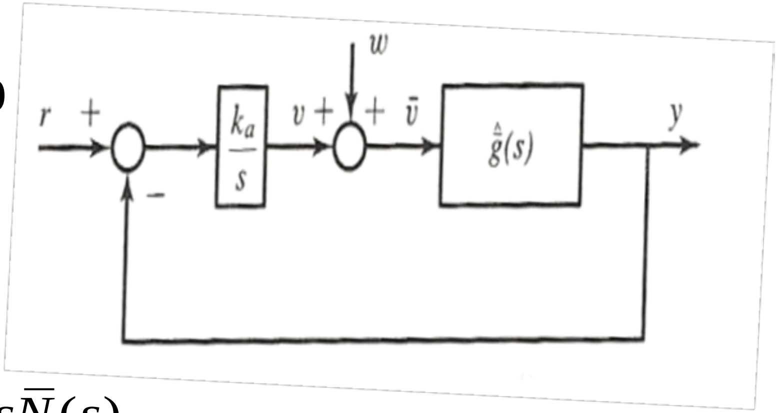
$$\leftarrow \bar{g}(s) := \frac{\bar{N}(s)}{\bar{D}(s)} := \mathbf{c}(s\mathbf{I}-\mathbf{A}-\mathbf{bk})^{-1}\mathbf{b}$$

$\Rightarrow$

$$\Delta_f(s) = s\bar{D}(s) + k_a \bar{N}(s)$$

Regulation and Tracking

Disturbance Rejection



$$y = \frac{\bar{N}(s)}{\bar{D}(s)}(v+w), \quad v = \frac{k_a}{s}(r-y), \quad r = 0$$

$$\left(1 + \frac{k_a \bar{N}(s)}{s\bar{D}(s)}\right) y = \frac{\bar{N}(s)}{\bar{D}(s)} w$$

$$g_{yw} = \frac{\dfrac{\bar{N}(s)}{\bar{D}(s)}}{1 + \dfrac{k_a \bar{N}(s)}{s\bar{D}(s)}} = \frac{s\bar{N}(s)}{s\bar{D}(s) + k_a \bar{N}(s)} = \frac{s\bar{N}(s)}{\Delta_f(s)}$$

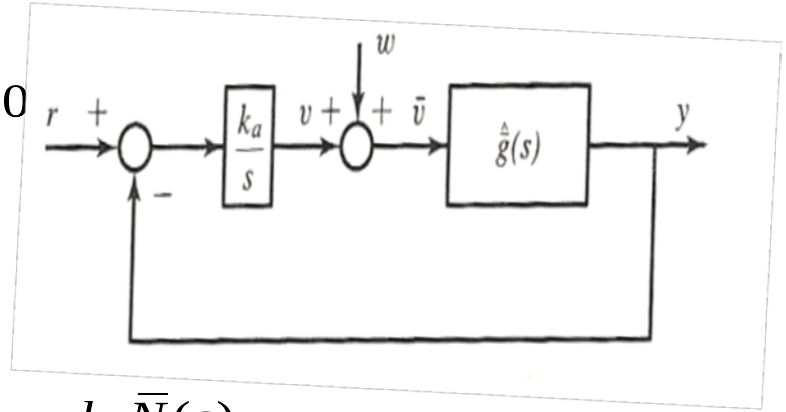
If $w(t) = \bar{w}, \ \mathcal{L}(w) = \bar{w}/s,$

$$y_w(s) = \frac{s\bar{N}(s)}{\Delta_f(s)} \frac{\bar{w}}{s} = \frac{\bar{w}\bar{N}(s)}{\Delta_f(s)} \to 0 \text{ by assigning poles in LHP.}$$

Regulation and Tracking

Tracking to step reference

$$y = \frac{\bar{N}(s)}{\bar{D}(s)}(v+w),\ \ v = \frac{k_a}{s}(r-y),\ \ w=0$$



$$(1+\frac{k_a\bar{N}(s)}{s\bar{D}(s)})y = \frac{k_a\bar{N}(s)}{s\bar{D}(s)}r$$

$$g_{yr}(s) = \frac{\frac{k_a}{s}\frac{\bar{N}(s)}{\bar{D}(s)}}{1+\frac{k_a}{s}\frac{\bar{N}(s)}{\bar{D}(s)}} = \frac{k_a\bar{N}(s)}{s\bar{D}(s)+k_a\bar{N}(s)} = \frac{k_a\bar{N}(s)}{\Delta_f(s)}$$

$$\text{If } r(t)=\bar{r},\ \ \mathcal{L}(w)=\bar{r}/s,$$

$$y_r(s) = \frac{k_a\bar{N}(s)}{\Delta_f(s)}\frac{\bar{r}}{s},$$

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} s y_r(s) = \frac{k_a\bar{N}(0)\bar{r}}{0\cdot\bar{D}(s)+k_a\bar{N}(0)} = \frac{k_a\bar{N}(0)\bar{r}}{k_a\bar{N}(0)} = \bar{r}$$

Regulation and Tracking

Stabilization

$$\begin{bmatrix} \dot{\bar{\mathbf{x}}}_c \\ \dot{\bar{\mathbf{x}}}_{\bar{c}} \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{A}}_c & \bar{\mathbf{A}}_{12} \\ \mathbf{0} & \bar{\mathbf{A}}_{\bar{c}} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{x}}_c \\ \bar{\mathbf{x}}_{\bar{c}} \end{bmatrix} + \begin{bmatrix} \bar{\mathbf{b}}_c \\ \mathbf{0} \end{bmatrix} u$$

$$u = r - \mathbf{k}\mathbf{x} = r - \bar{\mathbf{k}}\bar{\mathbf{x}} = r - \begin{bmatrix} \bar{\mathbf{k}}_1 & \bar{\mathbf{k}}_2 \end{bmatrix} \begin{bmatrix} \bar{\mathbf{x}}_c \\ \bar{\mathbf{x}}_{\bar{c}} \end{bmatrix}$$

$$\begin{bmatrix} \dot{\bar{\mathbf{x}}}_c \\ \dot{\bar{\mathbf{x}}}_{\bar{c}} \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{A}}_c - \bar{\mathbf{b}}_c \bar{\mathbf{k}}_1 & \bar{\mathbf{A}}_{12} - \bar{\mathbf{b}}_c \bar{\mathbf{k}}_2 \\ \mathbf{0} & \bar{\mathbf{A}}_{\bar{c}} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{x}}_c \\ \bar{\mathbf{x}}_{\bar{c}} \end{bmatrix} + \begin{bmatrix} \bar{\mathbf{b}}_c \\ \mathbf{0} \end{bmatrix} r$$

