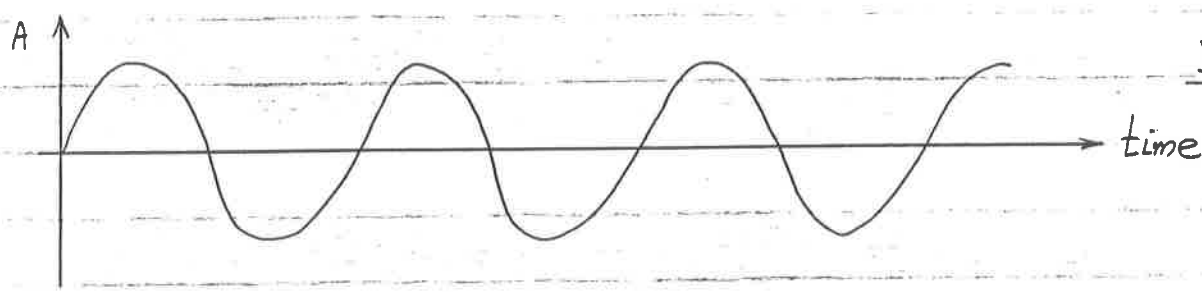
1. INTRODUCTION

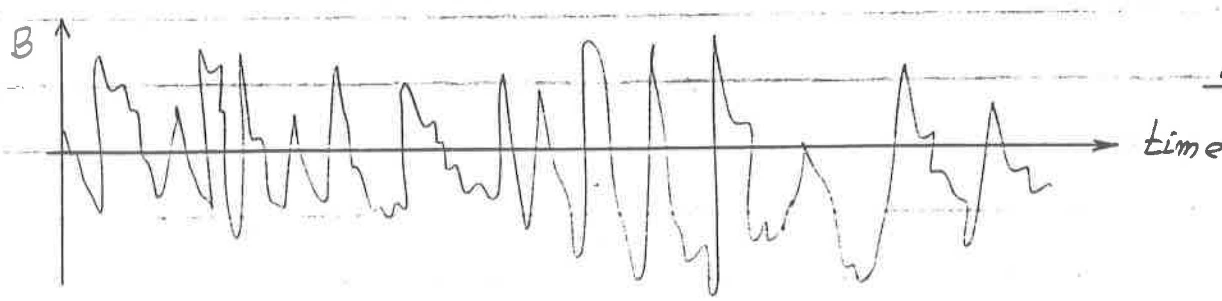
Excitation (Static or Dynamic) → Structure (Linear or Nonlinear) → Response

	Excitation	Structure	Response	
	Deterministic	Deterministic	Deterministic	
Dynamic Loading ⊛	Random	Deterministic	Random	Random Vibration
	Deterministic	Random	Random	System Stochasticity
	Random	Random	Random	Completely Random

- When there is no obvious pattern in a vibration, pressure, load e.t.c., the process is called random.
- If identical experiments are performed many times and the records obtained are always alike, then the process is deterministic.



Deterministic



Random

• Applications of Random Vibration Theory:

(a) Aeronautical Engineering

Gust Loading (air turbulence)
Jet Noise (noise of engine)
Fatigue Problems
Boundary Layer Turbulence
Maneuvers

(b) Civil Engineering - Engineering Mechanics

Wind Loading (on bridges, buildings e.t.c.)
Earthquake Loading
Ocean Waves

(c) Electrical Engineering

Random Signals
Communications

(d) Bio-engineering

Acoustics: noise, speech
Turbulence in Blood Flow
Brain Waves

(e) Mechanical Engineering

Vibration of Machinery
Response of Transportation Vehicles

2. RANDOM VARIABLES

- A random variable X is a function such that for every real number x there exists a probability $P[X \leq x]$.

- Probability Distribution Function:

$$F_X(x) = P[X \leq x]$$

  - Properties: $F_X(b) \geq F_X(a)$ for $b > a$, $F_X(-\infty) = 0$, $F_X(+\infty) = 1$

- Probability Density Function:

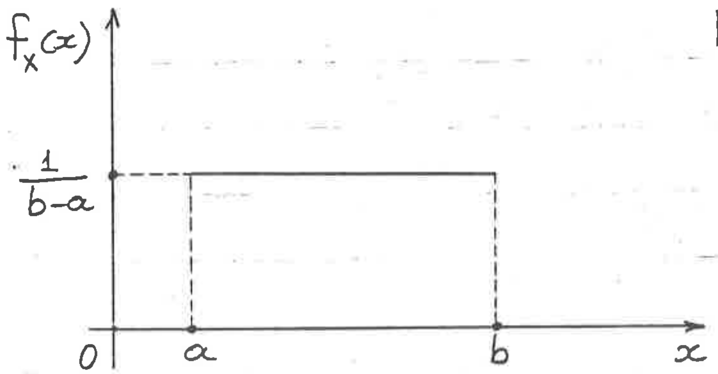
$$f_X(x) = \frac{dF_X(x)}{dx} \longrightarrow F_X(x) = \int_{-\infty}^{x} f_X(\xi)\, d\xi$$

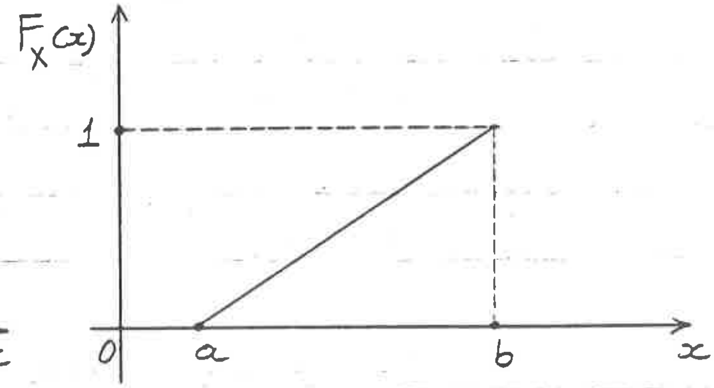
  - Property: $\int_{-\infty}^{\infty} f_X(x)\, dx = 1$

- Example: Uniform Distribution

$$f_X(x) = \begin{cases} \frac{1}{b-a} & \text{for } a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

$$F_X(x) = \begin{cases} 0 & \text{for } x < a \\ \dfrac{x-a}{b-a} & \text{for } a \leqq x \leqq b \\ 1 & \text{for } x > b \end{cases}$$





- Moments of Random Variables

  - Moments about the origin: $E\{X^n\} = \int_{-\infty}^{\infty} x^n \cdot f_X(x)\,dx$

    Mean Value: $E\{X\} = \int_{-\infty}^{\infty} x \cdot f_X(x)\,dx$

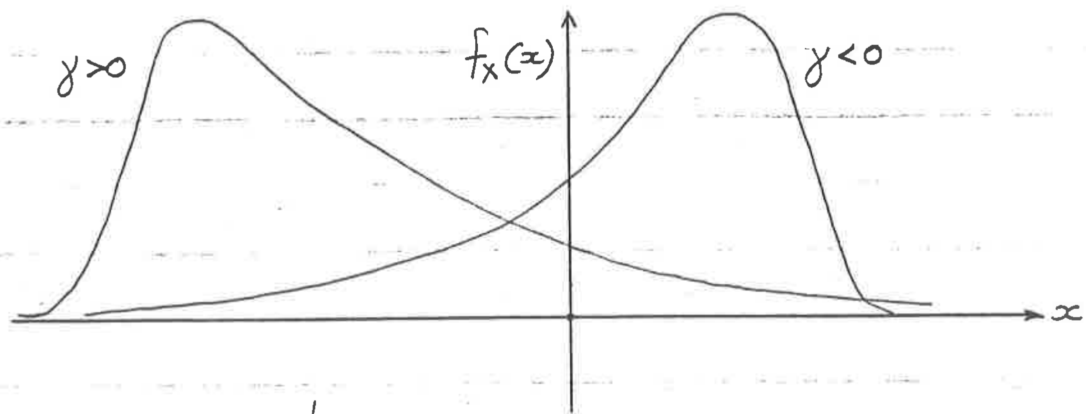
  - Central Moments: $E\{(X - E\{X\})^n\} = \int_{-\infty}^{\infty} (x - E\{X\})^n f_X(x)\,dx$

    Variance: $Var\{X\} = \int_{-\infty}^{\infty} (x - E\{X\})^2 f_X(x)\,dx = E\{X^2\} - E\{X\}$

    Standard Deviation: $\sigma_X = \sqrt{Var\{X\}}$

  - Coefficient of Variation: $V_X = \dfrac{\sigma_X}{E\{X\}}$ for $E\{X\} \neq 0$

- Skewness: $\gamma = \frac{1}{\sigma_X^3} \int_{-\infty}^{\infty} (x - E\{X\})^3 \cdot f_X(x)\, dx$



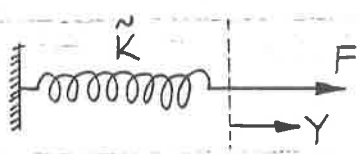
$\gamma = 0 \rightarrow$ symmetric

- <u>Functions of Random Variables</u>

- Consider that random variables X and Y are related as:

$$Y = G(X)$$

- <u>Example:</u>



$\tilde{K}$ = flexibility

Considering that F is a random variable $\rightarrow$ $Y = \tilde{K} \cdot F$

- <u>Example:</u> fluid pressure: $P = \frac{1}{2} \cdot g \cdot V^2 \cdot c_p$

V: velocity which is considered to be a random variable

- $F_Y(y) = P[Y \leq y] = P[G(X) \leq y]$

- 1-1 Mappings:

G increasing: $F_Y(y) = P[X \leq G^{-1}(y)] = F_X\{G^{-1}(y)\}$

G decreasing: $F_Y(y) = P[X \geq G^{-1}(y)] = 1 - F_X\{G^{-1}(y)\}$

- Example of a Not 1-1 Mapping:

Consider that $Y = X^2$; given $f_X(x)$ find $f_Y(y)$

$$F_Y(y) = P[Y \leq y] = P[X^2 \leq y] = P[-\sqrt{y} \leq X \leq \sqrt{y}] =$$

$$= \int_{-\sqrt{y}}^{\sqrt{y}} f_X(x)\,dx$$

Then: $$f_Y(y) = \frac{dF_Y(y)}{dy} = f_X(\sqrt{y})\cdot\tfrac{1}{2}\cdot y^{-\frac{1}{2}} - f_X(-\sqrt{y})\cdot(-\tfrac{1}{2})\cdot y^{-\frac{1}{2}}$$

or: $$f_Y(y) = \frac{f_X(\sqrt{y}) + f_X(-\sqrt{y})}{2\sqrt{y}}$$

- Leibnitz Rule:

$$\frac{d}{dy}\int_{a(y)}^{b(y)} f(x,y)\,dx = \int_{a(y)}^{b(y)} \frac{\partial}{\partial y}[f(x,y)]\,dx + f(b,y)\frac{db}{dy} - f(a,y)\frac{da}{dy}$$

- Moments of Functions of Random Variables

$$E\{Y\} = E\{G(X)\} = \int_{-\infty}^{\infty} G(x)\, f_X(x)\,dx$$

$$Var\{Y\} = Var\{G(X)\} = \int_{-\infty}^{\infty} [G(x) - E\{G(X)\}]^2 \, f_X(x)\, dx$$

- Jointly Distributed Random Variables

  - In many cases we need to know the joint behaviour of 2 or more random variables.
  - Consider X and Y to be two random variables.

  - Joint Probability Distribution Function:

$$F_{XY}(x,y) = P[(X \leq x) \cap (Y \leq y)]$$

  - Properties: $F_{XY}(-\infty,-\infty) = 0$, $F_{XY}(-\infty, y) = F_{XY}(x,-\infty) = 0$,
$F_{XY}(+\infty, +\infty) = 1$

  - Joint Probability Density Function:

$$f_{XY}(x,y) = \frac{\partial^2 F_{XY}(x,y)}{\partial x \, \partial y} \longrightarrow F_{XY}(x,y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f_{XY}(\xi_x, \xi_y)\, d\xi_x\, d\xi_y$$

  - When X and Y are statistically independent:

$$f_{XY}(x,y) = f_X(x) \cdot f_Y(y)$$

- **Moments about the origin of jointly distributed random variables**

$$\mu'_{\nu\nu'} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^{\nu} \cdot y^{\nu'} f_{XY}(x,y)\, dx\, dy$$

with: $\mu'_{00} = 1$; $\mu'_{10} = E\{X\}$; $\mu'_{01} = E\{Y\}$

- **Central Moments of jointly distributed random variables**

$$\mu_{\nu\nu'} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x - E\{X\})^{\nu} \cdot (y - E\{Y\})^{\nu'} f_{XY}(x,y)\, dx\, dy$$

with: $\mu_{00} = 1$; $\mu_{10} = \mu_{01} = 0$

$$\mu_{20} = Var\{X\} = \mu'_{20} - \mu'^{2}_{10}$$

$$\mu_{02} = Var\{Y\} = \mu'_{02} - \mu'^{2}_{01}$$

$$\mu_{11} = Cov\{X,Y\} = \mu'_{11} - \mu'_{10} \cdot \mu'_{01} \quad \text{(covariance)}$$

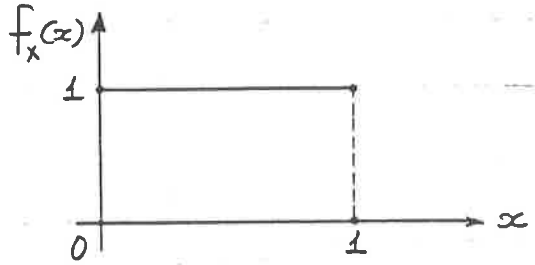
- **Correlation Coefficient:**

$$\rho = \frac{\mu_{11}}{\sqrt{\mu_{20} \cdot \mu_{02}}} \qquad -1 \leq \rho \leq +1$$

$$\rho = \pm 1 \iff Y = a \cdot X + b \quad \begin{pmatrix} a>0 : + \\ a<0 : - \end{pmatrix}$$

$$\rho = 0 \Longleftarrow X, Y \text{ statistically independent}$$

- <u>Example:</u> consider random variable X with density function $f_X(x) = 1$ for $0 \leq x \leq 1$.



<u>Let:</u> $Y = \sin(2\pi X)$
$Z = \cos(2\pi X)$

It is obvious that : $Y^2 + Z^2 = 1$, therefore Y and Z are statistically dependent (related)

$$E\{Y\} = E\{\sin(2\pi X)\} = \int_0^1 \sin(2\pi X) \cdot 1 \cdot dx = 0$$
$$E\{Z\} = E\{\cos(2\pi X)\} = \int_0^1 \cos(2\pi X) \cdot 1 \cdot dx = 0$$

$$\mathrm{Cov}\{Y, Z\} = E\{[Y - E\{Y\}] \cdot [Z - E\{Z\}]\} = E\{Y \cdot Z\} =$$
$$= \int_{-\infty}^{\infty} \sin(2\pi x) \cdot \cos(2\pi x) \cdot f_X(x) \cdot dx =$$
$$= \int_0^1 \frac{1}{2} \sin(4\pi x)\, dx = -\frac{1}{8\pi} [\cos(4\pi x)]_0^1 = 0$$

Therefore, although random variables Y and Z are statistically dependent, their correlation coefficient is equal to zero : $\rho = 0$. This means that Y and Z are uncorrelated random variables.

$$\left\{ \begin{matrix} Y \text{ and } Z \text{ statistical} \\ \text{independent} \end{matrix} \right\} \Longrightarrow \left\{ \begin{matrix} Y \text{ and } Z \\ \text{uncorrelated} \end{matrix} \right\}$$

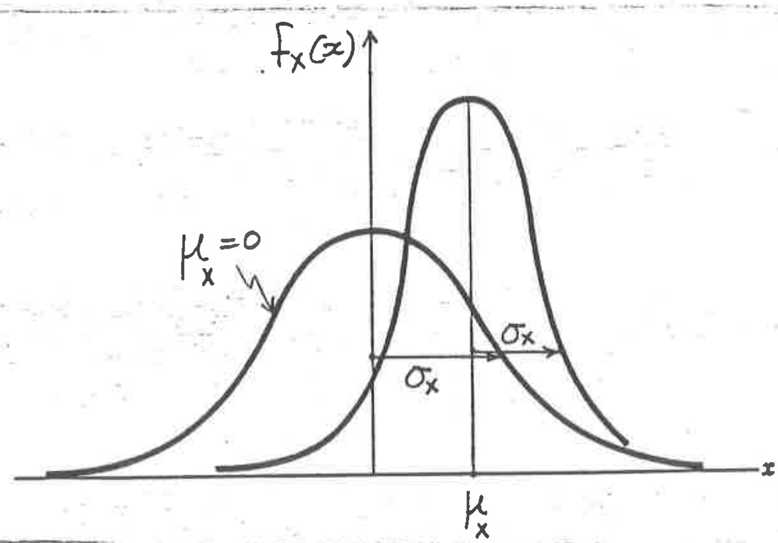
• Gaussian or Normal Random Variables

∘ A random variable X is said to be normally distributed if its probability density function is of the form:

$$f_X(x) = \frac{1}{\sqrt{2\pi}\cdot\sigma_X}\cdot \exp\left[\frac{(x-\mu_X)^2}{2\sigma_X^2}\right] \qquad -\infty < x < +\infty$$

where: μ_X = mean value and σ_X = standard deviation

∘ When $\mu_X = 0$ and $\sigma_X = 1$, random variable X is known as the unit Gaussian and $f_X(x)$ as the standardized normal density function.



∘ The importance of the Gaussian distribution comes from the Central Limit Theorem.

- <u>Central Limit Theorem</u> : the sum of n independent random variables X_k, $S_n = \sum_{k=1}^{n} X_k$, when centralized to zero mean and unit standard deviation, tends to a unit Gaussian random variable as $n \to \infty$, regardless of the individual distributions of X_k, provided the following conditions are satisfied:

(i) $E[|X_k|] < \infty$ and $E\left[|X_k - E[X_k]|^{2+\delta}\right] < \infty$

(ii) $$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} E\left[|X_k - E[X_k]|^{2+\delta}\right]}{\left(\sigma_{S_n}\right)^{2+\delta}} = 0$$

where $\delta > 0$ and $\sigma_{S_n}^2 = \sum_{k=1}^{n} \sigma_{X_k}^2$ is the variance of the sum

- <u>Properties of Gaussian Random Variables</u>.

1) Linear functions of Gaussian random variables remain Gaussian distributed.

If X is Gaussian, then $Y = a\cdot X + b$ is also Gaussian with mean: $a\cdot\mu_x + b$ and standard deviation: $a\cdot\sigma_x$

2) If X_1, X_2 are Gaussian with means μ_1, μ_2 and standard deviations σ_1, σ_2, then $X = X_1 + X_2$ is also Gaussian with mean: $\mu_1 + \mu_2$ and standard deviation: $\sqrt{\sigma_1^2 + \sigma_2^2}$
X_1 and X_2 have to be independent!

3) The n^{th} moment of the unit Gaussian random variable u can be computed from:

$$E[u^n] = \int_{-\infty}^{\infty} u^n f_u(u)\,du = \int_{-\infty}^{\infty} u^n \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}}\,du$$

$$\left\{ \begin{array}{ll} E[u^n] = 0 & \text{for } n = \text{odd} \\ E[u^n] = 2^{n/2} \cdot \pi^{-1/2} \cdot \Gamma\left(\frac{n+1}{2}\right) & n = \text{even} \end{array} \right\}$$

A Gaussian random variable X with mean μ_x and standard deviation σ_x can be obtained from:

$$X = \sigma_x \cdot u + \mu_x$$

Then, the central moments of X are:

$$E[(X-\mu_x)^n] = E[(\sigma_x \cdot u)^n] = \sigma_x^n \cdot E[u^n] =$$

$$= \begin{cases} 0 & \text{for } n = \text{odd} \\ \sigma_x^n \cdot 2^{n/2} \cdot \pi^{-1/2} \cdot \Gamma\left(\frac{n+1}{2}\right) & \text{for } n = \text{even} \end{cases}$$

- Jointly Distributed Gaussian Random Variables

Two random variables are said to be jointly Gaussian distributed, if the probability density function is given by:

$$f_{XY}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho_{XY}^2}} \cdot \exp\left\{-\frac{1}{2(1-\rho_{XY}^2)}\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 - 2\left(\frac{x-\mu_X}{\sigma_X}\right)\cdot\left(\frac{y-\mu_Y}{\sigma_Y}\right)\rho_{XY} + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right]\right\}$$

where ρ_{XY} = correlation coefficient of X and Y

- The higher joint moments of Gaussian random variables can be expressed in terms of the lower ones.