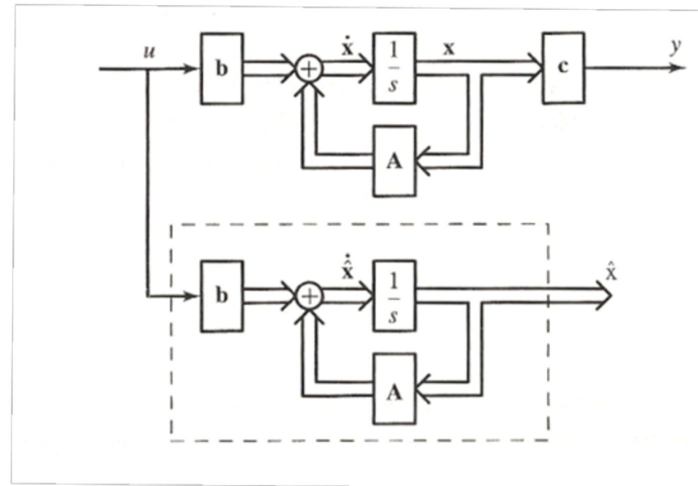
$\bar{\mathbf{A}}_{\bar{c}}$ is not affected by state feedback.

If $\bar{\mathbf{A}}_{\bar{c}}$ is stable and $(\bar{\mathbf{A}}_c, \ \bar{\mathbf{b}}_c)$ is controllable,

$(\mathbf{A}, \ \mathbf{b})$ is said to be **stabilizable**.

State Estimator Design

State Estimator



Open − loop state Estimator

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u$$

$$y = \mathbf{c}\mathbf{x}$$

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\mathbf{x} + \mathbf{b}u$$

Two problems

1. Initial state must be computed $\leftarrow$ {A, c} is observable
2. If **A** is unstable, the estimate error may diverge.

$$\dot{\mathbf{x}} - \dot{\hat{\mathbf{x}}} = \mathbf{A}(\mathbf{x} - \mathbf{x}),$$

$$\mathrm{e}(t) := \mathbf{x}(t) - \hat{\mathbf{x}}(t)$$

$$\dot{\mathrm{e}} = \mathbf{A}\mathrm{e}, \quad \mathrm{e} \to \infty \quad \text{if } \mathrm{Re}(\lambda(\mathbf{A})) > 0.$$

State Estimator Design

Closed-loop State Estimator

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\mathbf{x} + \mathbf{b}u + \mathbf{l}(y - \mathbf{c}\mathbf{x})$$

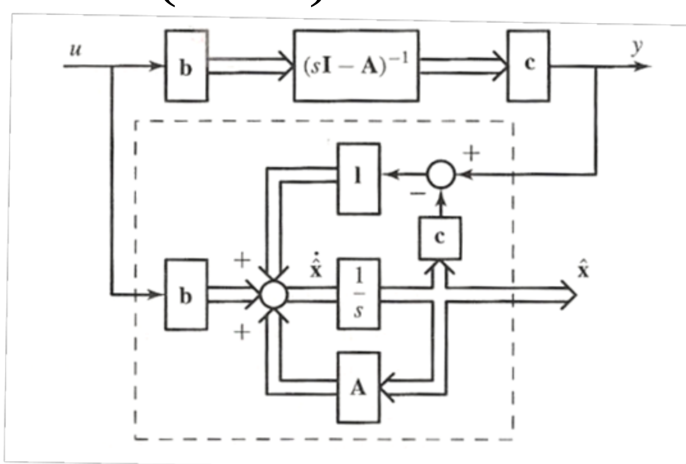
$$\dot{\hat{\mathbf{x}}} = (\mathbf{A} - \mathbf{l}\mathbf{c})\mathbf{x} + \mathbf{b}u + \mathbf{l}y$$

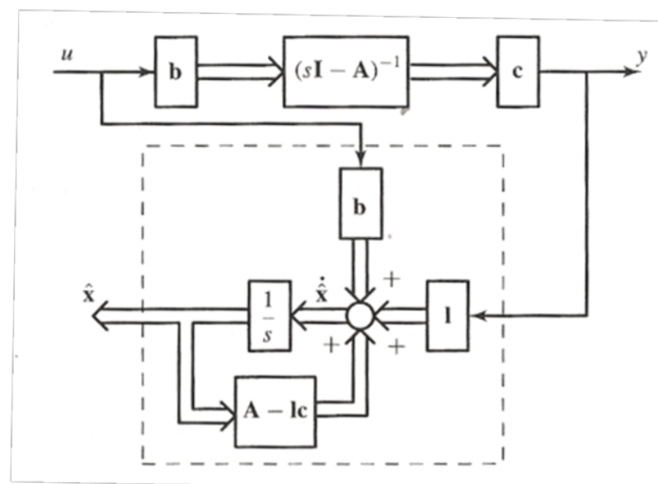
$$\mathbf{e}(t) := \mathbf{x}(t) - \hat{\mathbf{x}}(t)$$

$$\dot{\mathbf{e}} = \dot{\mathbf{x}} - \dot{\hat{\mathbf{x}}} = \mathbf{A}\mathbf{x} + \mathbf{b}u - (\mathbf{A} - \mathbf{l}\mathbf{c})\mathbf{x} - \mathbf{b}u - \mathbf{l}(\mathbf{c}\mathbf{x})$$

$$= (\mathbf{A} - \mathbf{l}\mathbf{c})\mathbf{x} - (\mathbf{A} - \mathbf{l}\mathbf{c})\hat{\mathbf{x}} = (\mathbf{A} - \mathbf{l}\mathbf{c})(\mathbf{x} - \mathbf{x})$$

$$\dot{\mathbf{e}} = (\mathbf{A} - \mathbf{l}\mathbf{c})\mathbf{e}$$





State Estimator Design

Theorem 8.03

Consider the pair $(\mathbf{A}, \mathbf{c})$. All eigenvalues of $(\mathbf{A} - \mathbf{lc})$ can be assigned arbitrarily by selecting a real constant vector $\mathbf{l}$ if and only if $(\mathbf{A}, \mathbf{c})$ is observable.

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u$$

$$y = \mathbf{c}\mathbf{x}$$

Procedure 8.01

1. Select an arbitrary $n \times n$ stable matrix $\mathbf{F}$ that has no eigenvalues in common with those of $\mathbf{A}$.
2. Select an arbitrary $n \times 1$ vector $\mathbf{l}$ such that $(\mathbf{F}, \mathbf{l})$ is controllable.

State Estimator Design

3. Solve the unique $\mathbf{T}$ in the Lyapunov equation $\mathbf{TA - FT = lc}$. This $\mathbf{T}$ is nonsingular following the dual of Theorem 8.4.
4. Then the state equation

$$\dot{\mathbf{z}} = \mathbf{Fz} + \mathbf{Tb}u + \mathbf{l}y$$
$$\hat{\mathbf{x}} = \mathbf{T}^{-1}\mathbf{z}$$

generates an estimate of $\mathbf{x}$.

Verify :

$$\mathbf{e} := \mathbf{z} - \mathbf{Tx}$$
$$\dot{\mathbf{e}} = \dot{\mathbf{z}} - \mathbf{T}\dot{\mathbf{x}} = \mathbf{Fz} + \mathbf{Tb}u + \mathbf{lcx} - \mathbf{TAx} - \mathbf{Tb}u$$
$$= \mathbf{Fz} + \mathbf{lcx} - (\mathbf{FT} + \mathbf{lc})\mathbf{x} = \mathbf{F}(\mathbf{z} - \mathbf{Tx}) = \mathbf{Fe}$$
$$\lim_{t\to\infty} \mathbf{e} = 0 \to \lim_{t\to\infty} \mathbf{z} = \mathbf{Tx} \to \lim_{t\to\infty} \mathbf{T}^{-1}\mathbf{z} = \mathbf{x}$$

State Estimator Design

Reduced-Dimensional State Estimator

Procedure 8.R1

1. Select an arbitrary $(n-1)\times(n-1)$ stable matrix $\mathbf{F}$ that has no eigenvalues in common with those of $\mathbf{A}$.
2. Select an arbitrary $(n-1)\times 1$ vector $\mathbf{l}$ such that $(\mathbf{F},\mathbf{l})$ is controllable.
3. Solve the unique $\mathbf{T}$ in the Lyapunov equation $\mathbf{TA}-\mathbf{FT}=\mathbf{lc}$. Note that $\mathbf{T}$ is an $(n-1)\times 1$ matrix.
4. Then the $(n-1)$-dimensional state equation

$$\dot{\mathbf{z}} = \mathbf{Fz} + \mathbf{Tb}u + \mathbf{l}y$$

$$\hat{\mathbf{x}} = \begin{bmatrix} \mathbf{c} \\ \mathbf{T} \end{bmatrix}^{-1} \begin{bmatrix} y \\ \mathbf{z} \end{bmatrix}$$

is an estimate of $\mathbf{x}$.

State Estimator Design

$$\begin{bmatrix} y \\ \mathbf{z} \end{bmatrix} = \begin{bmatrix} \mathbf{c} \\ \mathbf{T} \end{bmatrix} \hat{\mathbf{x}} =: \mathbf{P}\mathbf{x}$$

$$\mathbf{e} = \mathbf{z} - \mathbf{T}\mathbf{x}$$

$$\dot{\mathbf{e}} = \dot{\mathbf{z}} - \mathbf{T}\dot{\mathbf{x}} = \mathbf{F}\mathbf{z} + \mathbf{T}\mathbf{b}u + \mathbf{l}\mathbf{c}\mathbf{x} - \mathbf{T}\mathbf{A}\mathbf{x} - \mathbf{T}\mathbf{b}u = \mathbf{F}\mathbf{e}$$

Theorem 8.6

If $\mathbf{A}$ and $\mathbf{F}$ have no common eigenvalues,
then the square matrix

$$\mathbf{P} = \begin{bmatrix} \mathbf{c} \\ \mathbf{T} \end{bmatrix},$$

where $\mathbf{T}$ is the unique solution of $\mathbf{T}\mathbf{A} - \mathbf{F}\mathbf{T} = \mathbf{l}\mathbf{c}$,
is nonsingular if and only if $(\mathbf{A}, \mathbf{c})$ is observable
and $(\mathbf{F}, \mathbf{l})$ is controllable.

State Estimator Design

Proof :

Let $\Delta(s) = \det(s\mathbf{I} - \mathbf{A}) = s^4 + \alpha_1 s^3 + \alpha_2 s^2 + \alpha_3 s + \alpha_4$
Dual to (8.22),

$$-\mathbf{T}\Delta(\mathbf{F}) = \begin{bmatrix} \mathbf{l} & \mathbf{F}\mathbf{l} & \mathbf{F}^2\mathbf{l} & \mathbf{F}^3\mathbf{l} \end{bmatrix} \begin{bmatrix} \alpha_3 & \alpha_2 & \alpha_1 & 1 \\ \alpha_2 & \alpha_1 & 1 & 0 \\ \alpha_1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{c} \\ \mathbf{cA} \\ \mathbf{cA}^2 \\ \mathbf{cA}^3 \end{bmatrix}$$

$$:= C_4 \Lambda O$$

$\Delta(\mathbf{F})$ is nonsingular if $\mathbf{A}$ and $\mathbf{F}$ have no common eigenvalues.
Then $\mathbf{T} = -\Delta(\mathbf{F})^{-1} C_4 \Lambda O$ and $\mathbf{P}$ becomes

$$\mathbf{P} = \begin{bmatrix} \mathbf{c} \\ \mathbf{T} \end{bmatrix} = \begin{bmatrix} \mathbf{c} \\ -\Delta^{-1}(\mathbf{F}) C_4 \Lambda O \end{bmatrix} = \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & -\Delta^{-1}(\mathbf{F}) \end{bmatrix} \begin{bmatrix} \mathbf{c} \\ C_4 \Lambda O \end{bmatrix}$$

If $(\mathbf{F}, \mathbf{l})$ is not controllable, C_4 has rank 2 at most and $\mathbf{P}$ is singular.
If $(\mathbf{A}, \mathbf{c})$ is not observable, O has at least 1-dim. null space and there exists $\mathrm{r} \neq 0$ such that $O\mathrm{r}=0$ which implies $\mathbf{c}\mathrm{r}=0$ and $\mathbf{P}\,\mathrm{r} = 0$.
Thus $\mathbf{P}$ is singular. This is proof of the necessity part.