Assuming that all random variables have zero means:

$$E[X_1 \cdot X_2 \cdots X_{2n+1}] = 0$$
$$E[X_1 \cdot X_2 \cdots X_{2n}] = \sum E[X_j \cdot X_k] \cdot E[X_r \cdot X_s] \cdots$$

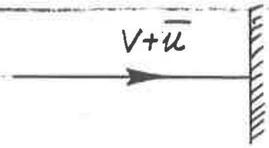
where the sum has to be taken over all different ways we can group $2n$ elements into n pairs. The number of terms N in this summation is:

$$N = \frac{(2n)!}{n!\,2^n} \quad \begin{cases} \text{For } n=2 \rightarrow N = \dfrac{4!}{2!\,2^2} = 3 \\ \text{For } n=4 \rightarrow N = \dfrac{8!}{4!\,2^4} = 105 \end{cases}$$

<u>Example:</u>

$$E[X_1 \cdot X_2 \cdot X_3 \cdot X_4] = E[X_1 X_2] \cdot E[X_3 \cdot X_4] + E[X_1 X_3] \cdot E[X_2 X_4] + E[X_1 X_4] \cdot E[X_2 X_3]$$

- <u>Example:</u>



V is a Gaussian random variable with $E[V]=0$ and standard deviation σ_v. $\bar{u}$ is the mean value of the velocity and is a constant.
The pressure is given by: $P = A\cdot(\bar{u}+V)^2 = A\cdot(\bar{u}^2+2\bar{u}\cdot V+V^2)$
where A is a constant.
<u>Find</u>: $E[P^2]$

$$E[P^2] = E\left[A\cdot(\bar{u}^2+2\bar{u}\cdot V+V^2)\cdot A\cdot(\bar{u}^2+2\bar{u}\cdot V+V^2)\right] =$$

$$= A^2\cdot E\left[\bar{u}^4+2\bar{u}^3 V+\bar{u}^2\cdot V^2+2\bar{u}^3\cdot V+4\cdot\bar{u}^2\cdot V^2+2\bar{u}\cdot V^3+ + \bar{u}^2\cdot V^2+2\cdot\bar{u}\cdot V^3+V^4\right]$$

It has been shown that: $E[V^3]=0$ and that:

$$E[V^4] = \sigma_v^4\cdot 2^{4/2}\cdot \pi^{-1/2}\cdot\Gamma\left(\tfrac{5}{2}\right) \longrightarrow E[V^4] = 3\cdot\sigma_v^4$$

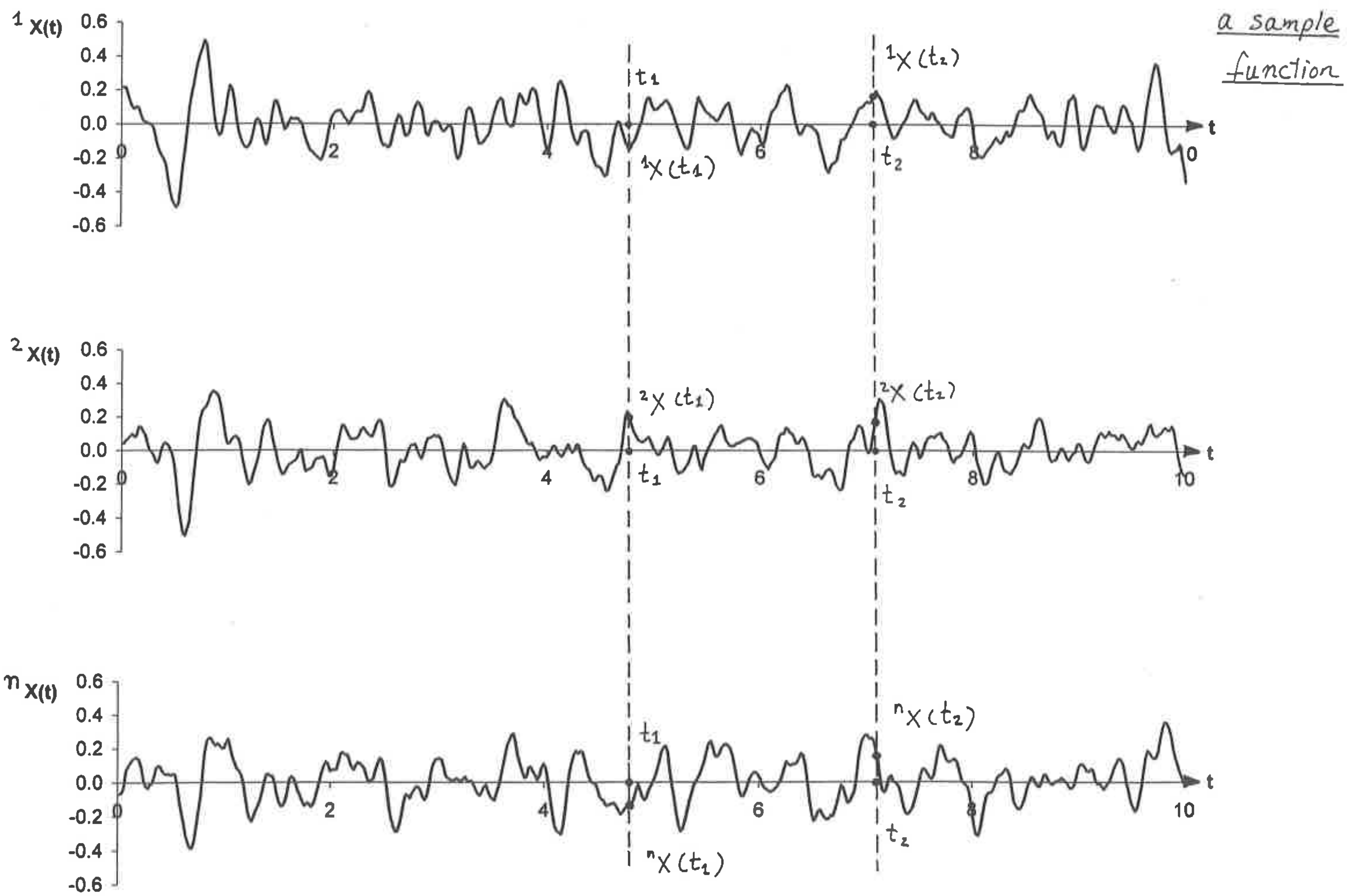
<u>Therefore:</u>

$$E[P^2] = A^2\cdot\left\{\bar{u}^4+\bar{u}^2\cdot E[V^2]+4\cdot\bar{u}^2\cdot E[V^2]+\bar{u}^2\cdot E[V^2]+ + E[V^4]\right\} = A^2\cdot\left\{\bar{u}^4+6\cdot\bar{u}^2\cdot\sigma_v^2+3\cdot\sigma_v^4\right\}$$

since: $E[V^2] = \sigma_v^2$

3. RANDOM PROCESSES

- A random process $X(t)$ is an ensemble of time functions that can be characterized statistically.
- A random process is a parametered family of random variables with the parameter belonging to an indexing set.



- When the sample function is a function of time → random process
- When the sample function is a function of space → random field
- The words random and stochastic are equivalent.

- In the increasing order of completeness, the probability structure of a random process is described by the following series of probability density functions:

$$\left\{\begin{array}{ll} f_{X(t_1)}(x_1) & 1^{st} \text{ order} \\ f_{X(t_1)X(t_2)}(x_1,x_2) & 2^{nd} \text{ order} \\ \vdots & \\ f_{X(t_1)X(t_2)\dots X(t_n)}(x_1,x_2,\dots,x_n) & \\ & n^{th} \text{ order} \\ \vdots & \end{array}\right\} \text{ or } \left\{\begin{array}{l} f_{\{X\}}(x_1,t_1) \text{ or } f_{\{X\}}(x,t) \\ f_{\{X\}}(x_1,t_1;x_2,t_2) \\ \vdots \\ f_{\{X\}}(x_1,t_1;x_2,t_2;\dots;x_n,t_n) \\ \vdots \end{array}\right.$$

- The lower probability density functions can be obtained from the higher ones from:

$$f_{\{X\}}(x_1,t_1;x_2,t_2;\dots;x_n,t_n) = \underbrace{\int\cdots\int}_{k\text{-fold}} f_{\{X\}}(x_1,t_1;\cdots;x_{n+k},t_{n+k})\,dx_{n+1}\cdots dx_{n+k}$$

- Example: for $n=1$ and $k=1$

$$f_{\{X\}}(x_1,t_1) = \int_{-\infty}^{\infty} f_{\{X\}}(x_1,t_1;x_2,t_2)\,dx_2$$

- A random process can also be described by moment functions of various orders:

1) First Moment or Mean:

random process

$$\mu_{\cdot}(t) = E[X(t)] = \int^{\infty} x\, f_{\{\cdot\}}(x,t)\,dx \qquad \text{(function of } t\text{ !)}$$

2) Second Moment or Mean Square:

$$E[X^2(t)] = \int_{-\infty}^{\infty} x^2 \cdot f_{\{X\}}(x,t)\, dx \qquad \text{(function of } t\text{ !)}$$

3) Variance:

$$E[\{X(t) - \mu_X(t)\}^2] = E[X^2(t)] - \mu_X^2(t) \qquad \text{(function of } t\text{ !)}$$

Standard deviation: $\sigma_X(t) = \sqrt{E[X^2(t)] - \mu_X^2(t)}$

4) Autocorrelation Function:

$$R_{XX}(t_1, t_2) = E[X(t_1)X(t_2)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 \cdot x_2 \cdot f_{\{X\}}(x_1, t_1; x_2, t_2)\, dx_1\, dx_2$$

- The autocorrelation function characterizes random process $X(t)$

5) Crosscorrelation Function:

$$R_{XY}(t_1, t_2) = E[X(t_1)Y(t_2)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x \cdot y \cdot f_{\{X\}\{Y\}}(x, t_1; y, t_2)\, dx\, dy$$

- The crosscorrelation function characterizes two random processes $X(t)$ and $Y(t)$.

6) Autocovariance Function:

$$K_{XX}(t_1, t_2) = E[\{X(t_1) - \mu_X(t_1)\} \cdot \{X(t_2) - \mu_X(t_2)\}] =$$

$$= R_{XX}(t_1, t_2) - \mu_X(t_1) \cdot \mu_X(t_2)$$

- Note that when $\mu_X(t_1) = \mu_X(t_2) = 0 \rightarrow K_{XX}(t_1, t_2) = R_{XX}(t_1, t_2)$

- Note that when $t_1 = t_2 = t \longrightarrow K_{XX}(t_1, t_2) = \sigma_X^2(t)$

7) Crosscovariance Function:

$$K_{XY}(t_1, t_2) = E\left[\{X(t_1) - \mu_X(t_1)\}\cdot\{Y(t_2) - \mu_Y(t_2)\}\right] =$$

$$= R_{XY}(t_1, t_2) - \mu_X(t_1)\cdot\mu_Y(t_2)$$

8) The normalized covariance functions are called Correlation Coefficients:

$$\rho_{XX}(t_1, t_2) = \frac{K_{XX}(t_1, t_2)}{\sigma_X(t_1)\sigma_X(t_2)}$$

$$\rho_{XY}(t_1, t_2) = \frac{K_{XY}(t_1, t_2)}{\sigma_X(t_1)\sigma_Y(t_2)}$$

- In order to completely describe the random process, moment functio of all orders are required.
- For most practical cases of random processes, only the mean and the variance or autocorrelation function are computed.
- Even though the first- and second-order moments do not characterize a random process completely, they still contain the most important information about the process.
- If the random process is Gaussian, all higher-order moments can be computed from mean and autocorrelation function.

- Stationary Random Processes

A random process is said to be strongly stationary if its complete probability structure is independent of a shift in the parametric origin.

$$\left\{\begin{array}{ll} f_{\{x\}}(x,t) = f_{\{x\}}(x,t+a) & (A) \\ f_{\{x\}}(x_1,t_1;x_2,t_2) = f_{\{x\}}(x_1,t_1+a;x_2,t_2+a) & (B) \\ \vdots & \\ f_{\{x\}}(x_1,t_1;\dots;x_n,t_n) = f_{\{x\}}(x_1,t_1+a;\dots;x_n,t_n+a) & \\ \vdots & \end{array}\right\}$$

- When only the first two conditions (A+B) are satisfied, the process is called weakly stationary.
- Random process (function of time) $\rightarrow$ stationary
- Random field (function of space) $\rightarrow$ homogeneous

- Properties of Correlation Functions

1) Symmetry: $\left\{\begin{array}{l} R_{XX}(t_1,t_2) = R_{XX}(t_2,t_1) \\ R_{XY}(t_1,t_2) = R_{YX}(t_2,t_1) \end{array}\right\}$

2) Non-negative Definite:

$$\sum_{j=1}^{n}\sum_{k=1}^{n} R_{XX}(t_j,t_k)\,h(t_j)\,h^*(t_k) \geq 0$$

where $h(t)$ = arbitrary function and n = any finite integer

- If the random process is stationary:

$$R_{XX}(t_1, t_2) = R_{XX}(t_1 - t_2) = R_{XX}(\tau) \quad \text{with} \quad \tau = t_1 - t_2$$

Properties:

1) Symmetry:
$$\left\{ \begin{array}{l} R_{XX}(\tau) = R_{XX}(-\tau) \\ R_{XY}(\tau) = R_{YX}(-\tau) \end{array} \right\}$$

2) Non-negative Definite:

$$\left\{ \begin{array}{l} \sum_{j=1}^{n} \sum_{k=1}^{n} R_{XX}(t_j - t_k) h(t_j) h^*(t_k) \geq 0 \\ \sum_{j=1}^{n} \sum_{k=1}^{n} R_{XY}(t_j - t_k) h(t_j) h^*(t_k) \geq 0 \end{array} \right\}$$

3) Inequality:
$$|R_{XY}(\tau)| \leqq \sqrt{R_{XX}(0) \cdot R_{YY}(0)}$$

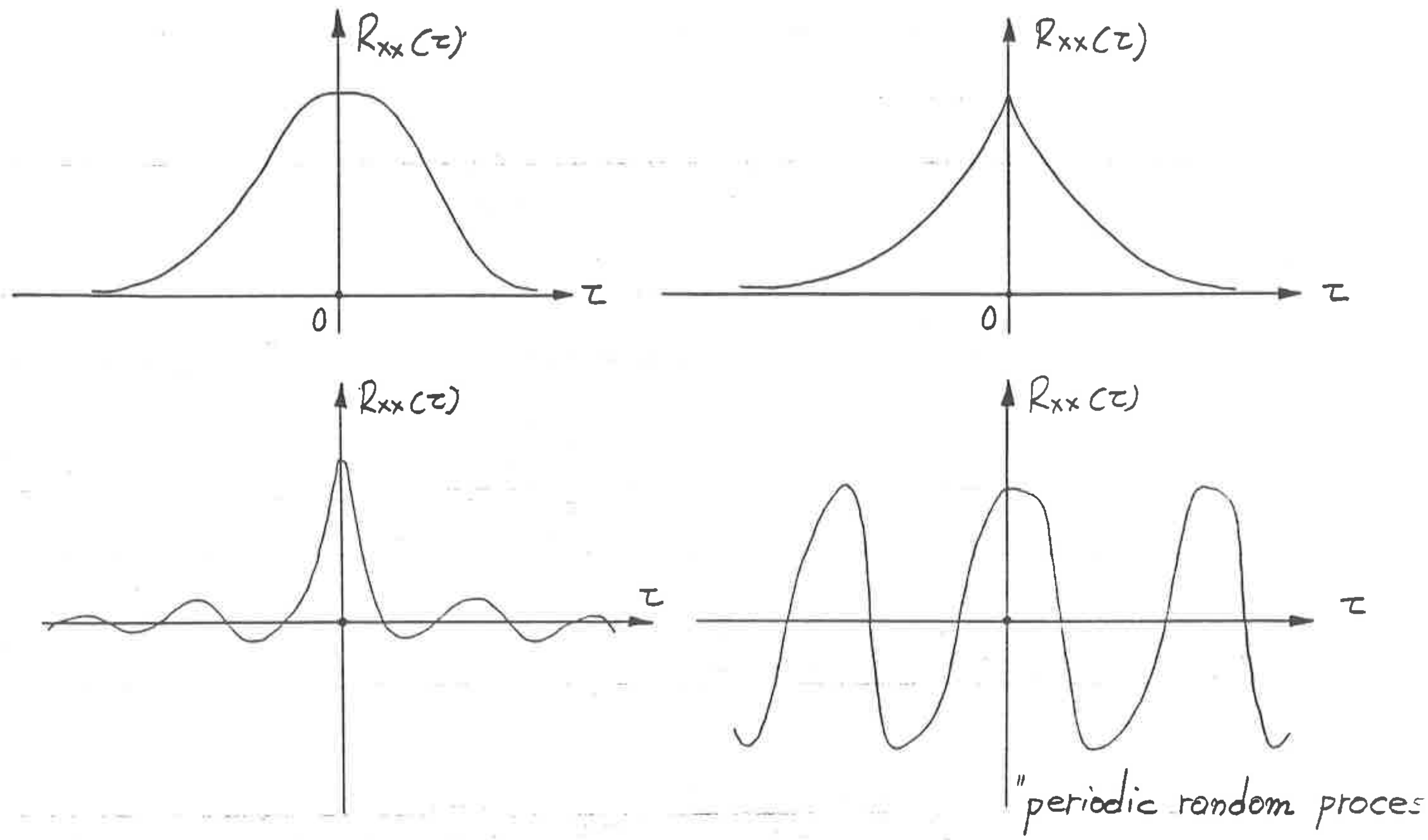
$$|R_{XX}(\tau)| \leqq R_{XX}(0)$$

The second inequality means that the autocorrelation function attains a maximum value at $\tau = 0$.

4) $\lim\limits_{\tau \to 0} R_{XX}(\tau) = E[X^2(t)] = \sigma_X^2$ if $\mu_X = 0$

5) $\lim\limits_{\tau \to \infty} R_{XX}(\tau) = 0$ (except for periodic processes!)

- Examples of autocorrelation functions:

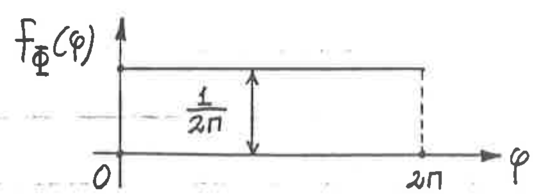


- Example: consider the stationary random process given by:

$$X(t) = A \cdot \sin(\omega \cdot t + \Phi) \quad \text{where: } A, \omega = \text{constants}$$

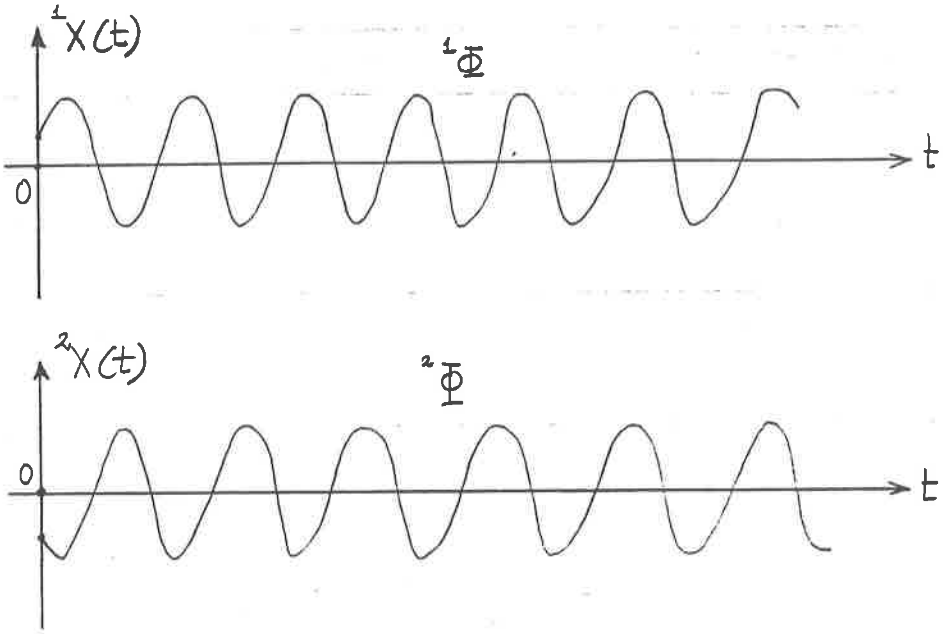
Φ = random phase angle uniformly distributed between 0 and 2π

$$f_\Phi(\varphi) = \begin{cases} \frac{1}{2\pi} & \text{for } 0 < \varphi < 2\pi \\ 0 & \text{otherwise} \end{cases}$$



Φ is a random variable.

Calculate the autocorrelation function of $X(t)$ and show that it is a function of $t_1 - t_2 = \tau$.



two realizations of random process $X(t)$

$\vdots$

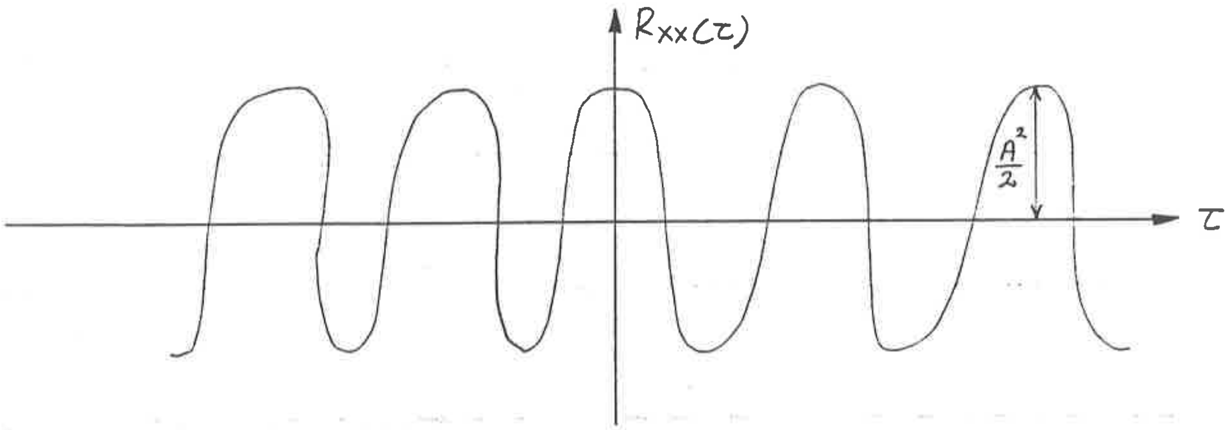
- $$E[X(t)] = \int_{-\infty}^{\infty} A \cdot \sin(\omega t + \varphi) \cdot f_{\Phi}(\varphi)\, d\varphi = \int_{0}^{2\pi} A \cdot \sin(\omega t + \varphi) \cdot \frac{1}{2\pi} \cdot d\varphi =$$
$$= -\frac{A}{2\pi} \cdot \left[\cos(\omega t + \varphi)\right]_0^{2\pi} = 0$$

- $$R_{XX}(t_1, t_2) = E[X(t_1)\, X(t_2)] = E[A \cdot \sin(\omega t_1 + \Phi) \cdot A \cdot \sin(\omega t_2 + \Phi)] =$$
$$= \int_0^{2\pi} A^2 \cdot \sin(\omega t_1 + \varphi) \cdot \sin(\omega t_2 + \varphi) \cdot \frac{1}{2\pi} \cdot d\varphi =$$
$$= \frac{A^2}{2\pi} \int_0^{2\pi} \frac{1}{2}\left[\cos(\omega t_1 + \varphi - \omega t_2 - \varphi) - \cos(\omega t_1 + \varphi + \omega t_2 + \varphi)\right] d\varphi =$$
$$= \frac{A^2}{4\pi} \int_0^{2\pi} \cos[\omega(t_1 - t_2)]\, d\varphi - \frac{A^2}{4\pi} \int_0^{2\pi} \cos[\omega(t_1 + t_2) + 2\varphi]\, d\varphi =$$
$$= \frac{A^2}{4\pi} \cdot \cos[\omega(t_1 - t_2)] \cdot 2\pi - 0$$

Therefore:

$$R_{xx}(t_1, t_2) = \frac{A^2}{2}\cdot\cos[\omega(t_1-t_2)] = \frac{A^2}{2}\cdot\cos(\omega\tau) = R_{xx}(\tau)$$

where: $\tau = t_1 - t_2$



- Properties of calculated autocorrelation function:

(i) $R_{xx}(\tau) = R_{xx}(-\tau)$

(ii) $R_{xx}(0) = \frac{A^2}{2} \geq \frac{A^2}{2}\cdot\cos(\omega\tau) = R_{xx}(\tau)$

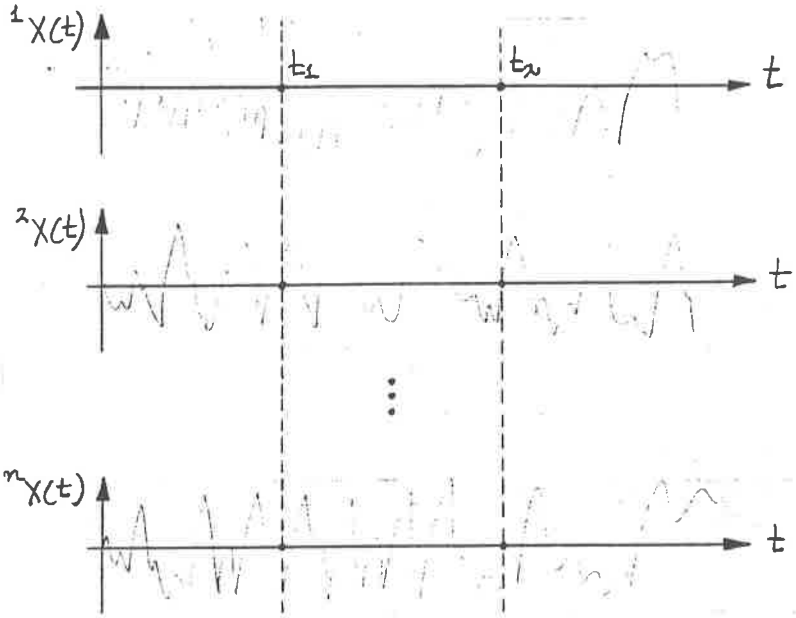
(iii) $R_{xx}(0) = \sigma_x^2 = \frac{A^2}{2}$ (since mean value is equal to zero)

and $\sigma_x = \frac{A}{\sqrt{2}} = 0.707\cdot A$

(iv) $\lim_{\tau\to\infty} R_{xx}(\tau) \neq 0$ since $X(t)$ is a periodic random process

• Ergodic Theorem

- For the theory of random processes to be useful, we need to estimate mean value and autocorrelation function by measurements.
- This estimation is done using a large number of sample functions or realizations of the random process:

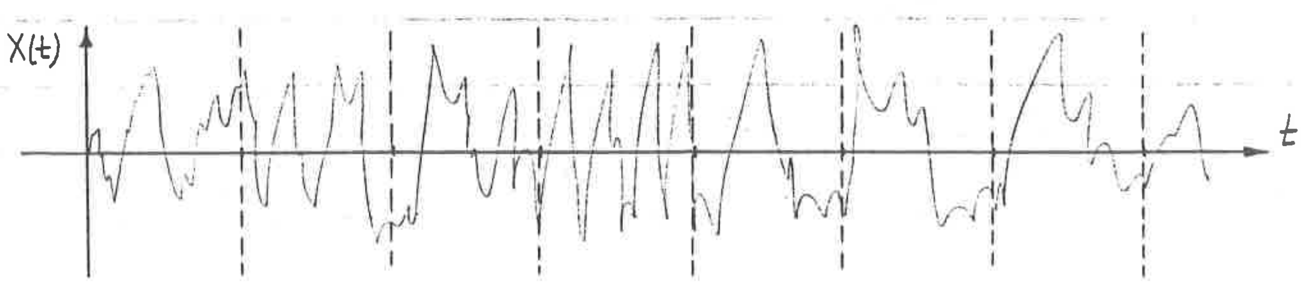


Ensemble Averages
(averages over the entire number of sample functions)

$$E[X(t_1)] = \mu_x(t_1)$$

$$E[X(t_1)X(t_2)] = R_{XX}(t_1, t_2)$$

- Unfortunately, we usually have only a very small number of sample functions of the random process.
- However, if a random process is stationary, <u>sometimes</u> it is possible to estimate the mean and autocorrelation function from just one sufficiently long sample function.