State Estimator Design

The sufficiency is proved by contraction.

Suppose P is singular, then there exists $\mathbf{r} \neq 0$ such that

$$\begin{bmatrix} \mathbf{c} \\ C_4 \Lambda O \mathbf{r} \end{bmatrix} \mathbf{r} = \begin{bmatrix} \mathbf{cr} \\ C_4 \quad O \end{bmatrix} = \mathbf{0}$$

Define $\mathbf{a} \text{:} \mathbf{A} \mathbf{r} \ O \ = [a_1 \quad a_2 \quad a_3 \quad a_4]' \ \mathbf{a} = [^- \quad a_4]$

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} \alpha_3 & \alpha_2 & \alpha_1 & 1 \\ \alpha_2 & \alpha_1 & 1 & 0 \\ \alpha_1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{cr} \\ \mathbf{cAr} \\ \mathbf{cA}^2\mathbf{r} \\ \mathbf{cA}^3\mathbf{r} \end{bmatrix} = \begin{bmatrix} x \\ x \\ x \\ \mathbf{cr} = 0 \end{bmatrix}$$

$C_4 \Lambda O \ = C a_4 \ = \mathbf{a} C^- \ \mathbf{0} \ \rightarrow \mathbf{a}^- = 0$ if $(\mathbf{F}, \mathbf{l}\ \}$ is controllable.

$\rightarrow \mathbf{a} = 0$ if $(\mathbf{F}, \mathbf{l}\}$ is controllable.

$\rightarrow \mathbf{a} \text{:} \mathbf{A} \mathbf{r} \ O \ = 0 \mathbf{r} \rightarrow \ = 0$ if $(\mathbf{A}, \mathbf{c}\ \}$ is observable.

$\rightarrow$ *contradict*.

HW 8-2

3. Find a reduced dimensional state estimator for the state equation and Verify the validity of the designed estimator through Matlab simulation

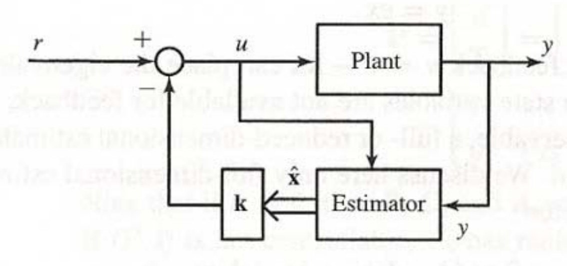
$$\dot{\mathbf{x}} = \begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \mathbf{x}.$$

Select the eigenvalues -3 and $-3 \pm 2j$.

State Feedback from Estimated States

Feedback from Estimated States



The state equation:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u$$

$$y = \mathbf{c}\mathbf{x}$$

If the state is not measurable,

$$\dot{\hat{\mathbf{x}}} = (\mathbf{A} - \mathbf{l}\mathbf{c})\mathbf{x} + \mathbf{b}u + \mathbf{l}y$$

we use the estimated state for feedback

$$u = r - \mathbf{k}\hat{\mathbf{x}}$$

Then

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} - \mathbf{b}\mathbf{k}\hat{\mathbf{x}} + \mathbf{b}r$$

$$\dot{\hat{\mathbf{x}}} = (\mathbf{A} - \mathbf{l}\mathbf{c})\mathbf{x} + \mathbf{b}(r - \mathbf{k}\mathbf{x}) + \mathbf{l}\mathbf{c}\mathbf{x}$$

Matrix form:

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\hat{\mathbf{x}}} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & -\mathbf{b}\mathbf{k} \\ \mathbf{l}\mathbf{c} & \mathbf{A} - \mathbf{l}\mathbf{c} - \mathbf{b}\mathbf{k} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \hat{\mathbf{x}} \end{bmatrix} + \begin{bmatrix} \mathbf{b} \\ \mathbf{b} \end{bmatrix} r$$

$$y = \begin{bmatrix} \mathbf{c} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \hat{\mathbf{x}} \end{bmatrix}$$

State Feedback from Estimated States

Separation Property

By selecting a equivalence transformation

$$\begin{bmatrix}\mathbf{x}\\ \mathbf{e}\end{bmatrix}=\begin{bmatrix}\mathbf{x}\\ \mathbf{x}-\hat{\mathbf{x}}\end{bmatrix}=\begin{bmatrix}\mathbf{I} & \mathbf{0}\\ \mathbf{I} & -\mathbf{I}\end{bmatrix}\begin{bmatrix}\mathbf{x}\\ \mathbf{x}\end{bmatrix}=:\mathbf{P}\begin{bmatrix}\mathbf{x}\\ \mathbf{x}\end{bmatrix}$$

The equivalent eq. is

$$\begin{bmatrix}\dot{\mathbf{x}}\\ \dot{\mathbf{e}}\end{bmatrix}=\begin{bmatrix}\mathbf{A}-\mathbf{bk} & \mathbf{bk}\\ \mathbf{0} & \mathbf{A}-\mathbf{lc}\end{bmatrix}\begin{bmatrix}\mathbf{x}\\ \mathbf{e}\end{bmatrix}+\begin{bmatrix}\mathbf{b}\\ \mathbf{0}\end{bmatrix}r$$

$$y=\begin{bmatrix}\mathbf{c} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{x}\\ \mathbf{e}\end{bmatrix}$$

Controllable part is given by

$$\dot{\mathbf{x}}=(\mathbf{A}-\mathbf{bk})\mathbf{x}+\mathbf{b}r \quad y=\mathbf{cx}$$

Overall transfer function becomes

$$g_f(s)=\mathbf{c}(s\mathbf{I}-\mathbf{A}+\mathbf{bk})^{-1}\mathbf{b}$$

State Feedback-Multivariable Case

State Feedback-Multivariable Case

Theorem 8.M1

The pair $(\mathbf{A}-\mathbf{BK}, \mathbf{B})$, for any $p\times n$ real constant matrix $\mathbf{K}$, is controllable if and only if $(\mathbf{A}, \mathbf{B})$ is controllable.

Proof :

The proof follows the proof of Theorem 8.1.

The only difference is that (8.4) is modified as:

$$C_f = C\begin{bmatrix} \mathbf{I}_p & -\mathbf{KB} & -\mathbf{K}(\mathbf{A}-\mathbf{BK})\mathbf{B} & -\mathbf{K}(\mathbf{A}-\mathbf{BK})^2\mathbf{B} \\ \mathbf{0} & \mathbf{I}_p & -\mathbf{KB} & -\mathbf{K}(\mathbf{A}-\mathbf{BK})\mathbf{B} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_p & -\mathbf{KB} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_p \end{bmatrix}$$

State Feedback-Multivariable Case

Theorem 8.M3

All eigenvalues $(\mathbf{A}-\mathbf{BK})$ can be assigned arbitrarily (provided complex conjugate eigenvalues are assigned in pairs) by selecting a real constatnt $\mathbf{K}$ if and only if $(\mathbf{A}, \mathbf{B})$ is controllable.

State Feedback-Multivariable Case

Cyclic Design

Theorem 8.7

If the n-dimensional p-input pair $(\mathbf{A}, \mathbf{B})$ is controllable and if $\mathbf{A}$ is cyclic, then for almost any $p \times 1$ vector $\mathbf{v}$, the single-input pair $(\mathbf{A}, \mathbf{Bv})$ is controllable.

$\mathbf{A}$ is cyclic if its characteristic polynomial equals its minimal polynomial.
$\mathbf{A}$ is cyclic iff its Jordan form has only one Jordan block for each distinct eigenvalue.

State Feedback-Multivariable Case

Intuitive Validation :

Controllability is invariant under any equivalence transformation, thus we assume $\mathbf{A}$ to be Jordan form

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 2 \\ 4 & 3 \\ 1 & 0 \end{bmatrix} \quad \mathbf{Bv} = \mathbf{B}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} x \\ x \\ \alpha \\ x \\ \beta \end{bmatrix}$$

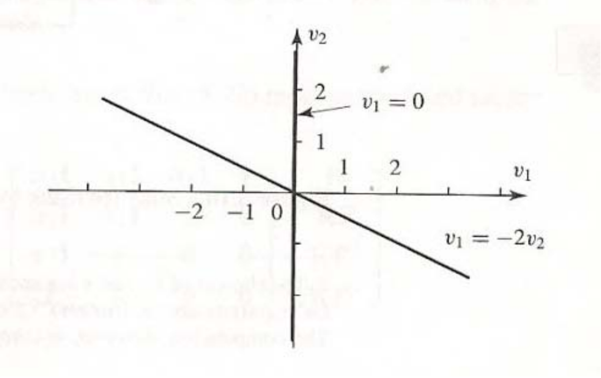
$\rightarrow (\mathbf{A}, \mathbf{Bv})$ is controllable iff $\alpha \neq 0,\ \beta \neq 0.$

$\leftarrow \alpha = v_1 + 2v_2 \neq 0$ and $\beta = v_1 \neq 0$

$\rightarrow$ *Almost controllable*

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \\ 1 & 0 \end{bmatrix}$$

$\rightarrow \exists$ no v s.t. $(\mathbf{A}, \mathbf{B}v)$ is controllable



State Feedback-Multivariable Case

Theorem 8.8

If $(\mathbf{A}, \mathbf{B})$ is controllable, then for almost any

$p \times n$ real constant matrix $\mathbf{K}$, the matrix $(\mathbf{A} - \mathbf{BK})$

has only distinct eigenvalues and is, consequently, cyclic.

Intuitive Verification:

$(\mathbf{A} - \mathbf{BK})$ has

$$\Delta_f(s) = s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4$$

where a_i are functions of the entries of $\mathbf{K}$. By differentiation

$$\Delta'_f(s) = 4s^3 + 3a_1 s^2 + 2a_2 s + a_3$$

If $\Delta_f(s)$ has repeated roots, then $\Delta_f(s)$ and $\Delta'_f(s)$ are not coprime.

Then $\exists$ a coprime fraction $\bar{\Delta}'_f(s) / \bar{\Delta}_f(s)$ such that

$$\Delta'_f(s) / \Delta_f(s) = \bar{\Delta}'_f(s) / \bar{\Delta}_f(s)$$

State Feedback-Multivariable Case

The sufficient and necessary condition is

Sylvester resultant is singular

$$\det\begin{bmatrix} a_4 & a_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_3 & 2a_2 & a_4 & a_3 & 0 & 0 & 0 & 0 \\ a_2 & 3a_1 & a_3 & 2a_2 & a_4 & a_3 & 0 & 0 \\ a_1 & 4 & a_2 & 3a_1 & a_3 & 2a_2 & a_4 & a_3 \\ 1 & 0 & a_1 & 4 & a_2 & 3a_1 & a_3 & 2a_2 \\ 0 & 0 & 1 & 0 & a_1 & 4 & a_2 & 3a_1 \\ 0 & 0 & 0 & 0 & 1 & 0 & a_1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} = b(k_{ij}) = 0$$

The solution space is a line(a very small portion)

in high dimensional space.

Thus Sylvester resultant is almost nonsingular, and

$\mathbf{A}-\mathbf{BK}$ is almost cyclic.

State Feedback-Multivariable Case

If $\mathbf{A}$ is not cyclic, we can choose $\mathbf{u} = \mathbf{w} - \mathbf{K}_1\mathbf{x},$ then

$$\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{B}\mathbf{K}_1)\mathbf{x} + \mathbf{B}\mathbf{w} =: \bar{\mathbf{A}}\mathbf{x} + \mathbf{B}\mathbf{w}$$

where $\bar{\mathbf{A}}$ is cyclic. $(\bar{\mathbf{A}}, \mathbf{B})$ is controllable since $(\mathbf{A}, \mathbf{B})$ is controllable.