<u>Temporal Averages</u> (averages over one sample function)

$$\left\{ \begin{aligned} \langle X(t) \rangle &= \frac{1}{T}\int_0^T X(t)\,dt \\ \langle X(t+\tau)X(t) \rangle &= \frac{1}{T-\tau}\int_0^{T-\tau} X(t+\tau)X(t)\,dt \end{aligned} \right\}$$ where T is the length of the sample function

- A random process is said to be <u>ergodic in the first moment</u> if:

$$\underset{\substack{\uparrow \\ \text{ensemble} \\ \text{average}}}{E[X(t)]} = \underset{\substack{\uparrow \\ \text{temporal} \\ \text{average}}}{\langle X(t) \rangle} \quad \text{for } T \to \infty$$

- The necessary and sufficient conditions for $X(t)$ to be ergodic in the first moment are:

(i) $E[X(t)] = \text{constant}$

(ii) $E[X(t+\tau)X(t)] = \text{function of } \tau \text{ only}$

(iii) $\lim\limits_{T\to\infty} \frac{1}{T}\int_0^T R_{xx}(\tau)\,d\tau = 0$

If the above conditions are satisfied, the mean value is computed

$$\mu_X = \frac{1}{T}\int_0^T X(t)\,dt = \text{independent of } t$$

- A random process is said to be <u>ergodic in correlation</u> if:

$$\underset{\substack{\uparrow \\ \text{ensemble} \\ \text{average}}}{R_{xx}(\tau)} = \underset{\substack{\uparrow \\ \text{temporal} \\ \text{average}}}{\langle X(t+\tau)X(t) \rangle} \quad \text{for } T \to \infty$$

- The necessary and sufficient conditions for $X(t)$ to be ergodic in correlation are:

$$\text{(i)}\quad E[X(t+\tau)X(t)] = \text{function of } \tau \text{ only}$$

$$\text{(ii)}\quad S_{XX}(\tau,u) = E\{[X(t+\tau)X(t) - R_{XX}(\tau)][X(t+\tau+u)X(t+u) - R_{XX}(\tau)]\} \text{ is independent of } t$$

$$\text{(iii)}\quad \lim_{T\to\infty} \frac{1}{T}\int_0^T S_{XX}(\tau,u)\,du = 0$$

If the above conditions are satisfied, the autocorrelation function is computed as:

$$R_{XX}(\tau) = \frac{1}{T-\tau}\int_0^{T-\tau} X(t+\tau)X(t)\,dt = \text{function of } \tau \text{ only}$$

- The property of ergodicity is very important since it permits the calculation of mean value and correlation function from just one sample function. This fact considerably reduces CPU time.

• Spectral Density Function of a Stationary Random Process

- The spectral density function or power spectrum of a stationary random process is defined as:

$$S_{XX}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_{XX}(\tau)\, e^{-i\omega\tau}\,d\tau \qquad (A)$$

where ω = frequency

- The inverse transformation is:

$$R_{XX}(\tau) = \int_{-\infty}^{\infty} S_{XX}(\omega)\, e^{+i\omega\tau}\, d\omega \qquad (B)$$

- Equations (A) and (B) are known as the <u>Wiener-Khintchine theorem.</u>

- <u>Properties of Spectral Density Function</u>

1) $S_{XX}(\omega)$ is a real and even function: $S_{XX}(\omega) = S_{XX}(-\omega)$

$$S_{XX}(\omega) = \frac{1}{\pi}\int_{0}^{\infty} R_{XX}(\tau) \cos\omega\tau \, d\tau$$

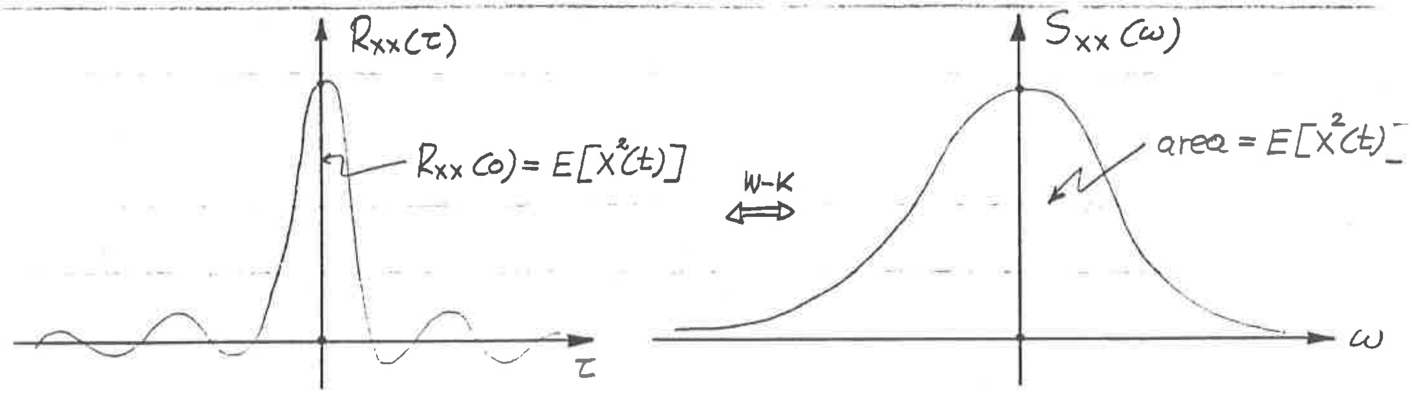
$$R_{XX}(\tau) = 2\int_{0}^{\infty} S_{XX}(\omega) \cos\omega\tau \, d\omega$$

2) $S_{XX}(\omega)$ is a non-negative function: $S_{XX}(\omega) \geq 0$

3) $E[X^2(t)] = R_{XX}(0) = \int_{-\infty}^{\infty} S_{XX}(\omega)\, d\omega \underset{\text{assuming zero mean}}{=} \sigma_X^2$

Therefore $S_{XX}(\omega)$ describes the distribution of the mean square over the frequency domain.

If $X(t)$ = displacement $\longrightarrow E[X^2(t)] \sim$ average potential energy
If $X(t)$ = velocity $\longrightarrow E[X^2(t)] \sim$ average kinetic energy



4) If $E[X^2(t)] < \infty$, then $S_{XX}(\omega)$ goes down to zero faster than $\frac{1}{\omega}$, as $\omega \to \pm\infty$.

- Cross-spectral Density Function:

For two random processes $X(t)$ and $Y(t)$, the cross-spectral density function is defined as:

$$S_{XY}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_{XY}(\tau)\, e^{-i\omega\tau}\, d\tau$$

$$R_{XY}(\tau) = \int_{-\infty}^{\infty} S_{XY}(\omega)\, e^{+i\omega\tau}\, d\omega$$

- Properties of Cross-spectral Density Function:

1) $S_{XY}(\omega)$ is complex in general

2) $S_{XY}(\omega) = S^*_{YX}(\omega) \longrightarrow$ Hermitian

3) If $E[X(t)Y(t)] < \infty$, then $S_{XY}(\omega)$ goes down to zero faster than $\frac{1}{\omega}$, as $\omega \to \pm\infty$.

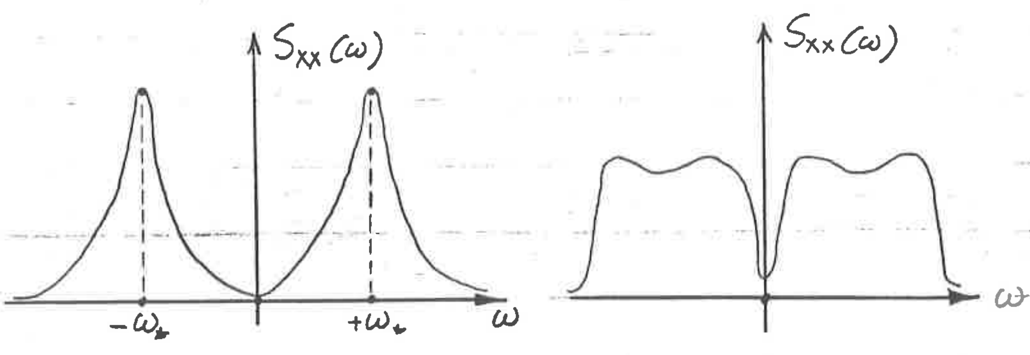
Estimation of Spectral Density Function from One Sample Function of the Random Process

Consider one sample function $x(t)$ of random process $X(t)$ having length T. Assuming ergodicity, the spectral density function of $X(t)$ can be estimated from:

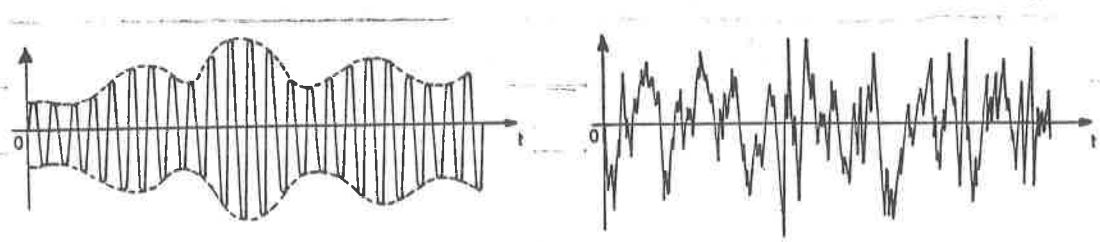
$$S_{XX}(\omega) = \frac{\left| \int_0^T x(t)\, e^{-i\omega t} dt \right|^2}{2\pi T}$$

- The computation shown above can be performed very efficiently using the FFT technique.
- The spectral density function shown above is the temporal spectral density function.

Two Types of Stationary Random Processes



"narrow-band r.p." "wide-band r.p."



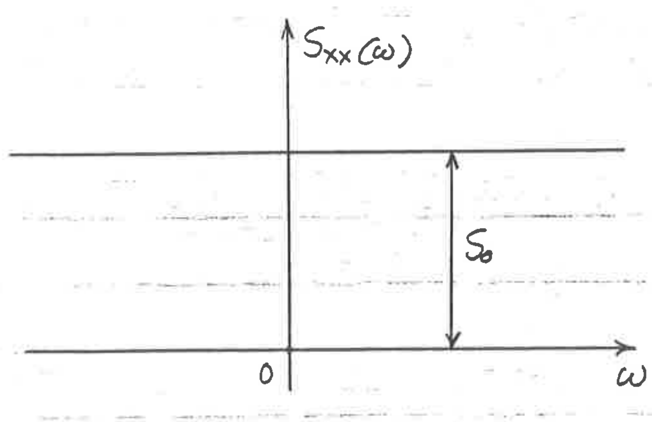
sample functions

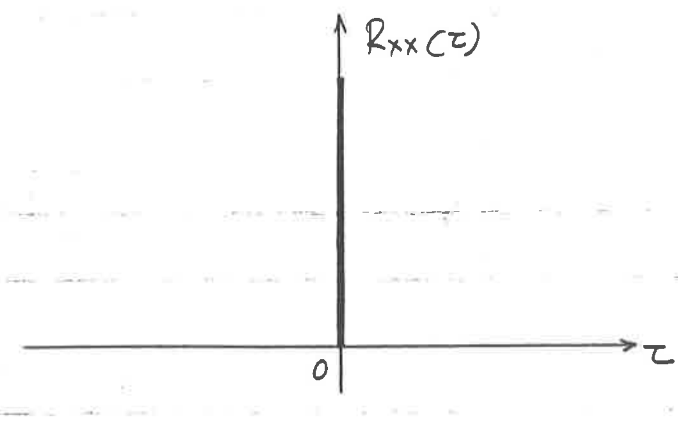
- Examples of typically wide-band random processes:
  - → pressure fluctuations on surface of a rocket missile due to acoustically transmitted jet noise or due to supersonic boundary layer turbulence
- Example of typically narrow-band random process:
  - → response of strongly resonant vibratory systems when the excitation is a wide-band process

• Some Commonly Used Stationary Random Processes

1) Idealized White-Noise:

$$\left\{ \begin{aligned} S_{XX}(\omega) &= S_0 \quad \text{for} \quad -\infty < \omega < +\infty \\ R_{XX}(\tau) &= \int_{-\infty}^{\infty} S_{XX}(\omega)\, e^{i\omega\tau}\, d\omega = 2\pi S_0 \cdot \delta(\tau) \end{aligned} \right\}$$





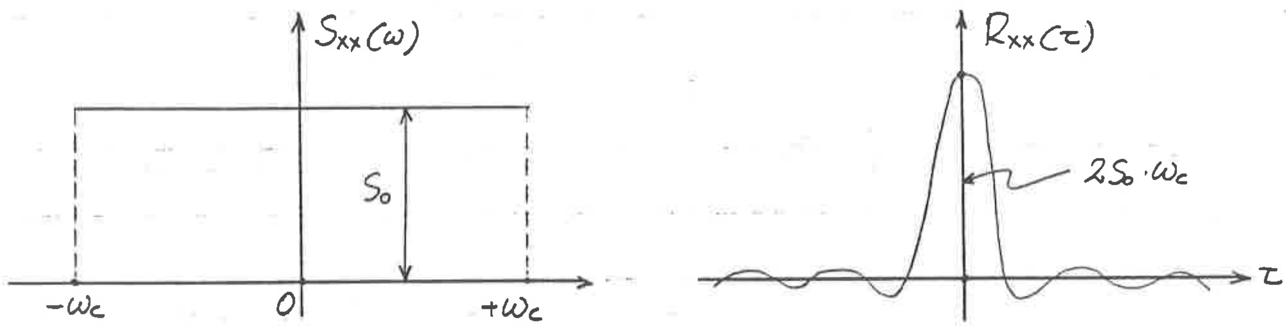
- Since $R_{XX}(\tau) = 0$ for $\tau \neq 0$, $X(t)$ and $X(t+\tau)$ are uncorrelated for any non-zero value of τ.
- Idealized white-noise is physically unrealizable, since its mean square is infinite: $E[X^2(t)] = R_{XX}(0) \rightarrow \infty$

2) Band limited White-Noise:

$$S_{xx}(\omega) = \begin{cases} S_0 & \text{for } |\omega| \leqq \omega_c \\ 0 & \text{otherwise} \end{cases} \qquad \text{where } \omega_c = \text{cutoff frequency}$$

$$R_{xx}(\tau) = \int_{-\infty}^{\infty} S_{xx}(\omega) \cdot e^{i\omega\tau} d\omega = \int_{-\omega_c}^{\omega_c} S_0 \cdot e^{i\omega\tau} d\omega = S_0 \int_{-\omega_c}^{\omega_c} \cos\omega\tau \, d\omega \Rightarrow$$

$$\Rightarrow R_{xx}(\tau) = 2 \cdot S_0 \cdot \frac{\sin\omega_c \cdot \tau}{\tau}$$



- Mean Square: $E[x^2(t)] = R_{xx}(0) = \int_{-\omega_c}^{\omega_c} S_0 \, d\omega = 2 S_0 \cdot \omega_c$

3) Kanai-Tajimi Spectrum:

- Used to describe ground acceleration during an earthquake

$$S_{xx}(\omega) = S_0 \cdot \frac{\omega_g^4 + 4\zeta_g^2 \cdot \omega_g^2 \cdot \omega^2}{(\omega^2 - \omega_g^2)^2 + 4\zeta_g^2 \cdot \omega_g^2 \cdot \omega^2}$$

S_0 is proportional to the earthquake magnitude

ω_g = ground resonance frequency

ζ_g depends on the attenuation of seismic waves in the ground

$$R_{xx}(\tau) = \int_{-\infty}^{\infty} S_{xx}(\omega)\, e^{i\omega\tau}\, d\omega = \frac{\pi S_0 \cdot \omega_g}{2\zeta_g} \cdot e^{-\zeta_g \omega_g |\tau|} \cdot$$

$$\cdot \left[(1+4\zeta_g^2) \cdot \cos\left(\sqrt{1-\zeta_g^2}\, \omega_g \cdot |\tau|\right) + (1-4\zeta_g^2) \cdot \frac{\zeta_g}{\sqrt{1-\zeta_g^2}} \cdot \sin\left(\sqrt{1-\zeta_g^2} \cdot \omega_g \cdot |\tau|\right) \right]$$

- Differentiation of a Random Process

  - Limit in the mean (l.i.m.):

$$\underset{h\to 0}{\text{l.i.m.}}\; X(t+h) = X(t) \quad \text{means that} \quad \lim_{h\to 0} E\left[\, |X(t+h) - X(t)|^2 \right] = 0$$

  - Continuity: a random process $X(t)$ is said to be continuous in the mean square if:

$$\underset{h\to 0}{\text{l.i.m.}}\; X(t+h) = X(t)$$

  - Mean square derivative of random process $X(t)$:

$$\frac{dX(t)}{dt} = \dot{X}(t) = \underset{h\to 0}{\text{l.i.m.}}\; \frac{X(t+h) - X(t)}{h}$$

  - The necessary and sufficient conditions for $X(t)$ to be differentiable in the mean square are:

$$\frac{\partial^2 R_{xx}(t_1, t_2)}{\partial t_1 \partial t_2} \text{ exists } (<\infty) \text{ and is continuous at } t_1 = t_2$$

If $X(t)$ is a stationary random process, these conditions become:

$$\frac{d^2 R_{XX}(\tau)}{d\tau^2} \text{ exists } (<\infty) \text{ and is continuous at } \tau = 0$$

- <u>If $X(t)$ is a stationary random process differentiable in the mean square, then:</u>

$$E[\dot{X}(t)] = \frac{d}{dt} E[X(t)]$$

$$R_{\dot{X}\dot{X}}(\tau) = -\frac{d^2 R_{XX}(\tau)}{d\tau^2} \rightsquigarrow S_{\dot{X}\dot{X}}(\omega) = \omega^2 \cdot S_{XX}(\omega)$$

<u>Proof:</u>

$$R_{XX}(\tau) = \int_{-\infty}^{\infty} S_{XX}(\omega) \cdot e^{+i\omega\tau} d\omega \Rightarrow \text{(by differentiating twice)}$$

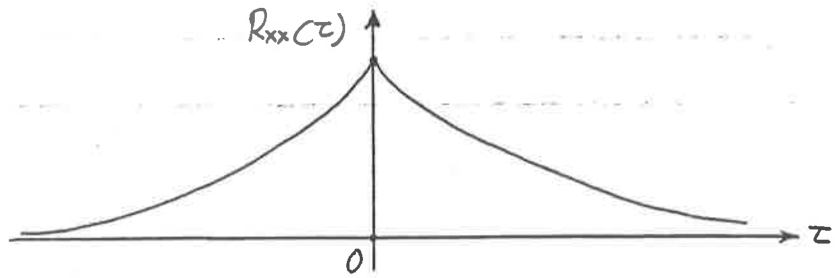
$$\Rightarrow \frac{d^2 R_{XX}(\tau)}{d\tau^2} = \int_{-\infty}^{\infty} S_{XX}(\omega) \cdot (i\omega)^2 e^{i\omega\tau} d\omega \Rightarrow$$

$$\left.\begin{aligned} &\Rightarrow -\frac{d^2 R_{XX}(\tau)}{d\tau^2} = R_{\dot{X}\dot{X}}(\tau) = \int_{-\infty}^{\infty} \omega^2 \cdot S_{XX}(\omega)\, e^{i\omega\tau} d\omega \\ &\text{But: } R_{\dot{X}\dot{X}}(\tau) = \int_{-\infty}^{\infty} S_{\dot{X}\dot{X}}(\omega)\, e^{i\omega\tau} d\omega \end{aligned}\right\} \Rightarrow S_{\dot{X}\dot{X}}(\omega) = \omega^2 \cdot S_{XX}(\omega)$$

<u>In general:</u>

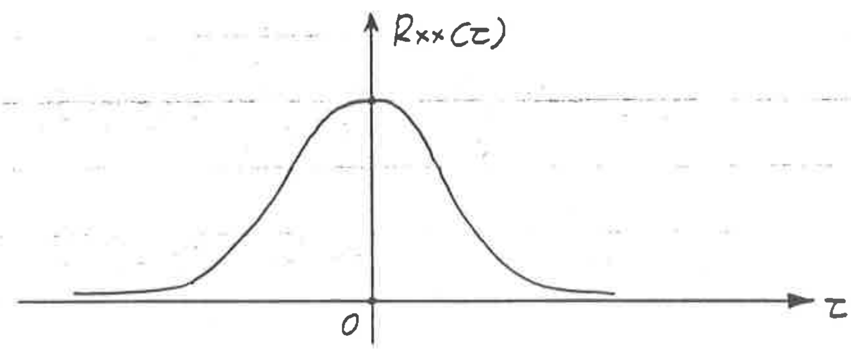
$$\left\{\begin{aligned} R_{X^{(n)}X^{(n)}}(\tau) &= (-1)^n \cdot R_{XX}^{(2n)}(\tau) \\ S_{X^{(n)}X^{(n)}}(\omega) &= (-1)^n \cdot (i\omega)^{2n}\, S_{XX}(\omega) \end{aligned}\right\}$$

Example: consider a stationary random process $X(t)$ with autocorrelation function : $R_{xx}(\tau) = \sigma^2 \cdot e^{-\alpha \cdot |\tau|}$; $\alpha > 0$



$X(t)$ is not differentiable since $\frac{d^2 R_{xx}(\tau)}{d\tau^2}$ is not continuous at $\tau = 0$.

Example: consider a stationary random process $X(t)$ with autocorrelation function : $R_{xx}(\tau) = \sigma^2 \cdot e^{-\alpha\tau^2}$; $\alpha > 0$



$X(t)$ is differentiable since $\frac{d^2 R_{xx}(\tau)}{d\tau^2}$ exists and is continuous at $\tau = 0$.