Thus $\exists\ \mathbf{v}$ such that $(\bar{\mathbf{A}}, \mathbf{B}\mathbf{v})$ is controllable. By choosing

$$\mathbf{w} = \mathbf{r} - \mathbf{K}_2\mathbf{x}, \text{ with } \mathbf{K}_2 = \mathbf{v}\mathbf{k}.$$

Then the system becomes

$$\dot{\mathbf{x}} = (\bar{\mathbf{A}} - \mathbf{B}\mathbf{K}_2)\mathbf{x} + \mathbf{B}\mathbf{r} = (\bar{\mathbf{A}} - \mathbf{B}\mathbf{v}\mathbf{k})\mathbf{x} + \mathbf{B}\mathbf{r}$$

The resulting state feedback control becomes

$$\mathbf{u} = \mathbf{r} - (\mathbf{K}_1 + \mathbf{K}_2)\mathbf{x} =: \mathbf{r} - \mathbf{K}\mathbf{x}.$$

The $\mathbf{K} = \mathbf{K}_1 + \mathbf{K}_2$ can achieve arbitrary eigenvalue assignment.

State Feedback-Multivariable Case

Lyapunov-Equation Method

1. Select an $n \times n$ matrix $\mathbf{F}$ with a set of desired eigenvalues that contains no eigenvalues of $\mathbf{A}$.
2. Select an arbitrary $p \times n$ matrix $\bar{\mathbf{K}}$ such that $(\mathbf{F}, \bar{\mathbf{K}})$ is obaservable.
3. Solve the unique $\mathbf{T}$ in the Lyapunov equation $\mathbf{AT} - \mathbf{TF} = \mathbf{B}\bar{\mathbf{K}}$.
4. If $\mathbf{T}$ is singular, select a different $\bar{\mathbf{K}}$ and repeat the process. If $\mathbf{T}$ is nonsingular, we compute $\mathbf{K} = \bar{\mathbf{K}}\mathbf{T}^{-1}$, and $(\mathbf{A} - \mathbf{BK})$ has the set of desired eigenvalues.

The Lyapunov equatuion becomes

$$(\mathbf{A} - \mathbf{BK})\mathbf{T} = \mathbf{TF} \quad \text{or} \quad \mathbf{A} - \mathbf{BK} = \mathbf{TFT}^{-1}$$

State Feedback-Multivariable Case

Theorem 8.M4

If $\mathbf{A}$ and $\mathbf{F}$ have no eigenvalues in common, then the unique solution $\mathbf{T}$ of $\mathbf{AT}-\mathbf{TF}=\mathbf{B}\bar{\mathbf{K}}$ is nonsingular only if $(\mathbf{A},\mathbf{B})$ is controllable and $(\mathbf{F},\bar{\mathbf{K}})$ is observable.

Proof :

The proof of Theorem 8.4 applies here except that (8.22) must be modified as

$$-\mathbf{T}\Delta(\mathbf{F})=\begin{bmatrix}\mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} & \mathbf{A}^3\mathbf{B}\end{bmatrix}\begin{bmatrix}\alpha_3\mathbf{I} & \alpha_2\mathbf{I} & \alpha_1\mathbf{I} & \mathbf{I}\\ \alpha_2\mathbf{I} & \alpha_1\mathbf{I} & \mathbf{I} & \mathbf{0}\\ \alpha_1\mathbf{I} & \mathbf{I} & \mathbf{0} & \mathbf{0}\\ \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\bar{\mathbf{K}}\\ \bar{\mathbf{K}}\mathbf{F}\\ \bar{\mathbf{K}}\mathbf{F}^2\\ \bar{\mathbf{K}}\mathbf{F}^3\end{bmatrix}$$

$$-\mathbf{T}\Delta(\mathbf{F})=C\Sigma O$$

where $\Delta(\mathbf{F})$ is nonsingular. If $(\mathbf{A},\mathbf{B})$ is uncontrollable or $(\mathbf{F},\bar{\mathbf{K}})$ is unobservable, then $\mathbf{T}$ is singular.
The contraction statement is true.

State Feedback-Multivariable Case

Canonical Form Method

$$\dot{\bar{\mathbf{x}}} = \begin{bmatrix} -\alpha_{111} & -\alpha_{112} & -\alpha_{113} & -\alpha_{114} & \vdots & -\alpha_{121} & -\alpha_{122} \\ 1 & 0 & 0 & 0 & \vdots & 0 & 0 \\ 0 & 1 & 0 & 0 & \vdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \vdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -\alpha_{211} & -\alpha_{212} & -\alpha_{213} & -\alpha_{214} & \vdots & -\alpha_{221} & -\alpha_{222} \\ 0 & 0 & 0 & 0 & \vdots & 1 & 0 \end{bmatrix} \bar{\mathbf{x}} + \begin{bmatrix} 1 & b_{12} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \cdots & \cdots \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{u}$$

$$y = \begin{bmatrix} \beta_{111} & \beta_{112} & \beta_{113} & \beta_{114} & \beta_{121} & \beta_{122} \\ \beta_{211} & \beta_{212} & \beta_{213} & \beta_{214} & \beta_{221} & \beta_{222} \end{bmatrix} \bar{\mathbf{x}}$$

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State Feedback-Multivariable Case

Desired characteristic polynomial is given by

$$\Delta_f(s) = (s^4 + \bar{\alpha}_{111}s^3 + \bar{\alpha}_{112}s^2 + \bar{\alpha}_{113}s + \bar{\alpha}_{114})(s^2 + \bar{\alpha}_{221}s + \bar{\alpha}_{222}).$$

Let us select $\bar{\mathbf{K}}$ as

$$\bar{\mathbf{K}} = \begin{bmatrix} 1 & b_{12} \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \bar{\alpha}_{111} - \alpha_{111} & \bar{\alpha}_{112} - \alpha_{112} & \bar{\alpha}_{113} - \alpha_{113} & \bar{\alpha}_{114} - \alpha_{114} & -\alpha_{121} & -\alpha_{122} \\ \bar{\alpha}_{211} - \alpha_{211} & \bar{\alpha}_{212} - \alpha_{212} & \bar{\alpha}_{213} - \alpha_{213} & \bar{\alpha}_{214} - \alpha_{214} & \bar{\alpha}_{221} - \alpha_{221} & \bar{\alpha}_{222} - \alpha_{222} \end{bmatrix}$$

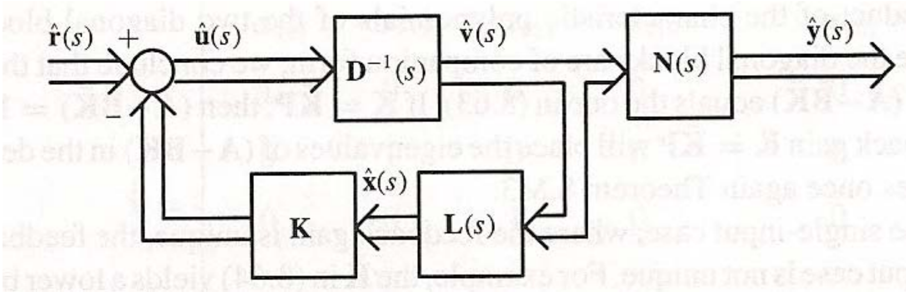
where $\bar{\alpha}_{21i}$ are arbitary real constants. Then we have

$$\bar{\mathbf{A}} - \bar{\mathbf{B}}\bar{\mathbf{K}} = \begin{bmatrix} -\bar{\alpha}_{111} & -\bar{\alpha}_{112} & -\bar{\alpha}_{113} & -\bar{\alpha}_{114} & \vdots & 0 & 0 \\ 1 & 0 & 0 & 0 & \vdots & 0 & 0 \\ 0 & 1 & 0 & 0 & \vdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \vdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -\bar{\alpha}_{211} & -\bar{\alpha}_{212} & -\bar{\alpha}_{213} & -\bar{\alpha}_{214} & \vdots & -\bar{\alpha}_{221} & -\bar{\alpha}_{222} \\ 0 & 0 & 0 & 0 & \vdots & 1 & 0 \end{bmatrix}.$$

State Feedback-Multivariable Case

Effects on Transfer Matrices

$$\mathbf{G}(s) = \mathbf{N}(s)\mathbf{D}^{-1}(s)$$



$$\mathbf{y}(s) = \mathbf{N}(s)\mathbf{D}^{-1}(s)\mathbf{u}(s)$$

$$\mathbf{D}(s)\mathbf{v}(s) = \mathbf{u}(s)$$

$$\mathbf{y}(s) = \mathbf{N}(s)\mathbf{v}(s)$$

$$\mathbf{x}(s) = \mathbf{L}(s)\mathbf{v}(s)$$

$$\mathbf{u}(s) = \mathbf{r}(s) - \mathbf{K}\mathbf{x}(s) = \mathbf{r}(s) - \mathbf{K}\mathbf{L}(s)\mathbf{v}(s)$$

$$\mathbf{D}(s) = \mathbf{D}_{hc}\mathbf{H}(s) + \mathbf{D}_{lc}\mathbf{L}(s)$$

$$\left[\mathbf{D}_{hc}\mathbf{H}(s) + \mathbf{D}_{lc}\mathbf{L}(s)\right]\mathbf{v}(s) = \mathbf{r}(s) - \mathbf{K}\mathbf{L}(s)\mathbf{v}(s)$$

$$\left[\mathbf{D}_{hc}\mathbf{H}(s) + (\mathbf{D}_{lc} + \mathbf{K})\mathbf{L}(s)\right]\mathbf{v}(s) = \mathbf{r}(s)$$

$$\mathbf{y}(s) = \mathbf{N}(s)\left[\mathbf{D}_{hc}\mathbf{H}(s) + (\mathbf{D}_{lc} + \mathbf{K})\mathbf{L}(s)\right]^{-1}\mathbf{r}(s)$$

$$\mathbf{G}_f(s) = \mathbf{N}(s)\left[\mathbf{D}_{hc}\mathbf{H}(s) + (\mathbf{D}_{lc} + \mathbf{K})\mathbf{L}(s)\right]^{-1} = \mathbf{N}(s)\left[\mathbf{D}(s) + \mathbf{K}\mathbf{L}(s)\right]^{-1}$$

State Estimators-Multivariable Case

State Estimators-Multivariable Case

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x}$$

$$\dot{\hat{\mathbf{x}}} = (\mathbf{A} - \mathbf{L}\mathbf{C})\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{L}\mathbf{y}$$

$$\mathbf{e}(t) := \mathbf{x}(t) - \hat{\mathbf{x}}(t)$$

$$\dot{\mathbf{e}} = (\mathbf{A} - \mathbf{L}\mathbf{C})\mathbf{e}$$

State Estimators-Multivariable Case

Procedure 8.MR1

Consider the n-dimensional q-output observable pair $(\mathbf{A}, \mathbf{C})$. It is assumed that $\mathbf{C}$ has rank q.

1. Select an arbitrary $(n-q)\times(n-q)$ stable matrix $\mathbf{F}$ that has no eigenvalues in common with those $\mathbf{A}$.
2. Select an arbitrary $(n-q)\times q$ matrix $\mathbf{L}$ such that $(\mathbf{F}, \mathbf{L})$ is controllable.
3. Solve the unique $(n-q)\times n$ matrix $\mathbf{T}$ in the Lyapunov equation $\mathbf{TA}-\mathbf{FT}=\mathbf{LC}$.
4. If the square matrix of order n

$$\mathbf{P}=\begin{bmatrix}\mathbf{C}\\ \mathbf{T}\end{bmatrix}$$

is singular, go back to Step 2 and repeat the process.

State Estimators-Multivariable Case

If $\mathbf{P}$ is nonsingular, then the $(n-q)$-dimensional state equation

$$\dot{\mathbf{z}} = \mathbf{Fz} + \mathbf{TBu} + \mathbf{Ly}$$

$$\mathbf{x} = \begin{bmatrix} \mathbf{C} \\ \mathbf{T} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{y} \\ \mathbf{z} \end{bmatrix}$$

generates an estimate of $\mathbf{x}$.