- Two stationary random processes $X(t)$ and $Y(t)$ are said to be orthogonal when:

$$E[X(t)Y(t)] = R_{XY}(0) = 0$$

If $X(t)$ is a stationary random process differentiable in the mean square, then:

$$R_{\dot{X}X}(\tau) = \frac{dR_{XX}(\tau)}{d\tau}$$

But since $X(t)$ is differentiable: $\frac{d^2R_{XX}(\tau)}{d\tau^2}$ exists and is continuous at $\tau=0$. In addition, it is known that $R_{XX}(\tau)$ is an even function of τ.. As a result:

$$R_{\dot{X}X}(\tau=0) = \left[\frac{dR_{XX}(\tau)}{d\tau}\right]_{\tau=0} = 0 = E[\dot{X}(t)X(t)]$$

Therefore, <u>a stationary random process and its derivative process</u> (if it exists) <u>are orthogonal.</u>

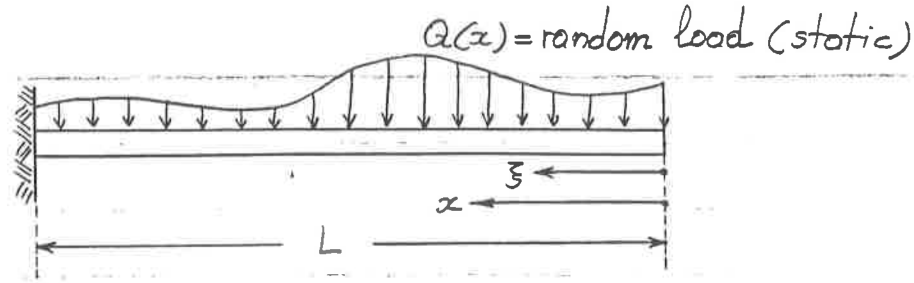
- <u>Conclusion:</u> if a stationary random process is differentiable in the mean square, then its autocorrelation function $R_{XX}(\tau)$ has a true maximum at $\tau=0$, in the sense that: $\left[\frac{dR_{XX}(\tau)}{d\tau}\right]_{\tau=0} = 0$ and $\left[\frac{d^2R_{XX}(\tau)}{d\tau^2}\right]_{\tau=0}$

- <u>Integration of a Random Process</u>

$$\underset{\text{random variable}}{Y} = \int_a^b \underset{\text{random process}}{X(t)}\, dt \qquad \underset{\text{random process}}{Z(t)} = \int_a^b \underset{\text{bounded deterministic function}}{h(t,t^*)}\, \underset{\text{random process}}{X(t^*)}\, dt^*$$

- Operations of expectation and integration can be interchanged

Example:



$Q(x)$ is a random field with given autocorrelation function $R_{QQ}(x_1, x_2)$. The bending moment is going to be a random field also. Find the autocorrelation function of the bending moment $M(x)$.

$$M(x) = \int_0^x Q(\xi)\,(x-\xi)\,d\xi$$

$$R_{MM}(x_1, x_2) = E[M(x_1)M(x_2)] = E\left[\int_0^{x_1}(x_1-\xi_1)\,Q(\xi_1)\,d\xi_1 \int_0^{x_2}(x_2-\xi_2)\,Q(\xi_2)\,d\xi_2\right]$$

$$= E\left[\int_0^{x_1}\!\!\int_0^{x_2}(x_1-\xi_1)(x_2-\xi_2)\,Q(\xi_1)\,Q(\xi_2)\,d\xi_1\,d\xi_2\right] =$$

$$= \int_0^{x_1}\!\!\int_0^{x_2}(x_1-\xi_1)(x_2-\xi_2)\,E[Q(\xi_1)\,Q(\xi_2)]\,d\xi_1\,d\xi_2 =$$

$$= \int_0^{x_1}\!\!\int_0^{x_2}(x_1-\xi_1)(x_2-\xi_2)\cdot R_{QQ}(\xi_1,\xi_2)\,d\xi_1\,d\xi_2$$

- Gaussian Random Processes

  - A random process is a parametered family of random variables with the parameter belonging to an indexing set
  - If all these random variables are Gaussian, then the random process is said to be Gaussian.

- A Gaussian random process is completely characterized by its mean value $\mu_x(t)$ and autocorrelation function $R_{xx}(t_1, t_2)$. For a stationary Gaussian random process, the following information is needed to completely describe it:

$$\left\{ \begin{matrix} \mu_x = \text{constant} \\ R_{xx}(\tau) \end{matrix} \right\}$$

- The higher moments of a Gaussian random process can be calculated from the lower ones as:

$$E\{[X(t_1)-\mu_x(t_1)]\cdot[X(t_2)-\mu_x(t_2)]\cdots[X(t_{2n+1})-\mu_x(t_{2n+1})]\} = 0$$

$$E\{[X(t_1)-\mu_x(t_1)]\cdot[X(t_2)-\mu_x(t_2)]\cdots[X(t_{2n})-\mu_x(t_{2n})]\} =$$

$$= \sum E\{[X(t_j)-\mu_x(t_j)]\cdot[X(t_k)-\mu_x(t_k)]\}\cdot$$

$$\cdot E\{[X(t_q)-\mu_x(t_q)]\cdot[X(t_s)-\mu_x(t_s)]\}\cdots$$

where the summation involves $\frac{(2n)!}{2^n \cdot n!}$ terms corresponding to all different ways by which $2n$ elements can be broken up into n pairs.

- Under linear operations, Gaussian random processes remain Gaussian.

If $X(t)$ is a Gaussian random process, then:

$$Z(t^*) = \int_a^b X(t) \cdot \underset{\substack{\uparrow \\ \text{deterministic} \\ \text{function}}}{h(t,t^*)}\, dt \quad \text{is also Gaussian}$$

$$\frac{dX(t)}{dt},\ \frac{d^2X(t)}{dt^2},\ \dots \quad \text{are also Gaussian (if they exist)}$$

$$Z(t) = a \cdot X(t) + b \quad \text{is also Gaussian}$$

But $X^2(t)$ or $X^3(t)$ are not Gaussian !

- Numerical Examples on Random Processes

Example 1: A random process $X(t)$ is given by:

$$X(t) = A \cdot \cos\omega t + B \cdot \sin\omega t$$

where ω is a constant and A and B are independen identically distributed random variables with zero means and standard deviations σ. Show that $X(t)$ is a stationary random process.

$$E[X(t)] = E[A \cdot \cos\omega t + B \cdot \sin\omega t] = E[A] \cdot \cos\omega t + E[B] \cdot \sin\omega t = c$$

$$R_{XX}(t_1,t_2) = E[X(t_1)X(t_2)] = E[(A \cdot \cos\omega t_1 + B \cdot \sin\omega t_1) \cdot (A \cdot \cos\omega t_2 + B \cdot \sin\omega t_2)] =$$

$$= E[A^2 \cdot \cos\omega t_1 \cdot \cos\omega t_2 + A \cdot B \cdot (\sin\omega t_1 \cdot \cos\omega t_2 + \cos\omega t_1 \cdot \sin\omega t_2) + B^2 \cdot \sin\omega t_1 \cdot \sin\omega t_2] =$$

$$= E[A^2] \cdot \cos\omega t_1 \cdot \cos\omega t_2 + E[A \cdot B] \cdot (\sin\omega t_1 \cdot \cos\omega t_2 + \cos\omega t_1 \cdot \sin\omega t_2) + E[B^2] \cdot \sin\omega t_1 \cdot \sin\omega t_2 =$$

Using $E[A^2] = E[B^2] = \sigma^2$ and $E[A \cdot B] = 0$ since A and B are independent random variables :

$$R_{xx}(t_1, t_2) = \sigma^2 \cdot \cos\omega(t_2 - t_1) = \sigma^2 \cdot \cos\omega\tau = R_{xx}(\tau)$$

where $\tau = t_2 - t_1$

Therefore $X(t)$ is a stationary random process.

__Example 2 :__ Show that the following autocorrelation function is not realistic :

$$R_{xx}(\tau) = \begin{cases} R_0 & \text{for } |\tau| \leq \tau_c \\ 0 & \text{otherwise} \end{cases}$$

The corresponding power spectral density function is given by :

$$S_{xx}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R_{xx}(\tau) \cdot e^{-i\omega\tau} d\tau = \frac{1}{2\pi} \int_{-\tau_c}^{\tau_c} R_0 \cdot \cos\omega\tau \, d\tau \Rightarrow$$

$$\Rightarrow S_{xx}(\omega) = \frac{R_0}{\pi\omega} \cdot \sin\omega\tau_c$$

The above power spectral density function can take negative values which violates its non-negative property. Therefore, the autocorrelation function is not realistic.

Example 3: Show that the stationary random process:

$X(t) = A \cdot \sin(\omega t + \Phi)$ where: A, ω = constants and Φ is a random phase angle uniformly distributed between 0 and 2π is ergodic in the first moment and in correlation.

It has been shown earlier that the ensemble averages are:

$$E[X(t)] = 0 \qquad \text{and} \qquad R_{xx}(\tau) = \frac{A^2}{2}\cos\omega\tau$$

Instead of trying to show that the three necessary and sufficient conditions are satisfied, we will show that:

$$E[X(t)] = \lim_{T\to\infty} \frac{1}{T}\int_0^T X(t)\,dt \qquad (A)$$

$$E[X(t+\tau)X(t)] = R_{xx}(\tau) = \lim_{T\to\infty} \frac{1}{T-\tau}\int_0^{T-\tau} X(t+\tau)X(t)\,dt \qquad (B)$$

If (A) is satisfied, then the random process is ergodic in the first moment. If (B) is satisfied, the process is ergodic in correlation.

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T X(t)\,dt = \lim_{T\to\infty} \frac{1}{T}\int_0^T A\cdot\sin(\omega t + \Phi)\,dt =$$

$$= \lim_{T\to\infty} \frac{1}{T}\cdot\frac{A}{\omega}\Big[-\cos(\omega t+\Phi)\Big]_0^T =$$

$$= \lim_{T\to\infty} \frac{A}{T\omega}\Big[-\cos(\omega T+\Phi) + \cos\Phi\Big] = 0 = E[X(t)]$$

$$\lim_{T\to\infty} \frac{1}{T-\tau}\int_0^{T-\tau} X(t+\tau)\,X(t)\,dt = \lim_{T\to\infty}\frac{1}{T-\tau}\int_0^{T-\tau} A\cdot\sin(\omega t+\omega\tau+\Phi)\cdot A\sin(\omega t+$$

$$+\Phi)\,dt =$$

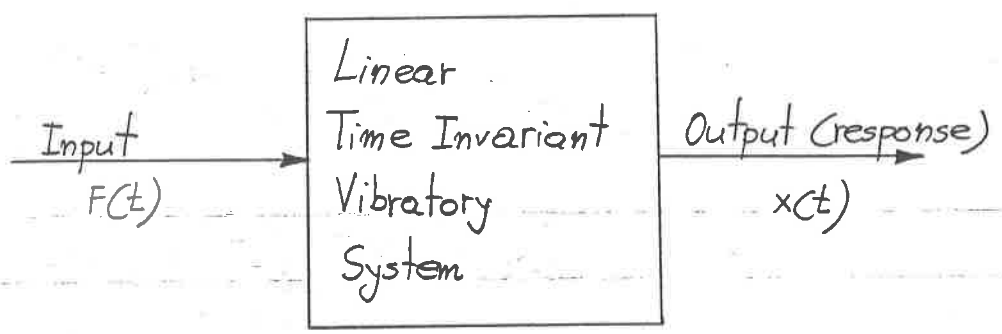
$$= \lim_{T\to\infty}\frac{1}{T-\tau}\cdot A^2\int_0^{T-\tau}\frac{1}{2}\left[\cos(\omega\tau)-\cos(2\omega t+\omega\tau+2\Phi)\right]dt =$$

$$= \lim_{T\to\infty}\frac{A^2}{2}\cdot\frac{1}{T-\tau}\left\{(T-\tau)\cdot\cos\omega\tau-\frac{1}{2\omega}\Big[\sin(2\omega t+\omega\tau+2\Phi)\Big]_0^{T-\tau}\right\} =$$

$$= \lim_{T\to\infty}\frac{A^2}{2}\cdot\cos\omega\tau-\lim_{T\to\infty}\frac{A^2}{4\omega}\cdot\frac{1}{T-\tau}\cdot\left[\sin(2\omega T-\omega\tau+2\Phi)-\sin(\omega\tau+2\Phi)\right]\Rightarrow$$

$$\Rightarrow \lim_{T\to\infty}\frac{1}{T-\tau}\int_0^{T-\tau} X(t+\tau)\,X(t)\,dt = \frac{A^2}{2}\cos\omega\tau = E[X(t+\tau)\,X(t)]$$

4. LINEAR STRUCTURES WITH SINGLE DEGREE OF FREEDOM

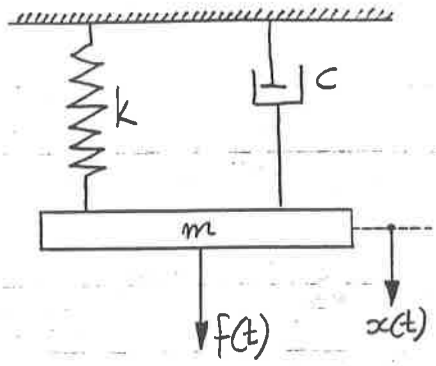


$$m\ddot{x}+c\dot{x}+kx = F(t)$$

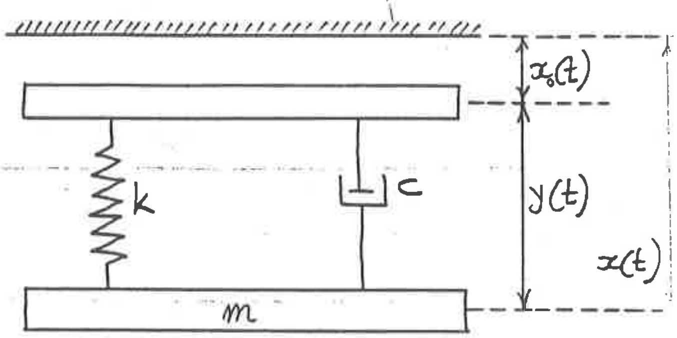
$F(t)$ is random $\rightarrow$ $x(t)$ will be random

Problem: if $F(t)$ is a given random process, find the statistical characteristics of random process $x(t)$.

• Review of Deterministic Theory



System (A)



System (B)

The governing differential equations of the two systems are:

(A) $m\ddot{x} + c\dot{x} + kx = f(t)$

(B) $m\ddot{x} + c(\dot{x}-\dot{x}_0) + k(x-x_0) = 0$

Let: $y = x - x_0 \Rightarrow x = y + x_0$ to get:

(B) $m\ddot{y} + c\dot{y} + ky = -m\ddot{x}_0 = f(t)$

x = absolute displacement

y = relative displacement

Introduce now: $\omega_0 = \sqrt{\frac{k}{m}}$ = undamped natural frequency

$$\zeta = \frac{c}{c_{cr.}} = \frac{c}{2\sqrt{k \cdot m}} = \frac{c}{2\omega_0 \cdot m} = \text{damping ratio}$$

Then the equation of motion can be written in the standard form:

$$\ddot{x} + 2\zeta\omega_0\dot{x} + \omega_0^2 x = \frac{f(t)}{m}$$

Standard Form

- Case I : Sinusoidal Input

$$f(t) = A \cdot e^{i\omega t} \quad ; \quad A = \text{constant}$$

Solution: $x(t) = B \cdot \exp[(-\zeta + i\sqrt{1-\zeta^2})\omega_0 t] + H(\omega) \cdot A \cdot e^{i\omega t} = T1 + T2$

To be complete, needs these two terms $\rightarrow + A \cdot \exp[(-\zeta - i\sqrt{1-\zeta^2})\omega_0 t]$

where: B = constant determined from initial conditions

$$H(\omega) = \frac{1}{m[\omega_0^2 - \omega^2 + 2i\zeta\omega_0\omega]} = \text{frequency response function or transfer function}$$

Usually the real part of $f(t)$ represents the exciting force. Then the real part of $x(t)$ represents the actual response of the syste[m].

$T1$ = complementary function representing the free vibration of the system

$T2$ = particular solution representing the forced vibration of the system.

- Case II : Impulsive Input

$$f(t) = C \cdot \delta(t) \quad ; \quad C = \text{constant}, \quad \delta(t) = \text{Dirac's delta function}$$

Solution: $x(t) = D \cdot \exp[(-\zeta + i\sqrt{1-\zeta^2})\omega_0 t] + C \cdot h(t)$

under $x(0) = 0$, $\dot{x}(0) = \frac{1}{m}$

where: D = constant determined from initial conditions

$$h(t) = \begin{cases} \dfrac{-i}{\sqrt{1-\zeta^2} \cdot \omega_0 \cdot m} \exp[(-\zeta + i\sqrt{1-\zeta^2})\omega_0 \cdot t] & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$

$h(t)$ = impulse response function

Since only the real part of $x(t)$ represents the actual motion, the real part of $h(t)$ is given by:

$$h(t) = \begin{cases} \frac{1}{\omega_d \cdot m} \cdot \exp(-\zeta\omega_0 t)\sin\omega_d t & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$

where $\omega_d = \sqrt{1-\zeta^2} \cdot \omega_0$ = damped natural frequency

- The frequency response function $H(\omega)$ and the impulse response function constitute a Fourier transform pair:

$$\left\{ \begin{aligned} h(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)\, e^{i\omega t}\, d\omega \\ H(\omega) &= \int_{-\infty}^{\infty} h(t)\, e^{-i\omega t}\, dt \end{aligned} \right\} \quad h(t) = 0 \text{ for } t<0$$

Note the difference with a Wiener-Khintchine transform pair!

- Case III: Arbitrary Input

An arbitrary forcing function $f(t)$ can be constructed from a sequence of impulses as:

$$f(t) = \int_0^{\infty} f(\tau) \cdot \delta(t-\tau)\, d\tau$$

Since the principle of superposition holds for a linear system, the response of the system to an arbitrary force $f(t)$ is given by:

$$x(t) = D \cdot \exp\left[(-\zeta + i\sqrt{1-\zeta^2})\omega_0 t\right] + \underbrace{\int_0^t f(\tau)\, h(t-\tau)\, d\tau}_{\text{Duhamel integral}}$$

When the influence of initial conditions on the response becomes negligible, the response is said to have reached steady-state. The steady-state response is given by:

$$x_{st}(t) = \int_{-\infty}^{\infty} f(\tau)\, h(t-\tau)\, d\tau = \int_{-\infty}^{\infty} H(\omega) \cdot \bar{F}(\omega)\, e^{i\omega t}\, d\omega \qquad (c)$$

where $\bar{F}(\omega)$ is the Fourier transform of $f(t)$

Equation (c) constitutes the convolution theorem of Fourier transformations.

- System Response to Random Excitations

$$\ddot{X}(t) + 2\zeta\omega_0 \dot{X}(t) + \omega_0^2 X(t) = \frac{F(t)}{m} \quad ; \ \zeta, \omega_0 = \text{constants}$$

$F(t)$ is a random process $\Rightarrow$ $X(t)$ is a random process

Solution: $$X(t) = \underbrace{D \cdot \exp\left[(-\zeta + i\sqrt{1-\zeta^2})\omega_0 t\right]}_{\substack{\text{deterministic part of} \\ \text{solution assuming} \\ \text{deterministic initial} \\ \text{conditions}}} + \underbrace{\int_0^t F(\tau)\, h(t-\tau)\, d\tau}_{\substack{\text{random part of} \\ \text{solution}}}$$

- Consider now only the random part of the solution:

$$X(t) = \int_0^t F(\tau)\, h(t-\tau)\, d\tau \tag{1}$$

The above expression gives the total displacement if the system was originally at rest prior to exposure to the excitation at $t=0$, but it gives the random deviation from the deterministic free vibration if the system was not originally at rest.

Moments of the response are calculated as:

$$E[X(t)] = \int_0^t E[F(\tau)]\; h(t-\tau)\, d\tau \tag{2}$$

$$E[X(t_1)X(t_2)] = \int_0^{t_1}\!\!\int_0^{t_2} E[F(\tau_1)F(\tau_2)]\; h(t_1-\tau_1)\, h(t_2-\tau_2)\, d\tau_1\, d\tau_2 \tag{3}$$

$$\vdots$$

$$E[X(t_1)\cdots X(t_n)] = \int_0^{t_1}\!\!\cdots\!\int_0^{t_n} E[F(\tau_1)\cdots F(\tau_n)]\; h(t_1-\tau_1)\cdots h(t_n-\tau_n)\, d\tau_1 \cdots d\tau_n \tag{4}$$

Note that if $F(t)$ is a Gaussian random process, $X(t)$ will also be Gaussian since $X(t)$ is obtained from $F(t)$ through a linear operation. In that case $E[X(t)]$ and $E[X(t_1)X(t_2)]$ completely describe the response random process.

- Stationary Random Excitation:

$$E[F(t)] = 0 \quad \text{(assumption)} \tag{5}$$

$$E[F(t_1)F(t_2)] = R_{FF}(t_2-t_1) = R_{FF}(\tau) \tag{6}$$

We know that: $R_{FF}(\tau) = R_{FF}(t_1 - t_2) = \int_{-\infty}^{\infty} S_{FF}(\omega)\, e^{i\omega(t_1-t_2)}\, d\omega$ (7)

Let the autocorrelation function of the response be:

$$E[X(t_1)X(t_2)] = R_{XX}(t_1, t_2) \tag{8}$$

We don't know at this stage whether the response $X(t)$ is a stationary or non-stationary random process.

Combining now Eqs. 3 and 7:

$$R_{XX}(t_1,t_2) = \int_0^{t_1}\int_0^{t_2}\int_{-\infty}^{\infty} S_{FF}(\omega)\, e^{i\omega(\tau_1-\tau_2)}\, d\omega\; h(t_1-\tau_1)\,h(t_2-\tau_2)\, d\tau_1\, d\tau_2 \tag{9}$$

Integrating Eq. 9 with respect to τ_1 and τ_2:

$$R_{XX}(t_1,t_2) = \int_{-\infty}^{\infty} S_{FF}(\omega)\; \mathcal{H}(\omega,t_1)\cdot \mathcal{H}^*(\omega,t_2)\; e^{i\omega(t_1-t_2)}\, d\omega \tag{10}$$

where: $\mathcal{H}(\omega,t) = \int_0^{t} h(u)\exp(-i\omega u)\, du$ (11)

and $*$ = complex conjugate

Note that Eq. 10 was obtained from Eq. 9 using:

$$\int_0^{t_1} h(t_1-\tau_1)\, e^{i\omega\tau_1}\, d\tau_1 = e^{i\omega t_1}\int_0^{t_1} h(t_1-\tau_1)\, e^{-i\omega(t_1-\tau_1)}\, d\tau_1 \tag{12}$$

Let now: $t_1 - \tau_1 = u \rightarrow du = -d\tau_1$ (13)

$$\int_0^{t_1} h(t_1-\tau_1)\, e^{i\omega\tau_1}\, d\tau_1 = -\,e^{i\omega t_1}\int_{t_1}^{0} h(u)\, e^{-i\omega u}\, du =$$

$$= e^{i\omega t_1} \int_0^{t_1} h(u)\, e^{-i\omega u}\, du = e^{i\omega t_1} \mathcal{H}(\omega, t_1) \tag{14}$$

Substitute now the following expression for $h(t)$:

$$h(t) = \frac{1}{m \cdot \omega_d} \cdot \exp(-\zeta\omega_0 t) \sin\omega_d t \tag{15}$$

into Eq. 11 to obtain:

$$\mathcal{H}(\omega, t) = \int_0^t \frac{1}{m\omega_d} \cdot \exp(-\zeta\omega_0 \cdot u) \cdot \sin(\omega_d \cdot u) \cdot \exp(-i\omega u)\, du \tag{16}$$

$$\text{where:} \quad \omega_d = \omega_0 \cdot \sqrt{1-\zeta^2} \tag{17}$$

Performing now the integration in Eq. 16:

$$\mathcal{H}(\omega, t) = H(\omega) \cdot \left[1 - \left(\cos\omega_d t + \frac{\zeta\omega_0 + i\omega}{\omega_d} \cdot \sin\omega_d t\right) \cdot \exp\{(-\zeta\omega_0 + i\omega)t\}\right] \tag{18}$$

where $H(\omega)$ is the frequency response function:

$$H(\omega) = \frac{1}{m[\omega_0^2 - \omega^2 + 2i\zeta\omega_0\omega]} \tag{19}$$

Note that:

$$\lim_{t\to\infty} \mathcal{H}(\omega, t) = H(\omega) = \int_{-\infty}^{\infty} h(u)\, e^{-i\omega u}\, du \tag{20}$$

Substitute now Eq. 18 into Eq. 10 to obtain:

$$
\begin{aligned}
R_{xx}(t_1, t_2) = \int_{-\infty}^{\infty} & S_{FF}(\omega) \cdot |H(\omega)|^2 \cdot \Big\{ \exp[i\omega(t_1 - t_2)] - \\
& - \exp(-\zeta\omega_0 t_1) \left[\left(\cos\omega_d t_1 + \frac{\zeta\omega_0}{\omega_d} \sin\omega_d t_1 \right) \cos\omega t_2 + \frac{\omega}{\omega_d} \sin\omega_d t_1 \sin\omega t_2 \right] \\
& - \exp(-\zeta\omega_0 t_2) \left[\left(\cos\omega_d t_2 + \frac{\zeta\omega_0}{\omega_d} \sin\omega_d t_2 \right) \cos\omega t_1 + \frac{\omega}{\omega_d} \sin\omega_d t_2 \sin\omega t_1 \right] \\
& + \exp[-\zeta\omega_0(t_1 + t_2)] \Bigg[\cos\omega_d t_1 \cos\omega_d t_2 + \frac{\zeta^2\omega_0^2 + \omega^2}{\omega_d^2} \sin\omega_d t_1 \sin\omega_d t_2 \\
& \qquad + \frac{\zeta\omega_0}{\omega_d} \sin\omega_d(t_1 + t_2) \Bigg] \Big\} \, d\omega
\end{aligned}
\tag{21}
$$

The mean-square response is obtained by letting $t_1 = t_2 = t$ in Eq. 21:

$$
\begin{aligned}
E[X^2(t)] = \int_{-\infty}^{\infty} & S_{FF}(\omega) \cdot |H(\omega)|^2 \cdot \Big\{ 1 + \\
& + \exp(-2\omega_0\zeta t) \Bigg[1 + \frac{2\omega_0\zeta}{\omega_d} \sin\omega_d t \cos\omega_d t \\
& - \exp(\omega_0\zeta t) \left(2\cos\omega_d t + \frac{2\omega_0\zeta}{\omega_d} \sin\omega_d t \right) \cos\omega t \\
& - \exp(\omega_0\zeta t) \left(\frac{2\omega}{\omega_d} \right) \sin\omega_d t \sin\omega t \\
& + \frac{(\omega_0\zeta)^2 - \omega_d^2 + \omega^2}{\omega_d^2} \sin^2\omega_d t \Bigg] \Big\} \, d\omega
\end{aligned}
\tag{22}
$$

o Important Conclusions:

1) In the beginning, the system response to a stationary excitation is nonstationary, since $R_{xx}(t_1, t_2)$ is not a function of $t_1 - t_2$ only and $E[X^2(t)]$ is a function of t, for small t.

2) For large t_1 and t_2:

$$R_{xx}(t_1, t_2) \longrightarrow R_{xx}(t_1 - t_2) = R_{xx}(\tau) = \int_{-\infty}^{\infty} S_{FF}(\omega) \cdot |H(\omega)|^2 \cdot e^{i\omega\tau} d\omega \quad (2$$

<u>For large t:</u>

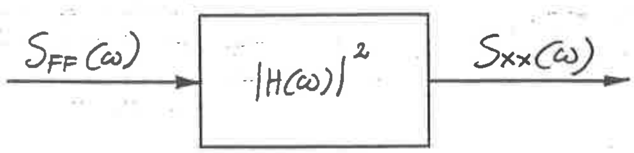
$$E[X^2(t)] \rightarrow R_{xx}(0) = \int_{-\infty}^{\infty} |H(\omega)|^2 \cdot S_{FF}(\omega)\, d\omega \qquad (24)$$

<u>The response eventually becomes stationary !</u>

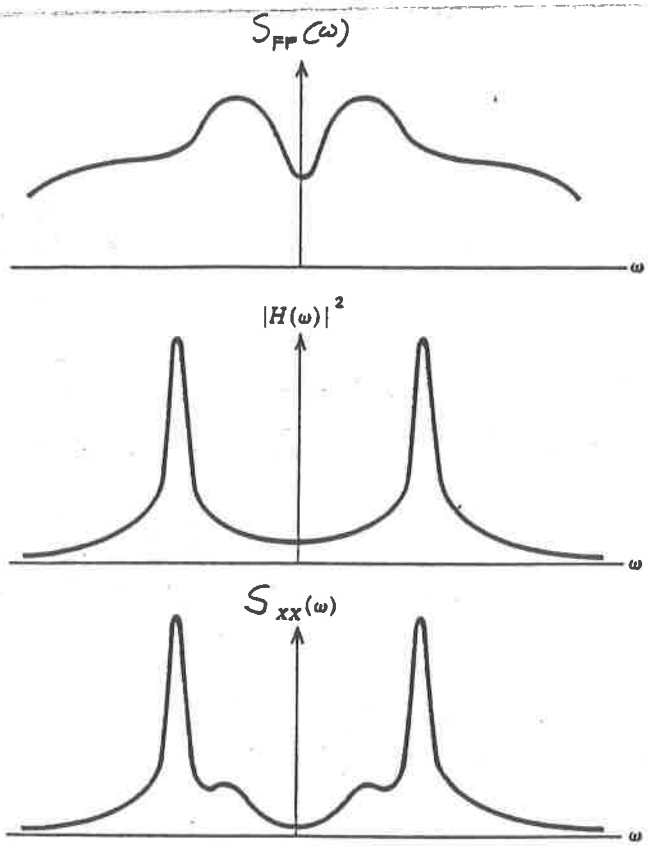
3) At $t=0 \longrightarrow E[X^2(t)] = 0$ as would be expected since the system is assumed to be initially at rest.

4) $$R_{xx}(\tau) = \int_{-\infty}^{\infty} S_{xx}(\omega)\, e^{i\omega\tau} d\omega \overset{(23)}{\Longrightarrow}$$

$$\boxed{S_{xx}(\omega) = S_{FF}(\omega) \cdot |H(\omega)|^2} \qquad (25)$$



$|H(\omega)|^2$ prescribes the fraction of energy to be transmitted through the system at various frequencies. For this reason a linear system is sometimes called a linear filter.



5) For practical applications, the response becomes stationary after

about 4 natural periods when $\zeta = 0.05$
about 20 natural periods when $\zeta = 0.01$

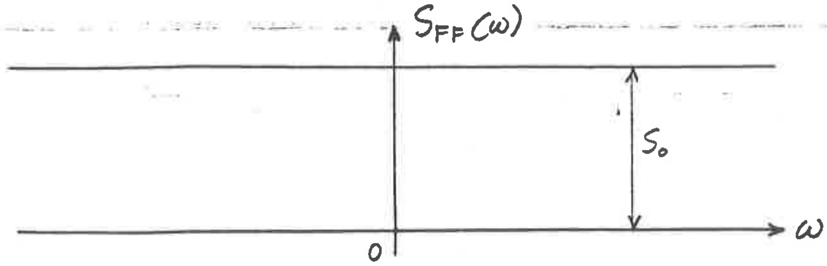
The stationary response is analogous to the steady-state response in deterministic vibration theory.

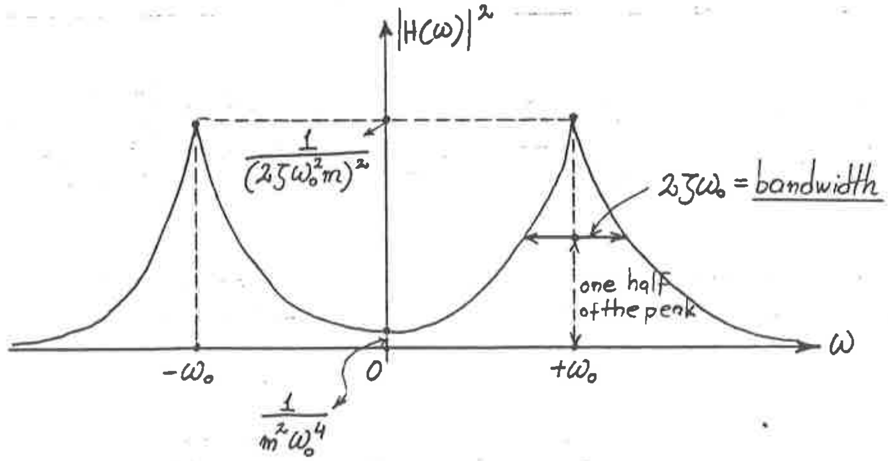
∘ Example: Consider that the excitation is white-noise

$$S_{FF}(\omega) = S_0 \qquad -\infty < \omega < \infty$$

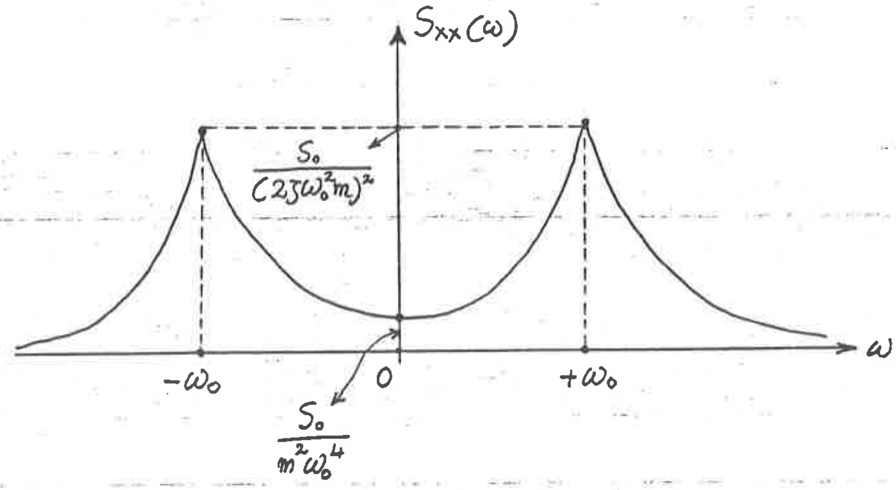
$$R_{FF}(\tau) = 2\pi S_0 \cdot \delta(\tau)$$

$$E[F^2(t)] = R_{FF}(0) \to \infty \qquad \text{(mean square value doesn't exist)}$$





$$|H(\omega)|^2 = \frac{1}{m^2[(\omega_o^2-\omega^2)^2+4\zeta^2\omega^2\omega_o^2]}$$



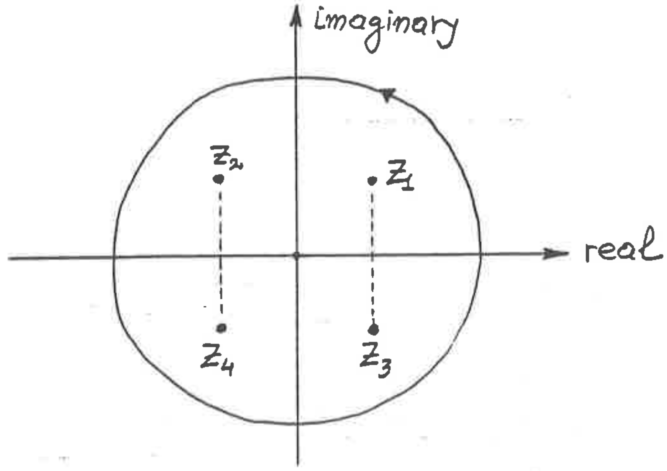
$$S_{XX}(\omega) = |H(\omega)|^2 \cdot S_{FF}(\omega)$$

The mean square value of the response $E[X^2(t)]$ exists, even though the mean square value of the white-noise input $E[F^2(t)]$ does not exist.

$$E[X^2(t)] = R_{XX}(0) = \int_{-\infty}^{\infty} S_{XX}(\omega)\, d\omega = \int_{-\infty}^{\infty} \frac{S_o}{m^2[(\omega_o^2-\omega^2)^2+4\zeta^2\omega^2\omega_o^2]}\, d\omega$$

The above integral can be calculated using the method of residues. The integrand, which is a function of the real variable ω, is treated as a function of the complex variable z:

$$I = \oint \frac{dz}{(\omega_0^2 - z^2)^2 + (2\zeta\omega_0 z)^2} = \begin{cases} 2\pi i \sum \text{Residues in the upper plane} \\ -2\pi i \sum \text{Residues in the lower plane} \end{cases}$$



The poles are located at the zeros of the denominator:

$$(\omega_0^2 - z^2)^2 + 4\zeta^2\omega_0^2 z^2 = (\omega_0^2 - z^2 + 2i\zeta\omega_0 z)(\omega_0^2 - z^2 - 2i\zeta\omega_0 z)$$

The poles are:
$$\begin{cases} z_1 = \omega_0 \cdot \sqrt{1-\zeta^2} + i\zeta\omega_0 \\ z_2 = -\omega_0 \cdot \sqrt{1-\zeta^2} + i\zeta\omega_0 \\ z_3 = \omega_0 \cdot \sqrt{1-\zeta^2} - i\zeta\omega_0 \\ z_4 = -\omega_0 \cdot \sqrt{1-\zeta^2} - i\zeta\omega_0 \end{cases}$$

Partial Fraction Expansion:

$$\frac{1}{(\omega_0^2 - z^2)^2 + 4\zeta^2\omega_0^2 z^2} = \frac{1}{(z-z_1)(z-z_2)(z-z_3)(z-z_4)} = \frac{R_1}{z-z_1} + \frac{R_2}{z-z_2} + \frac{R_3}{z-z_3} + \frac{R_4}{z-z_4}$$

where R_1, R_2, R_3, R_4 are the residues of z_1, z_2, z_3, z_4, respectively.

$$\begin{cases} R_1 = \dfrac{1}{(z_1-z_2)(z_1-z_3)(z_1-z_4)} = \dfrac{1}{2\omega_0\sqrt{1-\zeta^2}\,(2i\zeta\omega_0)(2)(\omega_0\sqrt{1-\zeta^2}+i\zeta\omega_0)} \\ R_2 = \dfrac{1}{(z_2-z_1)(z_2-z_3)(z_2-z_4)} = \dfrac{1}{-2\omega_0\sqrt{1-\zeta^2}\,(2i\zeta\omega_0)(-2)(\omega_0\sqrt{1-\zeta^2}-i\zeta\omega_0)} \end{cases}$$

$$R_1 + R_2 = \frac{1}{4i\zeta\omega_0^3}$$

And the integral is calculated as:

$$I = 2\pi i \cdot (R_1 + R_2) = \frac{2\pi i}{4i\zeta\cdot\omega_0^3} = \frac{\pi}{2\zeta\omega_0^3}$$

Finally, the mean square value of the response is given by:

$$\boxed{E[X^2(t)] = \frac{S_0\cdot\pi}{2\zeta\cdot m^2\cdot\omega_0^3}}$$

S_0 in $N^2\cdot sec$

m in $\dfrac{N\cdot sec^2}{m}$

ω_0 in $\dfrac{rad}{sec}$

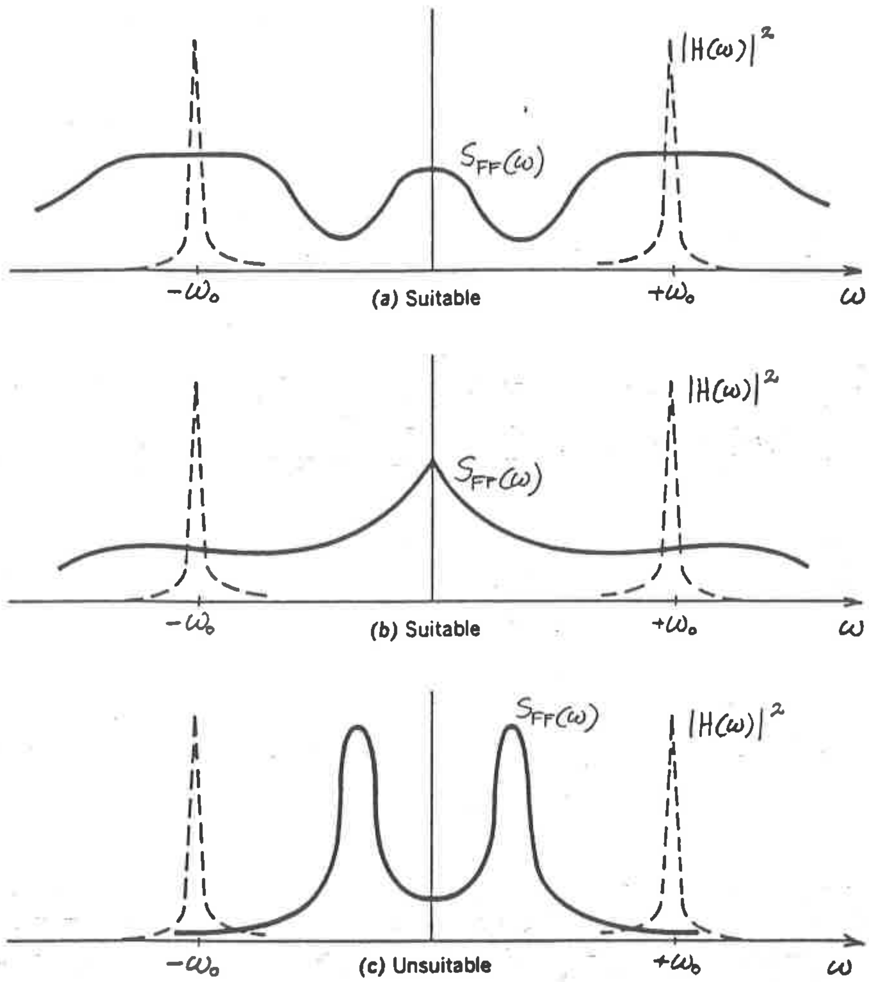
ζ is non-dimensional

$E[X^2(t)]$ in m^2 ($X(t) \rightarrow$ displacement)

The mean-square output is finite even though the mean-square input is infinite!

- White-noise Idealization:

Consider the following three cases of $S_{FF}(\omega)$ and $|H(\omega)|^2$:



$$E[X^2(t)] = R_{XX}(0) = \int_{-\infty}^{\infty} |H(\omega)|^2 \, S_{FF}(\omega)\, d\omega \qquad (A)$$

In cases like (a) and (b) where most of the value of the integral shown in Eq. A is calculated from the vicinities of $\pm\omega_o$ and in addition the spectral density $S_{FF}(\omega)$ is slowly varying in these vicinities, then the value of $S_{FF}(\omega)$ outside these vicinities is unimportant.

In such cases, the actual spectral density $S_{FF}(\omega)$ can be idealized as a white noise as follows:

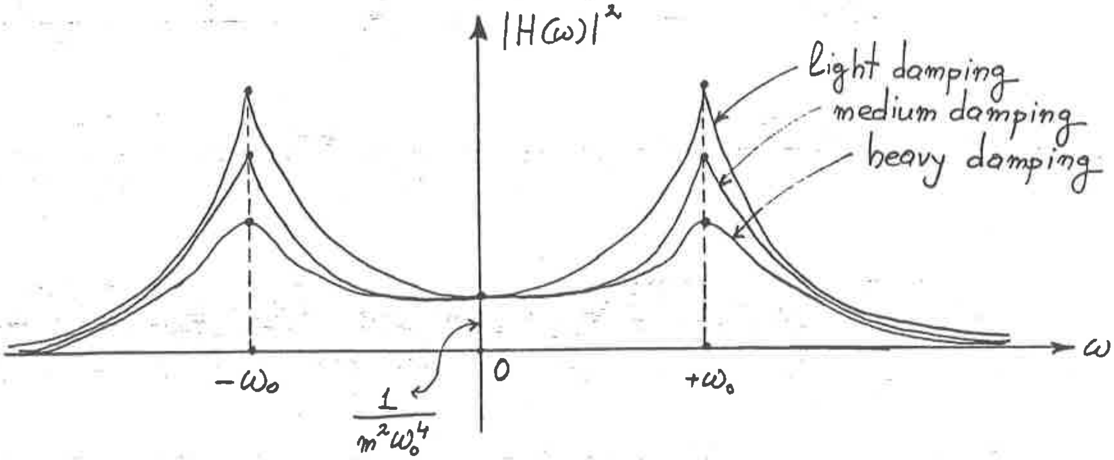
$$S_{FF}(\omega) = S_o = S_{FF}(\omega_o) \qquad -\infty < \omega < \infty$$

Then, Eq. A can be written as:

$$E[X^2(t)] = R_{XX}(0) = \int_{-\infty}^{\infty} |H(\omega)|^2 \cdot S_{FF}(\omega) d\omega \cong S_{FF}(\omega_0) \int_{-\infty}^{\infty} |H(\omega)|^2 d\omega =$$

$$= \frac{S_{FF}(\omega_0) \cdot \pi}{2\zeta \cdot m^2 \cdot \omega_0^3}$$

The white-noise idealization is more often valid when the damping ratio ζ is small (light damping):



Bandwidth = $2\zeta\omega_0$ is very small for cases of light damping

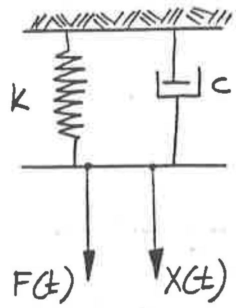
$\zeta < 0.02 \rightarrow$ light damping

$\zeta = 0.02 - 0.04 \rightarrow$ medium damping

$\zeta > 0.04 \rightarrow$ heavy damping

A lightly damped system is often called a narrow-band filter of bandwidth $2\zeta\omega_0$ since roughly only the excitation energy contained in the frequency bands $[\omega_0 - \zeta\omega_0, \omega_0 + \zeta\omega_0]$ and $[-\omega_0 - \zeta\omega_0, -\omega_0 + \zeta\omega_0]$ is transmitted by the filter.