To justify the procedure,

$$\begin{bmatrix} \mathbf{y} \\ \mathbf{z} \end{bmatrix} = \begin{bmatrix} \mathbf{C} \\ \mathbf{T} \end{bmatrix} \mathbf{x}$$

$$\mathbf{e} := \mathbf{z} - \mathbf{Tx}$$

$$\dot{\mathbf{e}} = \dot{\mathbf{z}} - \mathbf{T}\dot{\mathbf{x}} = \mathbf{Fz} + \mathbf{TBu} + \mathbf{LCx} - \mathbf{TAx} - \mathbf{TBu}$$

$$= \mathbf{Fz} + (\mathbf{LC} - \mathbf{TA})\mathbf{x} = \mathbf{F}(\mathbf{z} - \mathbf{Tx}) = \mathbf{Fe}.$$

Since $\mathbf{F}$ is selected as stable, $\mathbf{e} \to 0$ as $t \to \infty$.

State Estimators-Multivariable Case

Theorem 8.M6

If $\mathbf{A}$ and $\mathbf{F}$ have no common eigenvalues,
then the square matrix

$$\mathbf{P} := \begin{bmatrix} \mathbf{C} \\ \mathbf{T} \end{bmatrix},$$

where $\mathbf{T}$ is the unique solution of $\mathbf{TA} - \mathbf{FT} = \mathbf{LC}$,
is nonsingular only if $(\mathbf{A}, \mathbf{C})$ is observable and
$(\mathbf{F}, \mathbf{L})$ is controllable.

Feedback from Estimated States-Multivariable Case

Feedback from Estimated States-Multivariable Case

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x}$$

$$\begin{bmatrix} \mathbf{Q}_1 & \mathbf{Q}_2 \end{bmatrix} \begin{bmatrix} \mathbf{C} \\ \mathbf{T} \end{bmatrix} = \mathbf{Q}_1\mathbf{C} + \mathbf{Q}_2\mathbf{T} = \mathbf{I}$$

$$\dot{\mathbf{z}} = \mathbf{F}\mathbf{z} + \mathbf{T}\mathbf{B}\mathbf{u} + \mathbf{L}\mathbf{y}$$

$$\mathbf{x} = \mathbf{Q}_1\mathbf{y} + \mathbf{Q}_2\mathbf{z}$$

$$\mathbf{u} = \mathbf{r} - \mathbf{K}\mathbf{x} = \mathbf{r} - \mathbf{K}\mathbf{Q}_1\mathbf{y} - \mathbf{K}\mathbf{Q}_2\mathbf{z}$$

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}(\mathbf{r} - \mathbf{K}\mathbf{Q}_1\mathbf{C}\mathbf{x} - \mathbf{K}\mathbf{Q}_2\mathbf{z}) \\ &= (\mathbf{A} - \mathbf{B}\mathbf{K}\mathbf{Q}_1\mathbf{C})\mathbf{x} - \mathbf{B}\mathbf{K}\mathbf{Q}_2\mathbf{z} + \mathbf{B}\mathbf{r} \end{aligned}$$

$$\begin{aligned} \dot{\mathbf{z}} &= \mathbf{F}\mathbf{z} + \mathbf{T}\mathbf{B}(\mathbf{r} - \mathbf{K}\mathbf{Q}_1\mathbf{C}\mathbf{x} - \mathbf{K}\mathbf{Q}_2\mathbf{z}) + \mathbf{L}\mathbf{C}\mathbf{x} \\ &= (\mathbf{L}\mathbf{C} - \mathbf{T}\mathbf{B}\mathbf{K}\mathbf{Q}_1\mathbf{C})\mathbf{x} + (\mathbf{F} - \mathbf{T}\mathbf{B}\mathbf{K}\mathbf{Q}_2)\mathbf{z} + \mathbf{T}\mathbf{B}\mathbf{r} \end{aligned}$$

Feedback from Estimated States- Multivariable Case

$$\begin{bmatrix}\dot{\mathbf{x}}\\ \dot{\mathbf{z}}\end{bmatrix}=\begin{bmatrix}\mathbf{A}-\mathbf{BKQ}_1\mathbf{C} & -\mathbf{BKQ}_2\\ \mathbf{LC}-\mathbf{TBKQ}_1\mathbf{C} & \mathbf{F}-\mathbf{TBKQ}_2\end{bmatrix}\begin{bmatrix}\mathbf{x}\\ \mathbf{z}\end{bmatrix}+\begin{bmatrix}\mathbf{B}\\ \mathbf{TB}\end{bmatrix}\mathbf{r}$$

$$\mathbf{y}=[\mathbf{C}\quad \mathbf{0}]\begin{bmatrix}\mathbf{x}\\ \mathbf{z}\end{bmatrix}$$

$$\begin{bmatrix}\mathbf{x}\\ \mathbf{e}\end{bmatrix}=\begin{bmatrix}\mathbf{x}\\ \mathbf{z}-\mathbf{Tx}\end{bmatrix}=\begin{bmatrix}\mathbf{I}_n & \mathbf{0}\\ -\mathbf{T} & \mathbf{I}_{n-q}\end{bmatrix}\begin{bmatrix}\mathbf{x}\\ \mathbf{z}\end{bmatrix}$$

$$\begin{bmatrix}\dot{\mathbf{x}}\\ \dot{\mathbf{e}}\end{bmatrix}=\begin{bmatrix}\mathbf{A}-\mathbf{BK} & -\mathbf{BKQ}_2\\ \mathbf{0} & \mathbf{F}\end{bmatrix}\begin{bmatrix}\mathbf{x}\\ \mathbf{e}\end{bmatrix}+\begin{bmatrix}\mathbf{B}\\ \mathbf{0}\end{bmatrix}\mathbf{r}$$

$$y=[\mathbf{C}\quad \mathbf{0}]\begin{bmatrix}\mathbf{x}\\ \mathbf{e}\end{bmatrix}$$

$$\mathbf{G}_f(s)=\mathbf{C}(s\mathbf{I}-\mathbf{A}+\mathbf{BK})^{-1}\mathbf{B}$$

HW 8-3

4. Given

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -3 & 1 & 2 & 3 \\ 2 & 1 & 0 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 2 \\ 0 & 2 \end{bmatrix}$$

Find two different constant matrices $\mathbf{K}$ such that $\mathbf{(A-BK)}$ has eigenvalues $-4 \pm 3j$ and $-5 \pm 4j$.