- <u>Example:</u> consider a single-degree-of-freedom system with no mass



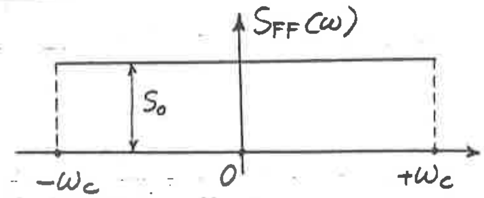
$$c \cdot \dot{X}(t) + k\,X(t) = F(t)$$

$F(t)$ is a stationary random process

<u>The frequency response function is now:</u> $H(\omega) = \dfrac{1}{k + ic\omega}$

Consider also that $F(t)$ is a <u>band-limited white-noise</u>:

$$S_{FF}(\omega) = \begin{cases} S_0 & \text{for } |\omega| \leq \omega_c \\ 0 & \text{otherwise} \end{cases}$$



<u>Then:</u> $S_{xx}(\omega) = |H(\omega)|^2 \cdot S_{FF}(\omega) = \dfrac{S_0}{k^2 + c^2\omega^2}$ for $|\omega| \leq \omega_c$

<u>The mean-square value of the response is calculated as:</u>

$$E[X^2(t)] = R_{xx}(0) = \int_{-\omega_c}^{\omega_c} \frac{S_0}{k^2 + c^2\omega^2}\, d\omega = \frac{2 \cdot S_0}{k \cdot c} \tan^{-1}\left(\frac{c \cdot \omega_c}{k}\right)$$

$$E[\dot{X}^2(t)] = R_{\dot{x}\dot{x}}(0) = \int_{-\omega_c}^{\omega_c} \frac{S_0 \cdot \omega^2}{k^2 + c^2\omega^2}\, d\omega = \frac{2 \cdot S_0}{c^2}\left[\omega_c - \frac{k}{c} \cdot \tan^{-1}\left(\frac{c\,\omega_c}{k}\right)\right]$$

since: $\left\{ \begin{array}{l} S_{\dot{x}\dot{x}}(\omega) = \omega^2 \cdot S_{xx}(\omega) \\ S_{\ddot{x}\ddot{x}}(\omega) = \omega^4 \cdot S_{xx}(\omega) \end{array} \right\}$

- <u>Example:</u> include now the mass effect in the previous example, keeping the band-limited white-noise excitation.

$$S_{xx}(\omega) = |H(\omega)|^2 \cdot S_{FF}(\omega) = \frac{S_0}{m^2[(\omega_0^2-\omega^2)^2+4\zeta^2\omega^2\omega_0^2]} \qquad \text{for } |\omega| \leq \omega_c$$

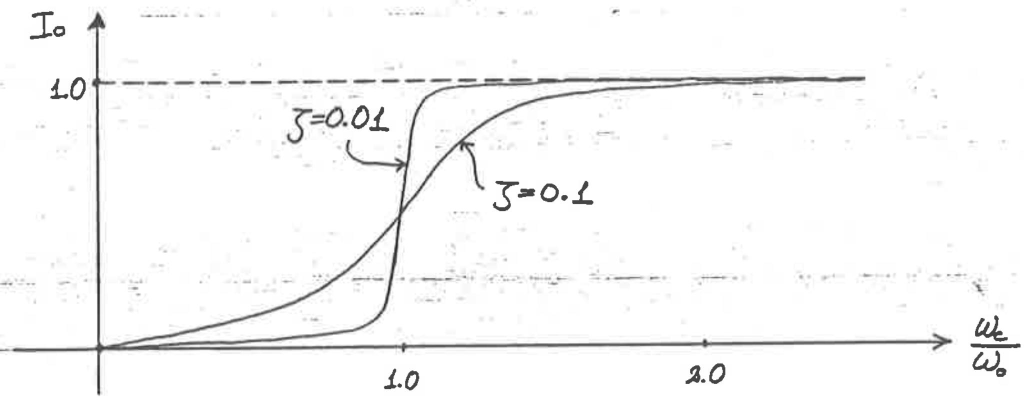
<u>The mean-square of the response is now calculated as:</u>

$$E[X^2(t)] = R_{xx}(0) = \int_{-\omega_c}^{\omega_c} \frac{S_0\, d\omega}{m^2[(\omega_0^2-\omega^2)^2+4\zeta^2\omega^2\omega_0^2]} = \frac{S_0 \cdot \pi}{2m^2\zeta\omega_0^3} \cdot I_0\left(\frac{\omega_c}{\omega_0}, \zeta\right)$$

where:

$$I_0\left(\frac{\omega_c}{\omega_0}, \zeta\right) = \frac{\zeta}{2\pi\sqrt{1-\zeta^2}} \cdot \ln \frac{\left(\frac{\omega_c}{\omega_0}+\sqrt{1-\zeta^2}\right)^2+\zeta^2}{\left(\frac{\omega_c}{\omega_0}-\sqrt{1-\zeta^2}\right)^2+\zeta^2} +$$

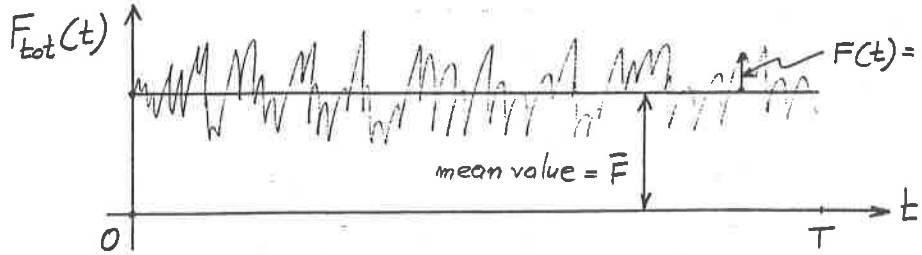
$$+ \frac{1}{\pi}\left[\tan^{-1}\left(\frac{\frac{\omega_c}{\omega_0}-\sqrt{1-\zeta^2}}{\zeta}\right) + \tan^{-1}\left(\frac{\frac{\omega_c}{\omega_0}+\sqrt{1-\zeta^2}}{\zeta}\right)\right]$$



As $\frac{\omega_c}{\omega_0} \to \infty$ the band-limited white-noise becomes idealized white-noise and $I_0 \to 1$, leading to:

$$\lim_{\frac{\omega_c}{\omega_0}\to\infty} E[X^2(t)] = \frac{S_0 \cdot \pi}{2m^2\zeta\omega_0^3}$$

- General Problem: consider a single-degree-of-freedom system (ζ, ω_0) subjected to a force $F_{tot}(t)$.



$$F_{tot}(t) = \bar{F} + F(t)$$

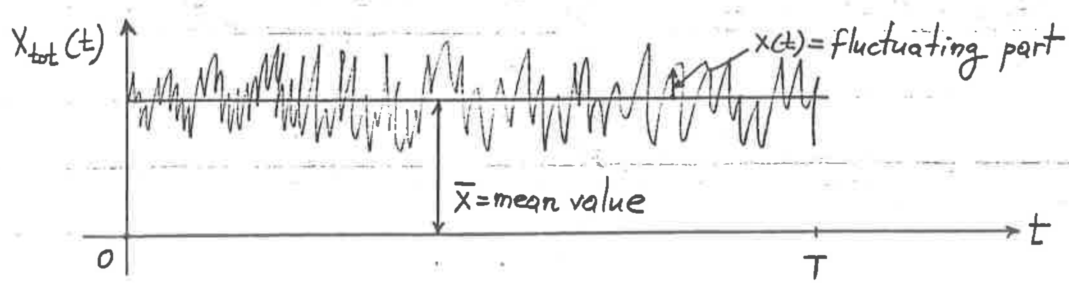
Assume that $F_{tot}(t)$ shown above is one sample function of a stationary and ergodic random process.
Find the mean value and the mean square of the response $X_{tot}(t)$, given the sample function of $F_{tot}(t)$ shown above.

1) Time Domain Analysis:

Since the time-history of the input force $F_{tot}(t)$ is given (either in digital or analogue form), the differential equation:

$$\ddot{X}_{tot} + 2\zeta\omega_0 \dot{X}_{tot} + \omega_0^2 X_{tot} = \frac{F_{tot}(t)}{m} = \frac{\bar{F} + F(t)}{m}$$

can be solved for $X_{tot}(t)$ using a numerical integration algorithm.
After integrating:



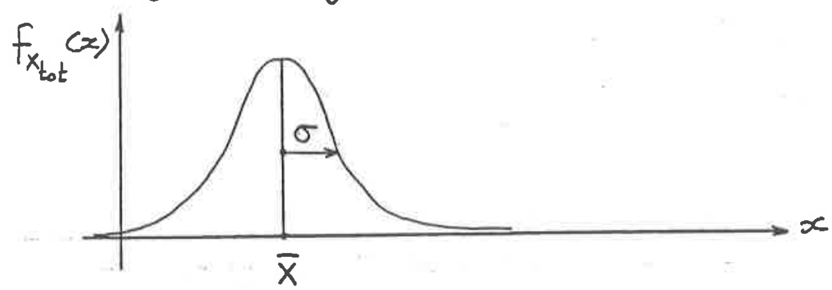
$$\underline{\text{Mean Value}} = \bar{X} = \langle X_{tot}(t) \rangle = \frac{1}{T}\int_0^T X_{tot}(t)\,dt \cong \frac{1}{N}\sum_{i=1}^{N} X_i$$

$$\underline{\text{Mean Square}} = \langle X_{tot}^2(t) \rangle = \frac{1}{T}\int_0^T X_{tot}^2(t)\,dt \cong \frac{1}{N-1}\sum_{i=1}^{N} X_i^2$$

if the response $X_{tot}(t)$ is calculated at N points: X_i ; $i=1,\ldots,$

$$\underline{\text{Standard Deviation}} = \sigma = \sqrt{\langle X_{tot}^2(t) \rangle - \langle X_{tot}(t) \rangle^2}$$

IF $F_{tot}(t)$ is a Gaussian r.p., then $X_{tot}(t)$ will also be Gaussian with probability density function:



2) <u>Frequency Domain Analysis</u> :

$$X_{tot}(t) = \bar{X} + X(t)$$

(i) <u>Mean response:</u> (considered to be a static problem)

$$F(t) = \ddot{X}_{tot} = \dot{X}_{tot} = 0 \rightarrow \omega_0^2 \cdot \bar{X} = \frac{\bar{F}}{m} \Rightarrow \bar{X} = \frac{\bar{F}}{\omega_0^2 \cdot m} = \frac{\bar{F}}{\frac{k}{m}\cdot m} = \frac{\bar{F}}{k} \Rightarrow$$

$$\Rightarrow \bar{X} = \frac{\bar{F}}{k}$$

(ii) Dynamic response:

$$\ddot{X}(t) + 2\zeta\omega_0 \dot{X}(t) + \omega_0^2 X(t) = \frac{F(t)}{m}$$

The first step is to calculate the power spectral density function or autocorrelation function of the input force F(t) (the fluctuating random part of the excitation) : $S_{FF}(\omega)$ or $R_{FF}(\tau)$. Then, the response spectral density is obtained as:

$$S_{XX}(\omega) = |H(\omega)|^2 \cdot S_{FF}(\omega)$$

Finally, the mean-square value of the response is calculated as:

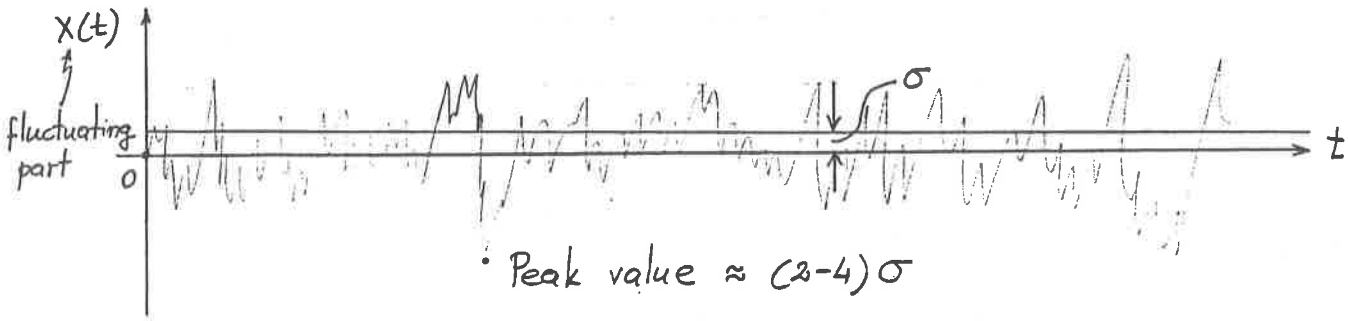
$$E[X^2(t)] = R_{XX}(0) = \int_{-\infty}^{\infty} S_{XX}(\omega)\, d\omega = \int_{-\infty}^{\infty} |H(\omega)|^2 \cdot S_{FF}(\omega)\, d\omega =$$

assume white-no
idealization is
good

$$= \frac{S_{FF}(\omega_0)\cdot \pi}{2m^2 \zeta \omega_0^3}$$

Standard Deviation: $\sigma = \sqrt{E[X^2(t)] - \bar{X}^2}$

Again, if $F_{tot}(t)$ is a Gaussian r.p., then $X_{tot}(t)$ will also be Gaussian with mean value $\bar{X}$ and standard deviation σ.



· Peak value ≈ $(2-4)\sigma$

Joint Behaviour of the Response and the Time-Derivative of the Response

The motion at a particular time instant is characterized by the displacement $X(t)$ and its time-derivative: $\frac{dX(t)}{dt} = \dot{X}(t)$
Let $\dot{X}(t)$ be the mean-square derivative. Then:

$$E[X(t_1)\cdots X(t_m)\dot{X}(t_{m+1})\cdots \dot{X}(t_{m+n})] =$$

$$= \frac{\partial^n}{\partial t_{m+1}\cdots\partial t_{m+n}}\left\{E[X(t_1)X(t_2)\cdots X(t_{m+n})]\right\}$$

Therefore, the joint moments of $X(t)$ and $\dot{X}(t)$ can be obtained from those of $X(t)$.

- If $X(t)$ is a stationary r.p., then $X(t)$ and $\dot{X}(t)$ are orthogonal when evaluated at the same time instant:

$$E[X(t)\dot{X}(t)] = 0$$

- In addition if $X(t)$ is a Gaussian r.p. with mean value equal to zero, then $X(t)$ and $\dot{X}(t)$ are independent. Their joint probability density function is:

$$f_{\{X\}\{\dot{X}\}}(x,t;\dot{x},t) = \frac{1}{2\pi\sigma_X\sigma_{\dot{X}}}\cdot \exp\left[-\frac{1}{2}\left(\frac{x^2}{\sigma_X^2}+\frac{\dot{x}^2}{\sigma_{\dot{X}}^2}\right)\right]$$

where σ_X^2 and $\sigma_{\dot{X}}^2$ are the variances of $X(t)$ and $\dot{X}(t)$ since the mean values of $X(t)$ and $\dot{X}(t)$ are equal to zero.

$$\sigma_X^2 \overset{\text{since } E[X(t)]=0}{=} E[X^2(t)] = \int_{-\infty}^{\infty} S_{XX}(\omega)\, d\omega$$

$$\sigma_{\dot X}^2 = E[\dot X^2(t)] = \int_{-\infty}^{\infty} S_{\dot X \dot X}(\omega)\, d\omega = \int_{-\infty}^{\infty} \omega^2 \cdot S_{XX}(\omega)\, d\omega$$

since $E[\dot X(t)]=0$

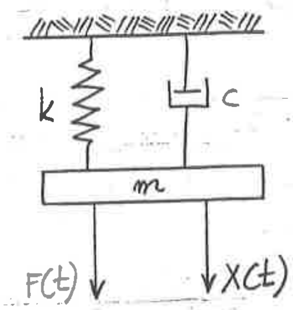
<u>since:</u> $\boxed{S_{X^{(n)}X^{(n)}}(\omega) = \omega^{2n} \cdot S_{XX}(\omega)}$

and: $R_{X^{(n)}X^{(n)}}(\tau) = (-1)^n \cdot R_{XX}^{(2n)}(\tau)$

$$\sigma_{\ddot X}^2 = E[\ddot X^2(t)] = \int_{-\infty}^{\infty} S_{\ddot X \ddot X}(\omega)\, d\omega = \int_{-\infty}^{\infty} \omega^4 \cdot S_{XX}(\omega)\, d\omega$$

since $E[\ddot X(t)]=0$

- <u>Example:</u> consider idealized white-noise as the excitation of a linear single-degree-of-freedom system



$S_{FF}(\omega) = S_0$ for $-\infty < \omega < +\infty$

Assume that $E[F(t)]=0$

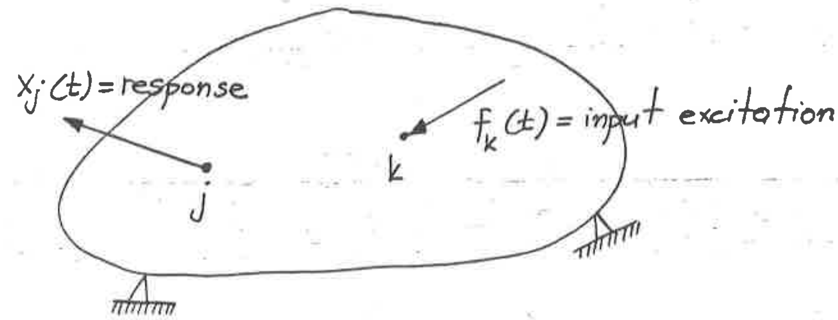
<u>Calculate</u>: σ_X, $\sigma_{\dot X}$ and $\sigma_{\ddot X}$

$$\sigma_X^2 = \int_{-\infty}^{\infty} S_{XX}(\omega)\, d\omega = \int_{-\infty}^{\infty} |H(\omega)|^2 S_{FF}(\omega)\, d\omega = S_0 \int_{-\infty}^{\infty} |H(\omega)|^2 d\omega = \frac{S_0 \cdot \pi}{2m^2 \zeta \omega_0^3}$$

$$\sigma_{\dot X}^2 = \int_{-\infty}^{\infty} \omega^2 \cdot S_{XX}(\omega)\, d\omega = S_0 \int_{-\infty}^{\infty} \omega^2 \cdot |H(\omega)|^2 d\omega = \frac{S_0 \cdot \pi}{2m^2 \zeta \omega_0}$$

$$\sigma_{\ddot X}^2 = \int_{-\infty}^{\infty} \omega^4 \cdot S_{XX}(\omega)\, d\omega = \frac{S_0}{m^2} \int_{-\infty}^{\infty} \frac{\omega^4\, d\omega}{[(\omega_0^2-\omega^2)^2 + 4\zeta^2 \omega_0^2 \omega^2]} \longrightarrow \infty$$

5. LINEAR STRUCTURES WITH FINITELY MANY DEGREES OF FREEDOM



- The dynamic characteristics of a linear n-degrees-of-freedom structure are specified by the matrix of frequency response functions $[H(\omega)]$:

$$[H(\omega)] = \begin{bmatrix} H_{11} & H_{12} & \cdots & H_{1n} \\ H_{21} & H_{22} & \cdots & H_{2n} \\ \vdots & & \ddots & \vdots \\ H_{n1} & H_{n2} & \cdots & H_{nn} \end{bmatrix}$$

- Physical Meaning of $H_{jk}(\omega)$:

  - Neglecting the effect of initial conditions, we assume that the response has reached steady-state.
  - Excitation at point k: $f_k(t) = A \cdot e^{i\omega t}$; A = constant
  - Response at point j: $x_j(t) = A \cdot H_{jk}(\omega) \cdot e^{i\omega t}$

frequency response function (response at j, due to input at k)

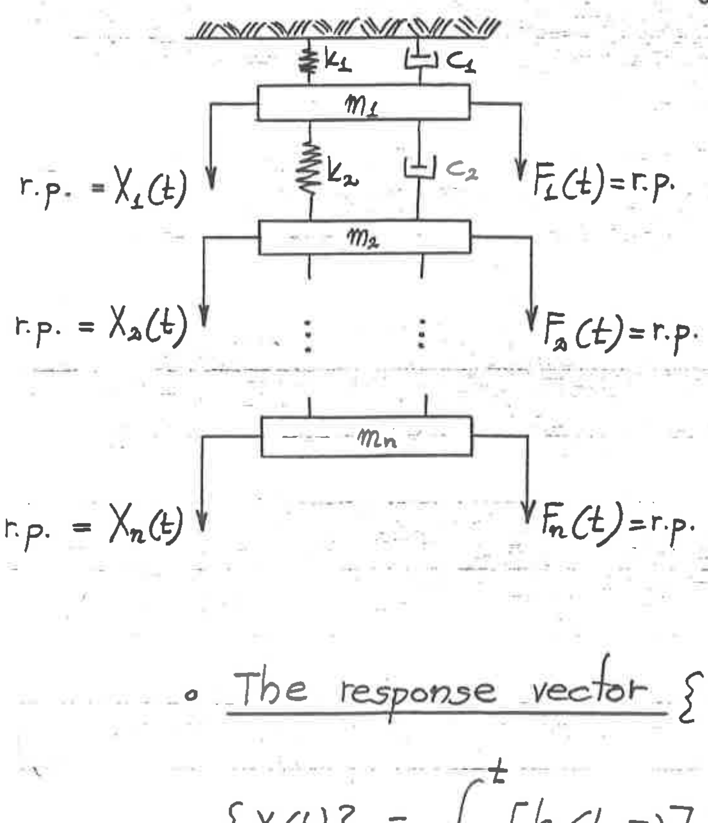
$$H_{jk}(\omega) = \frac{[x_j(t)]_{\text{steady-state}}}{[f_k(t)]_{\text{sinusoidal}}}$$

- Matrix of Impulse Response Functions $[h(t)]$:

$$[h(t)] = \begin{bmatrix} h_{11}(t) & h_{12}(t) & \cdots & h_{1n}(t) \\ h_{21}(t) & h_{22}(t) & \cdots & h_{2n}(t) \\ \vdots & & \ddots & \vdots \\ h_{n1}(t) & h_{n2}(t) & \cdots & h_{nn}(t) \end{bmatrix} \quad \text{with } h_{jk}(t)=0 \text{ for } t<0$$

$$\left\{ \begin{aligned} [h(t)] &= \frac{1}{2\pi}\int_{-\infty}^{\infty} [H(\omega)]\, e^{i\omega t}\, d\omega \\ [H(\omega)] &= \int_{-\infty}^{\infty} [h(t)]\, e^{-i\omega t}\, dt \end{aligned} \right\}$$

- Consider now the following n-degrees-of-freedom system:



Excitation vector: (random)

$$\{F(t)\} = [\, F_1(t) \;\; F_2(t) \;\; \cdots \;\; F_n(t) \,]^T$$

Response vector: (random)

$$\{X(t)\} = [\, X_1(t) \;\; X_2(t) \;\; \cdots \;\; X_n(t) \,]^T$$

- The response vector $\{X(t)\}$ is obtained as:

$$\{X(t)\} = \int_0^t [h(t-\tau)]\,\{F(\tau)\}\, d\tau$$

- Consider a particular element of the response vector $\{X(t)\}$:

$$X_j(t) = \int_0^t \underbrace{h_{jk}(t-\tau) F_k(\tau)}_{\substack{\text{summation with}\\ \text{respect to } k=1,n\,!}} d\tau = \sum_{k=1}^{m} \int_0^t h_{jk}(t-\tau) F_k(\tau) d\tau$$

- <u>Mean Value:</u>

$$E[\{X(t)\}] = \int_0^t [h(t-\tau)] \cdot E[\{F(\tau)\}] d\tau \qquad \underline{\text{or}}$$

$$E\left[\begin{Bmatrix} X_1(t) \\ X_2(t) \\ \vdots \\ X_n(t) \end{Bmatrix}\right] = \int_0^t \underset{\substack{\uparrow \\ n\times n \text{ matrix}}}{[h(t-\tau)]} \cdot E\left[\begin{Bmatrix} F_1(\tau) \\ F_2(\tau) \\ \vdots \\ F_n(\tau) \end{Bmatrix}\right] d\tau$$

- <u>Cross-correlation Matrix</u> (contains auto- and cross-correlation functions):

$$E[\{X(t_1)\}\{X^*(t_2)\}^T] =$$

$$= \int_0^{t_1}\int_0^{t_2} [h(t_1-\tau_1)] \cdot E[\{F(\tau_1)\}\{F^*(\tau_2)\}^T] \cdot [h(t_2-\tau_2)]^T d\tau_1 d\tau_2 =$$

[using: $(A\cdot B)^T = B^T \cdot A^T$ where A, B = matrices]

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [h(t_1-\tau_1)] \cdot E[\{F(\tau_1)\}\{F^*(\tau_2)\}^T] \cdot [h(t_2-\tau_2)]^T d\tau_1 d\tau_2 \quad (1)$$

<u>since:</u> $[h(t)] = 0$ for $t<0$ (upper limit extension)
$\{F(t)\} = 0$ for $t<0$ (lower limit extension)

- <u>Cross-spectral Density Matrix</u> (contains power spectral density and cross-spectral density functions):

<u>We know that:</u>

$$[S_{XX}(\omega_1,\omega_2)] = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E[\{X(t_1)\}\cdot\{X^*(t_2)\}^T]\cdot e^{-i\omega_1 t_1 + i\omega_2 t_2}\, dt_1\, dt_2 \quad (2)$$

$$[S_{FF}(\omega_1,\omega_2)] = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E[\{F(\tau_1)\}\cdot\{F^*(\tau_2)\}^T]\cdot e^{-i\omega_1 \tau_1 + i\omega_2 \tau_2}\, d\tau_1\, d\tau_2 \quad (3)$$

<u>Substituting Eq. 1 into Eq. 2:</u>

$$[S_{XX}(\omega_1,\omega_2)] = \frac{1}{2\pi}\iiiint_{-\infty}^{+\infty}[h(t_1-\tau_1)]\cdot E[\{F(\tau_1)\}\cdot\{F^*(\tau_2)\}^T]\cdot[h(t_2-\tau_2)]^T d\tau_1\, d\tau_2\, \cdot e^{-i\omega_1 t_1}\cdot e^{+i\omega_2 t_2}\, dt_1\, dt_2 =$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[h(t_1-\tau_1)]\, e^{-i\omega_1(t_1-\tau_1)}\, d(t_1-\tau_1)\cdot e^{-i\omega_1\tau_1}\cdot E[\{F(\tau_1)\}\cdot\{F^*(\tau_2)\}^T]\cdot \int_{-\infty}^{\infty}[h(t_2-\tau_2)]^T\cdot e^{+i\omega_2(t_2-\tau_2)}\, d(t_2-\tau_2)\cdot e^{+i\omega_2\tau_2}\, d\tau_1\, d\tau_2 =$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[H(\omega_1)]\cdot E[\{F(\tau_1)\}\cdot\{F^*(\tau_2)\}^T]\cdot e^{-i\omega_1\tau_1 + i\omega_2\tau_2}\cdot[H^*(\omega_2)]^T d\tau_1\, d\tau_2$$

<u>And using finally Eq. 3 into the above equation:</u>

$$[S_{XX}(\omega_1,\omega_2)] = [H(\omega_1)]\cdot[S_{FF}(\omega_1,\omega_2)]\cdot[H^*(\omega_2)]^T$$

- Stationary Excitation:

$$E\left[\{F(\tau_1)\}\cdot\{F(\tau_2)\}^T\right] = \left[R_{FF}(\tau_1-\tau_2)\right]$$

$$\left\{\begin{array}{l} \left[R_{FF}(\tau_1-\tau_2)\right] = \displaystyle\int_{-\infty}^{\infty}\left[S_{FF}(\omega)\right]\cdot e^{i\omega(\tau_1-\tau_2)}\,d\omega \\ \left[S_{FF}(\omega)\right] = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty}\left[R_{FF}(\tau)\right]\cdot e^{-i\omega\tau}\,d\tau \end{array}\right\}$$

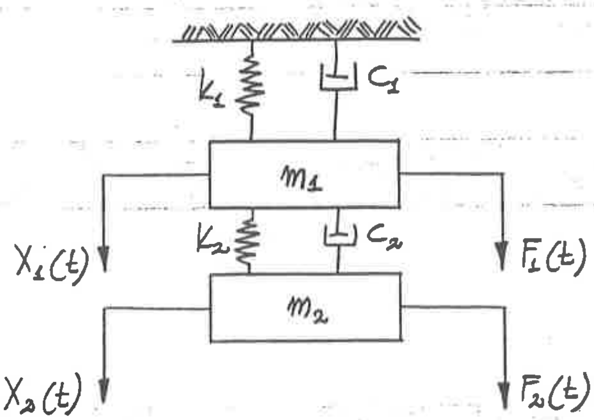
where : $[R_{FF}(\tau)]$ = cross-correlation matrix
$[S_{FF}(\omega)]$ = cross-spectral density matrix

The response which is stationary is calculated as:

$$\left[S_{xx}(\omega)\right] = \left[H(\omega)\right]\cdot\left[S_{FF}(\omega)\right]\cdot\left[H^*(\omega)\right]^T$$

$$\left\{\begin{array}{l} \left[R_{xx}(\tau)\right] = \displaystyle\int_{-\infty}^{\infty}\left[S_{xx}(\omega)\right]\cdot e^{i\omega\tau}\,d\omega \\ \left[S_{xx}(\omega)\right] = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty}\left[R_{xx}(\tau)\right]\cdot e^{-i\omega\tau}\,d\tau \end{array}\right\}$$

Example: consider a two-degrees-of-freedom system:



$F_1(t)$, $F_2(t)$ = stationary random processe

Then, $X_1(t)$ and $X_2(t)$ will be stationary random processes

$$[S_{xx}(\omega)] = [H(\omega)] \cdot [S_{FF}(\omega)] \cdot [H^*(\omega)]^T$$

$$\begin{bmatrix} S_{x_1x_1}(\omega) & S_{x_1x_2}(\omega) \\ S_{x_2x_1}(\omega) & S_{x_2x_2}(\omega) \end{bmatrix} = \begin{bmatrix} H_{11}(\omega) & H_{12}(\omega) \\ H_{21}(\omega) & H_{22}(\omega) \end{bmatrix} \cdot \begin{bmatrix} S_{F_1F_1}(\omega) & S_{F_1F_2}(\omega) \\ S_{F_2F_1}(\omega) & S_{F_2F_2}(\omega) \end{bmatrix} \cdot \begin{bmatrix} H_{11}^*(\omega) & H_{21}^*(\omega) \\ H_{12}^*(\omega) & H_{22}^*(\omega) \end{bmatrix}$$

$$= \begin{bmatrix} H_{11} \cdot S_{F_1F_1} + H_{12} \cdot S_{F_2F_1} & H_{11} \cdot S_{F_1F_2} + H_{12} \cdot S_{F_2F_2} \\ H_{21} \cdot S_{F_1F_1} + H_{22} S_{F_2F_1} & H_{21} \cdot S_{F_1F_2} + H_{22} \cdot S_{F_2F_2} \end{bmatrix} \cdot \begin{bmatrix} H_{11}^*(\omega) & H_{21}^*(\omega) \\ H_{12}^*(\omega) & H_{22}^*(\omega) \end{bmatrix}$$

$$S_{x_1x_1}(\omega) = |H_{11}|^2 \cdot S_{F_1F_1} + H_{12} \cdot H_{11}^* \cdot S_{F_2F_1} + H_{11} \cdot H_{12}^* \cdot S_{F_1F_2} + |H_{12}|^2 S_{F_2F_2}$$

Note that $S_{F_1F_2}(\omega)$ and $S_{F_2F_1}(\omega)$ are rarely known since it is very difficult to estimate cross-spectral density functions.

- <u>The Stiffness-Coefficient Representation</u>

  - A linear structure is represented by the following matrices:

    [k] - stiffness matrix (element: k_{ij}) symmetric
    [c] - damping matrix (element: c_{ij}) usually symmetric
    [m] - mass matrix (element: m_{ij}) symmetric

  - When the structure has n degrees of freedom, the above three matrices have dimension $n \times n$.

  - <u>Strain or Potential Energy:</u>

$$V = \frac{1}{2} \underset{1\times n}{\{x\}^T} \cdot \underset{n\times n}{[k]} \cdot \underset{n\times 1}{\{x\}} = \frac{1}{2} \sum_{j=1}^{n} \sum_{\ell=1}^{n} k_{j\ell}\, x_j \cdot x_\ell \qquad (1)$$

where $\{x\} = [x_1(t)\ x_2(t) \cdots x_n(t)]^T$ = vector of generalized displacements

- Dissipation Function (associated with damping):

$$F_d = \frac{1}{2} \underset{1\times n}{\{\dot{x}\}^T} \cdot \underset{n\times n}{[c]} \cdot \underset{n\times 1}{\{\dot{x}\}} = \frac{1}{2} \sum_{j=1}^{n} \sum_{\ell=1}^{n} c_{j\ell}\, \dot{x}_j \cdot \dot{x}_\ell \qquad (2)$$

- Kinetic Energy:

$$T = \frac{1}{2} \{\dot{x}\}^T \cdot [m] \cdot \{\dot{x}\} = \frac{1}{2} \sum_{j=1}^{n} \sum_{\ell=1}^{n} m_{j\ell}\, \dot{x}_j \cdot \dot{x}_\ell \qquad (3)$$

- Lagrange's Equations:

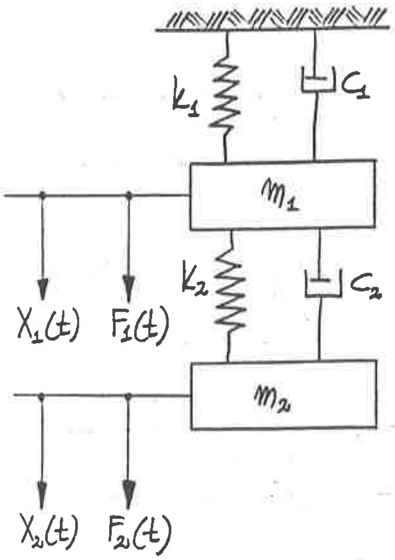
$$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{x}_j}\right] + \frac{\partial F_d}{\partial \dot{x}_j} + \frac{\partial V}{\partial x_j} = F_j \qquad (4)$$

where F_j is the generalized force

- Substituting now Eqs. 1, 2, 3 into Eq. 4:

$$[m] \cdot \{\ddot{x}\} + [c] \cdot \{\dot{x}\} + [k] \cdot \{x\} = \{F\} \qquad (5)$$

- *Example:*



$$[m] = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \qquad \{X\} = \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}$$

$$[k] = \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \qquad \{F\} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

$$[c] = \begin{bmatrix} c_1+c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix}$$

- *Matrix of Frequency Response Functions:*

In Eq. 5 set: $\{F\} = \{F_0\} \cdot e^{i\omega t}$; $\{F_0\} = \text{constant}$

$$\{X\} = [H(\omega)] \cdot \{F_0\} \cdot e^{i\omega t}$$

$$\{\dot{X}\} = [H(\omega)] \cdot \{F_0\} \cdot (i\omega) \cdot e^{i\omega t}$$

$$\{\ddot{X}\} = [H(\omega)] \cdot \{F_0\} \cdot (-\omega^2) \cdot e^{i\omega t}$$

to obtain: $[m] \cdot [H(\omega)] (-\omega^2) + [c] \cdot [H(\omega)] \cdot (i\omega) + [k] \cdot [H(\omega)] = [I$

or: $\{-\omega^2 [m] + i\omega [c] + [k]\} [H(\omega)] = [I]$

Therefore: $$\boxed{[H(\omega)] = \{-\omega^2 [m] \pm i\omega [c] + [k]\}^{-1}}$$

VALID FOR ANY NUMBER OF DEGREES OF FREEDOM !

where $[I]$ = identity matrix

- Going back to the 2-D.O.F. example:

$$[H(\omega)] = \left[\begin{bmatrix} -\omega^2 m_1 & 0 \\ 0 & -\omega^2 m_2 \end{bmatrix} + \begin{bmatrix} i\omega(c_1+c_2) & -i\omega c_2 \\ -i\omega c_2 & i\omega c_2 \end{bmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}\right]^{-1} =$$

$$= \begin{bmatrix} -\omega^2 m_1 + i\omega(c_1+c_2) + k_1 + k_2 & -i\omega c_2 - k_2 \\ -i\omega c_2 - k_2 & -\omega^2 m_2 + i\omega c_2 + k_2 \end{bmatrix}^{-1}$$

Calculation of the inverse of a 2×2 Matrix:

$$[A] = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad ; \quad [A]^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \cdot \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

Check by: $[A]\cdot[A]^{-1} = [I]$

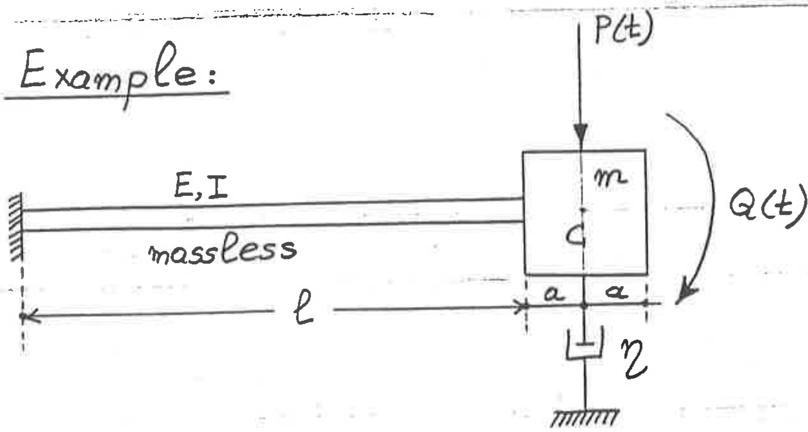
Finally the Cross-spectral Density Matrix is Obtained as:

$$[S_{XX}(\omega)] = [H(\omega)]\cdot[S_{FF}(\omega)][H^*(\omega)]^T = \begin{bmatrix} S_{X_1X_1}(\omega) & S_{X_1X_2}(\omega) \\ S_{X_2X_1}(\omega) & S_{X_2X_2}(\omega) \end{bmatrix}$$

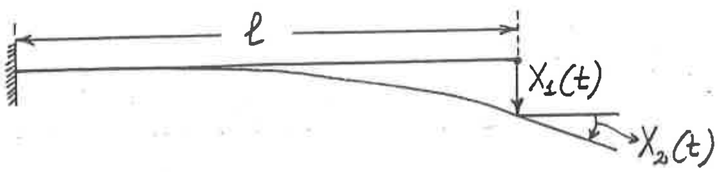
And of course:

$$\left\{\begin{aligned} E[X_1^2(t)] &= \int_{-\infty}^{\infty} S_{X_1X_1}(\omega)\,d\omega \\ E[X_2^2(t)] &= \int_{-\infty}^{\infty} S_{X_2X_2}(\omega)\,d\omega \end{aligned}\right\}$$

Example:



Consider a massless cantilever beam supporting a heavy rigid block.



$P(t)$ = force r.p.
$Q(t)$ = moment r.p.
$X_1(t)$ = displacement r.p.
$X_2(t)$ = rotation r.p.

From strength of materials it can be shown that:

$$V = \frac{6EI}{\ell^3} \cdot X_1^2 - \frac{6EI}{\ell^2} X_1 X_2 + \frac{2EI}{\ell} \cdot X_2^2 \quad \text{(strain energy)}$$

$$T = \frac{1}{2} m \dot{X}_c^2 + \frac{1}{2} I_c \cdot \dot{X}_2^2 \quad \text{(kinetic energy)}$$

where: I_c = moment of inertia of block around centroid C
$X_c = X_1 + a \cdot X_2$

Therefore: $T = \frac{1}{2} m \dot{X}_1^2 + ma \dot{X}_1 \dot{X}_2 + \frac{1}{2}(I_c + ma^2) \cdot \dot{X}_2^2$

$$F_d = -\frac{1}{2} \eta \left(\dot{X}_1^2 + 2a \dot{X}_1 \dot{X}_2 + a^2 \dot{X}_2^2 \right)$$

$$\left\{ \begin{aligned} F_1(t) &= P(t) \\ F_2(t) &= a \cdot P(t) + Q(t) \end{aligned} \right\}$$

Applying now Lagrange's equations:

$$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{x}_j}\right] + \frac{\partial F_d}{\partial \dot{x}_j} + \frac{\partial V}{\partial x_j} = F_j \quad ; \quad j = 1, 2$$

the following equations of motion are obtained:

$$\begin{bmatrix} m & ma \\ ma & I_c + ma^2 \end{bmatrix} \begin{Bmatrix} \ddot{X}_1 \\ \ddot{X}_2 \end{Bmatrix} + \begin{bmatrix} \eta & a\eta \\ a\eta & a^2\eta \end{bmatrix} \cdot \begin{Bmatrix} \dot{X}_1 \\ \dot{X}_2 \end{Bmatrix} + \begin{bmatrix} \frac{12EI}{\ell^3} & -\frac{6EI}{\ell^2} \\ -\frac{6EI}{\ell^2} & \frac{4EI}{\ell} \end{bmatrix} \cdot \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} =$$

$$= \begin{Bmatrix} P(t) \\ a \cdot P(t) + Q(t) \end{Bmatrix}$$

$[H(\omega)]$ is now calculated as:

$$[H(\omega)] = \begin{bmatrix} \frac{12EI}{\ell^3} - \omega^2 m + i\omega\eta & -\frac{6EI}{\ell^2} - \omega^2 ma + i\omega\eta a \\ -\frac{6EI}{\ell^2} - \omega^2 ma + i\omega\eta a & \frac{4EI}{\ell} - \omega^2 (I_c + ma^2) + i\omega a^2\eta \end{bmatrix}^{-1} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} =$$

$$= \frac{1}{\Delta} \cdot \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

where: $\Delta = m I_c \cdot \omega^4 - \frac{4EI\omega^2}{\ell} \left[m + \frac{3}{\ell} ma + \frac{3}{\ell^2} (I_c + ma^2) \right] +$

$$+ \frac{12(EI)^2}{\ell^4} + i\omega\eta \left(-I_c \cdot \omega^2 + \frac{12EIa^2}{\ell^3} + \frac{12EIa}{\ell^2} + \frac{4EI}{\ell} \right)$$

And the cross-spectral density matrix of the response is:

$$[S_{xx}(\omega)] = [H(\omega)] \cdot \begin{bmatrix} S_{F_1F_1}(\omega) & S_{F_1F_2}(\omega) \\ S_{F_2F_1}(\omega) & S_{F_2F_2}(\omega) \end{bmatrix} \cdot [H^*(\omega)]^T$$

The elements of $[S_{FF}(\omega)]$ are calculated as:

$$\left\{\begin{array}{ll} S_{F_1F_1}(\omega) = S_{PP}(\omega) & (1) \\ S_{F_2F_2}(\omega) = a^2 S_{PP}(\omega) + 2a\, S_{PQ}(\omega) + S_{QQ}(\omega) & (2) \\ S_{F_1F_2}(\omega) = S_{F_2F_1}(\omega) = a \cdot S_{PP}(\omega) + S_{PQ}(\omega) & (3) \end{array}\right\}$$

Proof of Eq. 2:

$$E[F_2(t)F_2(t+\tau)] = E[\{a\cdot P(t) + Q(t)\}\cdot\{a\cdot P(t+\tau) + Q(t+\tau)\}] =$$

$$= E[a^2\cdot P(t)P(t+\tau) + a\cdot P(t)\,Q(t+\tau) + a\cdot Q(t)\cdot P(t+\tau) + $$
$$+ Q(t)\cdot Q(t+\tau)] \Rightarrow$$

$$\Rightarrow R_{F_2F_2}(\tau) = a^2\cdot R_{PP}(\tau) + 2a\cdot R_{PQ}(\tau) + R_{QQ}(\tau) \Rightarrow$$

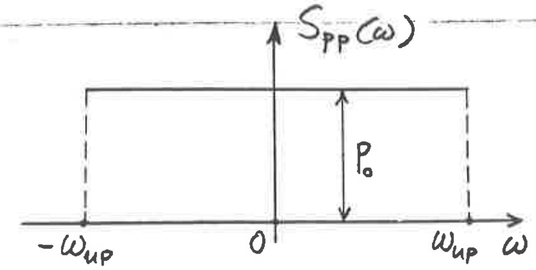
$$\Rightarrow S_{F_2F_2}(\omega) = a^2\cdot S_{PP}(\omega) + 2a\cdot S_{PQ}(\omega) + S_{QQ}(\omega)$$

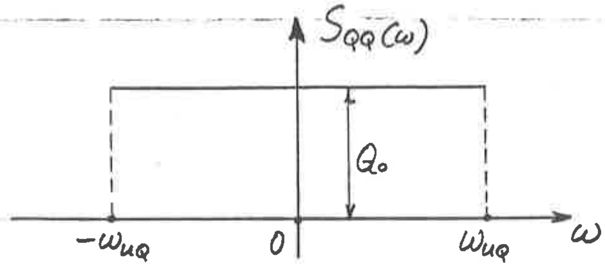
Important Note:

When $P(t)$ and $Q(t)$ are two independent r.p., then:

$$S_{PQ}(\omega) = 0 \quad !$$

Calculate now $E[X_1^2(t)]$ assuming that $P(t)$ and $Q(t)$ are independent r.p. and that:





Since now $S_{PQ}(\omega) = 0$, we have:

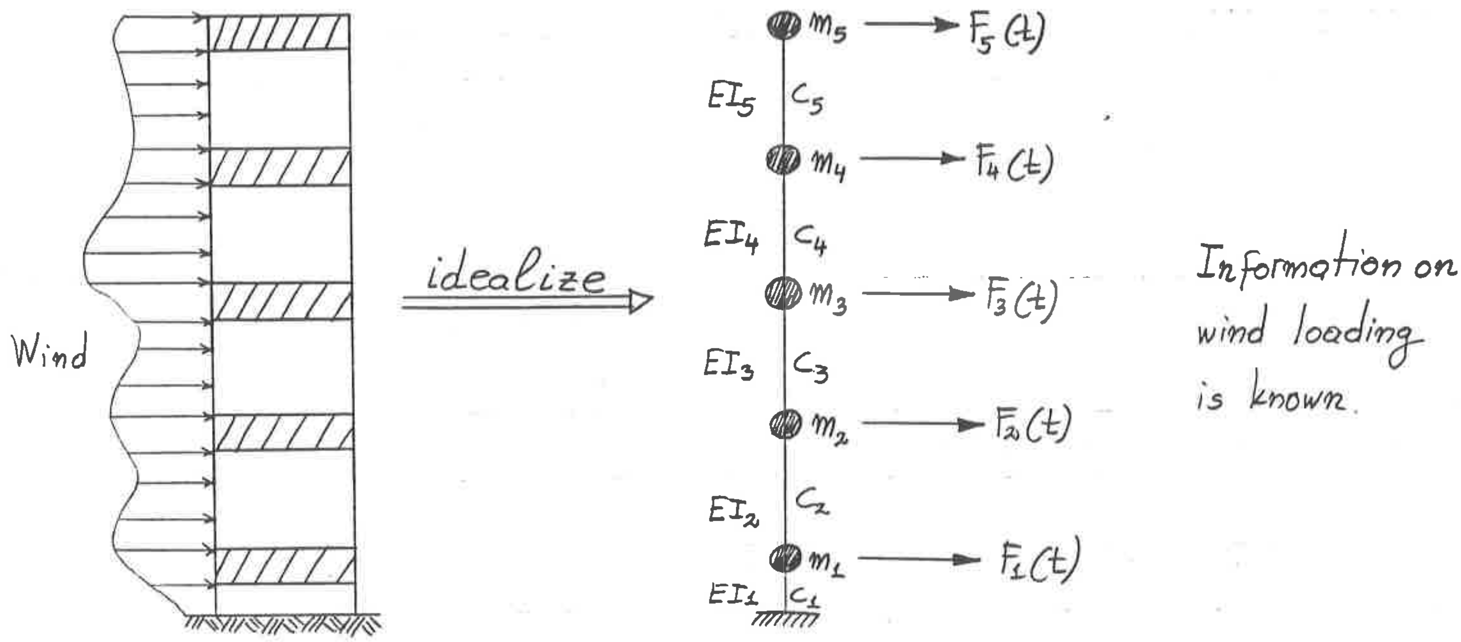
$$\left\{ \begin{aligned} S_{F_1F_1}(\omega) &= S_{PP}(\omega) \\ S_{F_2F_2}(\omega) &= a^2 \cdot S_{PP}(\omega) + S_{QQ}(\omega) \\ S_{F_1F_2}(\omega) &= S_{F_2F_1}(\omega) = a \cdot S_{PP}(\omega) \end{aligned} \right\}$$

$$S_{X_1X_1}(\omega) = |H_{11}(\omega)|^2 \cdot S_{F_1F_1}(\omega) + H_{12}(\omega) \cdot H_{11}^*(\omega) \cdot S_{F_2F_1}(\omega) + $$
$$+ H_{11}(\omega) H_{12}^*(\omega) \cdot S_{F_1F_2}(\omega) + |H_{12}(\omega)|^2 \cdot S_{F_2F_2}(\omega)$$

And: $E[X_1^2(t)] = \int_{-\infty}^{\infty} S_{X_1X_1}(\omega)\, d\omega =$

$$= \int_{-\omega_{uP}}^{\omega_{uP}} \Big[|H_{11}(\omega)|^2 \cdot P_o + H_{12}(\omega) \cdot H_{11}^*(\omega) \cdot a \cdot P_o + H_{11}(\omega) \cdot H_{12}^*(\omega) \cdot a \cdot P_o + $$
$$+ |H_{12}(\omega)|^2 a^2 \cdot P_o \Big] d\omega + \int_{-\omega_{uQ}}^{\omega_{uQ}} |H_{12}(\omega)|^2 \cdot Q_o \, d\omega$$

- General Procedure: Response of Tall Buildings to Random Wind Forces



Step 1: Find Matrix of Frequency Response Functions $[H(\omega)]$

Step 2: Determine Cross-Spectral Density Matrix $[S_{FF}(\omega)]$ of the wind forces

Step 3: Calculate Cross-Spectral Density Matrix $[S_{xx}(\omega)]$ of the response as:

$$[S_{xx}(\omega)] = [H(\omega)] \cdot [S_{FF}(\omega)] \cdot [H^*(\omega)]^T$$

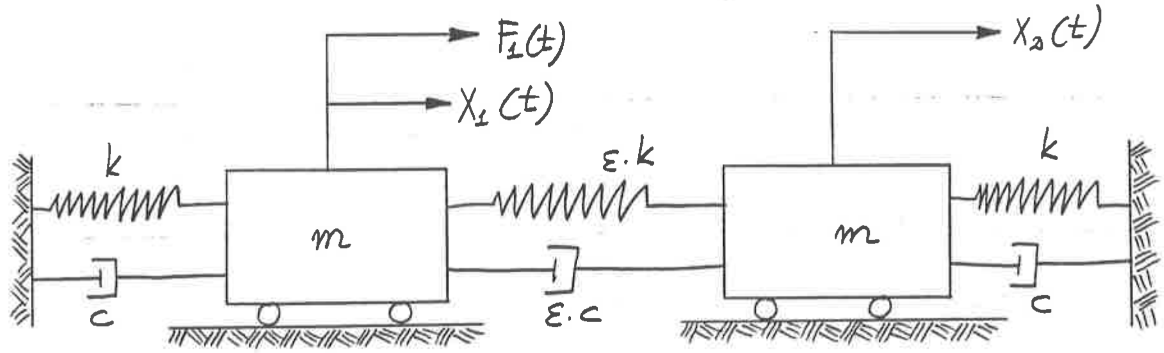
Step 4: Find $E[X_j^2]$ as:

$$E[X_j^2] = \sigma_{X_j}^2 = \int_{-\infty}^{\infty} S_{X_jX_j}(\omega)\,d\omega$$

mean value zero

where X_j is the response of the j-th story

<u>Example:</u> consider the following two-degrees-of-freedom system subjected to a stationary random process force $F_1(t)$.



Find the spectral densities : $S_{X_1X_1}(\omega)$ and $S_{\dot{X}_1\dot{X}_1}(\omega)$.

<u>Equations of Motion:</u>

$$\left.\begin{aligned}\Sigma F_{X_1}=0 &\Rightarrow m\ddot{X}_1 + c\dot{X}_1 - \varepsilon\cdot c(\dot{X}_2-\dot{X}_1) + k\cdot X_1 - \varepsilon k(X_2-X_1) = F_1\\ \Sigma F_{X_2}=0 &\Rightarrow m\ddot{X}_2 + c\dot{X}_2 - \varepsilon\cdot c(\dot{X}_1-\dot{X}_2) + k\cdot X_2 - \varepsilon k(X_1-X_2) = 0\end{aligned}\right\} \Rightarrow$$

$$\Rightarrow [m]\{\ddot{X}\} + [c]\cdot\{\dot{X}\} + [k]\cdot\{X\} = \{F\}$$

where: $[m] = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}$; $[c] = \begin{bmatrix} c(1+\varepsilon) & -c\varepsilon \\ -c\varepsilon & c(1+\varepsilon) \end{bmatrix}$

$$[k] = \begin{bmatrix} k(1+\varepsilon) & -k\cdot\varepsilon \\ -k\varepsilon & k(1+\varepsilon) \end{bmatrix} ; \quad \{X\} = \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} ; \quad \{F\} = \begin{Bmatrix} F_1 \\ 0 \end{Bmatrix}$$

Matrix of Frequency Response Functions:

$$[H(\omega)] = \begin{bmatrix} H_{11}(\omega) & H_{12}(\omega) \\ H_{21}(\omega) & H_{22}(\omega) \end{bmatrix} = \left\{ -\omega^2 [m] + i\omega [c] + [k] \right\}^{-1} =$$

$$= \begin{bmatrix} -\omega^2 m + i\omega c(1+\varepsilon) + k(1+\varepsilon) & -i\omega c\varepsilon - k\varepsilon \\ -i\omega c\varepsilon - k\varepsilon & -\omega^2 m + i\omega c(1+\varepsilon) + k(1+\varepsilon) \end{bmatrix}^{-1} =$$

$$= \frac{1}{\Delta} \begin{bmatrix} -\omega^2 m + i\omega c(1+\varepsilon) + k(1+\varepsilon) & +i\omega c\varepsilon + k\varepsilon \\ +i\omega c\varepsilon + k\varepsilon & -\omega^2 m + i\omega c(1+\varepsilon) + k(1+\varepsilon) \end{bmatrix}$$

where: $\Delta = \omega^4 \cdot m^2 - \omega^2 c^2 (1+\varepsilon)^2 + k^2 (1+\varepsilon)^2 - 2i\omega^3 mc(1+\varepsilon) +$
$+ 2i\omega ck(1+\varepsilon)^2 - 2\omega^2 mk(1+\varepsilon) + \omega^2 c^2 \varepsilon^2 - k^2 \varepsilon^2 - 2i\omega ck\varepsilon^2$

Neglecting now the ε^2 terms (assuming that $\varepsilon \ll 1$):

$$\Delta = \omega^4 \cdot m^2 - \omega^2 c^2 (1+2\varepsilon) + k^2 (1+2\varepsilon) - 2i\omega^3 mc(1+\varepsilon) + 2i\omega ck(1+2\varepsilon) - 2\omega^2 mk(1+\varepsilon)$$

Assume now that $F_1(t)$ is a Gaussian white-noise excitation with intensity S_0:

$$S_{F_1F_1}(\omega) = S_0 \quad \text{for} \quad -\infty < \omega < +\infty$$

Then: $[S_{FF}(\omega)] = \begin{bmatrix} S_{F_1F_1}(\omega) & 0 \\ 0 & 0 \end{bmatrix}$

And the cross-spectral density matrix of the response is:

$$\begin{bmatrix} S_{X_1X_1}(\omega) & S_{X_1X_2}(\omega) \\ S_{X_2X_1}(\omega) & S_{X_2X_2}(\omega) \end{bmatrix} = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \cdot \begin{bmatrix} S_{F_1F_1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} H_{11}^* & H_{21}^* \\ H_{12}^* & H_{22}^* \end{bmatrix} \Longrightarrow$$

$$\left.\begin{aligned} \Longrightarrow \quad S_{X_1X_1}(\omega) &= H_{11} \cdot H_{11}^* \cdot S_{F_1F_1} = |H_{11}(\omega)|^2 \cdot S_{F_1F_1}(\omega) \\ S_{\dot{X}_1\dot{X}_1}(\omega) &= \omega^2 \cdot |H_{11}(\omega)|^2 \cdot S_{F_1F_1}(\omega) \end{aligned}\right\} \Longrightarrow$$

$$\Longrightarrow \left\{ \begin{aligned} S_{X_1X_1}(\omega) &= S_0 \cdot |H_{11}(\omega)|^2 \\ S_{\dot{X}_1\dot{X}_1}(\omega) &= \omega^2 \cdot S_0 \cdot |H_{11}(\omega)|^2 \end{aligned} \right\}$$

Find the mean square of X_1 and $\dot{X}_1$:

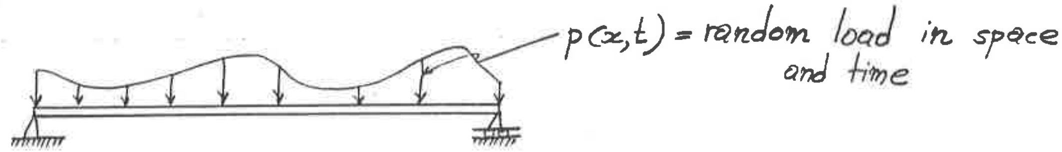
$$E[X_1^2] = \int_{-\infty}^{\infty} S_{X_1X_1}(\omega)\,d\omega = \frac{\pi \cdot S_0}{4kc} \left[1 + \frac{1}{(1+2\varepsilon)^2} + \frac{2 \cdot \left(\frac{c^2}{k \cdot m}\right)}{\frac{\varepsilon^2}{1+\varepsilon} + (1+2\varepsilon)\frac{c^2}{km}} \right]$$

$$E[\dot{X}_1^2] = \int_{-\infty}^{\infty} \omega^2 \cdot S_{X_1X_1}(\omega)\,d\omega = \frac{\pi \cdot S_0}{4mc} \left[1 + \frac{1}{(1+2\varepsilon)} - \frac{2\left(\frac{c^2}{km}\right)}{\frac{\varepsilon^2}{1+\varepsilon} + (1+\varepsilon) \cdot \frac{c^2}{km}} \right]$$

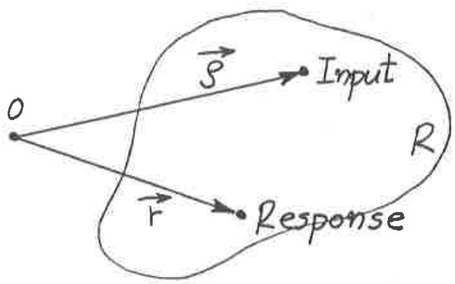
6. LINEAR CONTINUOUS STRUCTURES

• **General Formulation:**

Beams:



Plates: $p(x,y,t)$ = random load in 2 space dimensions and time



The linear continuous structure R is characterized by:

$h(\vec{r},\vec{\rho},t)$ = impulse influence function

or

$H(\vec{r},\vec{\rho},\omega)$ = frequency influence function

$h(\vec{r},\vec{\rho},t)$ = generalization of matrix of impulse response functions
$H(\vec{r},\vec{\rho},\omega)$ = generalization of matrix of frequency response functions

$\vec{r}$ and $\vec{\rho}$ are position vectors

$h(\vec{r},\vec{\rho},t)$ describes the motion at location $\vec{r}$ due to a unit impulse applied at location $\vec{\rho}$ at time $t=0$

$H(\vec{r},\vec{\rho},\omega)$ is based on the steady-state response due to a single sinusoidal excitation:
If $A\cdot e^{i\omega t}$ is applied at location $\vec{\rho}$, then the steady-state response at location $\vec{r}$ is $A\cdot H(\vec{r},\vec{\rho},\omega)\cdot e^{i\omega t}$

$h(\vec{r},\vec{\rho},t) = 0$ for either $t<0$ or $\vec{r} \notin R$, $\vec{\rho} \notin R$.
$H(\vec{r},\vec{\rho},\omega) = 0$ for either $\vec{r} \notin R$ or $\vec{\rho} \notin R$

Note that R is an uncountable set (continuous structure)

The above two functions are Fourier transform pairs:

$$\left\{ \begin{aligned} \vec{h}(\vec{r},\vec{\rho},t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\vec{r},\vec{\rho},\omega)\, e^{i\omega t}\, d\omega \qquad (1) \\ \vec{H}(\vec{r},\vec{\rho},\omega) &= \int_{-\infty}^{\infty} h(\vec{r},\vec{\rho},t)\, e^{-i\omega t}\, dt \qquad (2) \end{aligned} \right\}$$

- Denote now the random excitation by $P(\vec{\rho},t)$ and the random response by $W(\vec{r},t)$. Assume that the initial conditions are:

$$W(\vec{r},0) = 0 \quad \text{and} \quad \dot{W}(\vec{r},0) = 0$$

- The response is then obtained as:

$$W(\vec{r},t) = \int_0^t \int_R h(\vec{r},\vec{\rho},t-\tau)\cdot P(\vec{\rho},\tau)\, d\vec{\rho}\, d\tau \qquad (3)$$

- The n-th moment of the response is obtained as:

$$E[W(\vec{r}_1,t_1)\cdots W(\vec{r}_n,t_n)] =$$

$$= \int_0^{t_1}\cdots\int_0^{t_n}\int_R \overset{n\text{-fold}}{\cdots\cdots}\int_R h(\vec{r}_1,\vec{\rho}_1,t_1-\tau_1)\cdots h(\vec{r}_n,\vec{\rho}_n,t_n-\tau_n)\cdot$$

$$\cdot E[P(\vec{\rho}_1,\tau_1)\cdot P(\vec{\rho}_2,\tau_2)\cdots P(\vec{\rho}_n,\tau_n)]\,d\vec{\rho}_1\cdots d\vec{\rho}_n\,d\tau_1\cdots d\tau_n \qquad (4)$$

- The first two moments (mean and autocorrelation) are given by:

$$E[W(\vec{r},t)] = \int_0^t\int_R h(\vec{r},\vec{\rho},t-\tau)\cdot E[P(\vec{\rho},\tau)]\,d\vec{\rho}\,d\tau \qquad (5)$$

$$E[W(\vec{r}_1,t_1)\cdot W(\vec{r}_2,t_2)] = R_{WW}(\vec{r}_1,t_1;\vec{r}_2,t_2) =$$

$$= \int_0^{t_1}\int_0^{t_2}\int_R\int_R h(\vec{r}_1,\vec{\rho}_1,t_1-\tau_1)\cdot h(\vec{r}_2,\vec{\rho}_2,t_2-\tau_2)\cdot \underbrace{E[P(\vec{\rho}_1,\tau_1)\cdot P(\vec{\rho}_2,\tau_2)]}_{R_{PP}(\vec{\rho}_1,\tau_1;\vec{\rho}_2,\tau_2)}\cdot$$

$$\cdot\, d\vec{\rho}_1\,d\vec{\rho}_2\,d\tau_1\,d\tau_2 \qquad (6)$$

- Assuming now that the excitation is stationary:

$$R_{PP}(\vec{\rho}_1,\tau_1;\vec{\rho}_2,\tau_2) = R_{PP}(\vec{\rho}_1,\vec{\rho}_2;\tau_1-\tau_2) = R_{PP}(\vec{\rho}_1,\vec{\rho}_2,\tau) \qquad (7)$$

the autocorrelation of the response is given by:

$$R_{WW}(\vec{r}_1,t_1;\vec{r}_2,t_2) = \int_0^{t_1}\int_0^{t_2}\int_R\int_R h(\vec{r}_1,\vec{\rho}_1,t_1-\tau_1)\cdot h(\vec{r}_2,\vec{\rho}_2,t_2-\tau_2)\cdot$$

$$\cdot R_{PP}(\vec{\rho}_1,\vec{\rho}_2;\tau_1-\tau_2)\,d\vec{\rho}_1\,d\vec{\rho}_2\,d\tau_1\,d\tau_2 \qquad (8)$$

- Define now:

$$\mathcal{H}(\vec{r},\vec{\rho},\omega,t) = \int_0^t h(\vec{r},\vec{\rho};u)\,e^{-i\omega u}\,du = \int_{-\infty}^t h(\vec{r},\vec{\rho},u)\cdot e^{-i\omega u}\,du \qquad (9)$$

- Note that: $\lim_{t\to\infty}\mathcal{H}(\vec{r},\vec{\rho},\omega,t) = H(\vec{r},\vec{\rho},\omega)$ (10)

- Considering now that $t_1 \to \infty$, $t_2 \to \infty$ but $t_1 - t_2 = $ finite, Eq. 8 can be written after some algebra as:

$$R_{ww}(\vec{r}_1, \vec{r}_2; t_1 - t_2) = \int_{-\infty}^{\infty} \iint_{R\,R} S_{pp}(\vec{\rho}_1, \vec{\rho}_2; \omega) \cdot H(\vec{r}_1, \vec{\rho}_1, \omega) \cdot H^*(\vec{r}_2, \vec{\rho}_2, \omega) \cdot e^{i\omega(t_1 - t_2)}\, d\vec{\rho}_1\, d\vec{\rho}_2\, d\omega \qquad (11)$$

where <u>the cross-spectral density of the input</u> $S_{pp}(\vec{\rho}_1, \vec{\rho}_2; \omega)$ is defined as:

$$\left\{ \begin{aligned} S_{pp}(\vec{\rho}_1, \vec{\rho}_2; \omega) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} R_{pp}(\vec{\rho}_1, \vec{\rho}_2; \tau) \cdot e^{-i\omega\tau}\, d\tau \qquad (12) \\ R_{pp}(\vec{\rho}_1, \vec{\rho}_2; \tau) &= \int_{-\infty}^{\infty} S_{pp}(\vec{\rho}_1, \vec{\rho}_2; \omega) \cdot e^{+i\omega\tau}\, d\omega \qquad (13) \end{aligned} \right\}$$

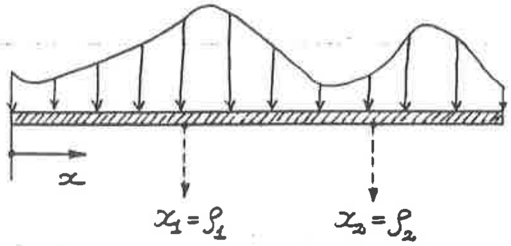
- <u>The cross-spectral density of the response</u> $S_{ww}(\vec{r}_1, \vec{r}_2; \omega)$ is defined as:

$$\left\{ \begin{aligned} S_{ww}(\vec{r}_1, \vec{r}_2; \omega) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} R_{ww}(\vec{r}_1, \vec{r}_2; \tau)\, e^{-i\omega\tau}\, d\tau \qquad (14) \\ R_{ww}(\vec{r}_1, \vec{r}_2; \tau) &= \int_{-\infty}^{\infty} S_{ww}(\vec{r}_1, \vec{r}_2; \omega)\, e^{+i\omega\tau}\, d\omega \qquad (15) \end{aligned} \right\}$$

- Finally, combining Eqs. 11 and 15, the cross-spectral density of the response can be written as:

$$\boxed{S_{ww}(\vec{r}_1, \vec{r}_2; \omega) = \iint_{R\,R} S_{pp}(\vec{\rho}_1, \vec{\rho}_2; \omega) \cdot H(\vec{r}_1, \vec{\rho}_1; \omega) \cdot H^*(\vec{r}_2, \vec{\rho}_2; \omega)\, d\vec{\rho}_1\, d\vec{\rho}_2} \qquad (16)$$

◦ Example:



$S_{pp}(x_1, x_2; \omega) =$ given

• For many practical applications the forcing function can be assumed to be a homogeneous random field in space:

$$\left\{ \begin{array}{l} S_{pp}(\vec{\rho}_1, \vec{\rho}_2; \omega) = S_{pp}(\vec{\rho}_1 - \vec{\rho}_2; \omega) \\ R_{pp}(\vec{\rho}_1, \vec{\rho}_2; \tau) = R_{pp}(\vec{\rho}_1 - \vec{\rho}_2; \tau) \end{array} \right\} \quad \begin{array}{r} (17) \\ (18) \end{array}$$

◦ Setting now: $\vec{\rho}_1 - \vec{\rho}_2 = \vec{\rho}$, the following relations are obtained

$$\left\{ \begin{array}{l} S_{pp}(\vec{\rho}, \omega) = \int \Psi_{pp}(\vec{k}, \omega)\, e^{i\vec{k}\cdot\vec{\rho}}\, d\vec{k} \\ \Psi_{pp}(\vec{k}, \omega) = \frac{1}{(2\pi)^\alpha} \int S_{pp}(\vec{\rho}, \omega) \cdot e^{-i\vec{k}\cdot\vec{\rho}}\, d\vec{\rho} \end{array} \right\} \quad \begin{array}{r} (19) \\ (20) \end{array}$$

where: $\vec{k}$ = wavenumber vector
α = dimension of $\vec{k}$ and $\vec{\rho}$

◦ $\Psi_{pp}(\vec{k}, \omega)$ is the so-called frequency-wave number spectrum that describes the energy distribution in both the frequency and wave number domains

3-D Case:

$$\vec{\rho}_1 = \vec{i}\cdot x_1 + \vec{j}\cdot y_1 + \vec{k}\cdot z_1 \quad ; \quad \vec{k} = \vec{i}\cdot k_1 + \vec{j}\, k_2 + \vec{k}\cdot k_3$$

$$\vec{\rho}_1 - \vec{\rho}_2 = \vec{\rho} = \vec{i}\cdot\xi + \vec{j}\cdot\eta + \vec{k}\cdot\sigma$$

where: $\xi = x_1 - x_2$, $\eta = y_1 - y_2$, $\sigma = z_1 - z_2$

$$\left\{ \begin{aligned} S_{pp}(\xi,\eta,\sigma;\omega) &= \iiint \Psi_{pp}(k_1,k_2,k_3;\omega)\, e^{i(k_1\xi + k_2\eta + k_3\sigma)}\, dk_1 dk_2 dk_3 \\ \Psi_{pp}(k_1,k_2,k_3;\omega) &= \frac{1}{(2\pi)^3}\iiint S_{pp}(\xi,\eta,\sigma;\omega)\cdot e^{-i(k_1\xi + k_2\eta + k_3\sigma)}\, d\xi\, d\eta\, d\sigma \end{aligned} \right\} \quad \begin{aligned} &(22 \\ &(22 \end{aligned}$$

2-D Case:

$$\vec{\rho}_1 = \vec{i}\cdot x_1 + \vec{j}\cdot y_1 \; ; \quad \vec{k} = \vec{i}\,k_1 + \vec{j}\,k_2 \quad ; \quad \vec{\rho} = \vec{i}\cdot\xi + \vec{j}\cdot\eta$$

where: $\xi = x_1 - x_2$, $\eta = y_1 - y_2$

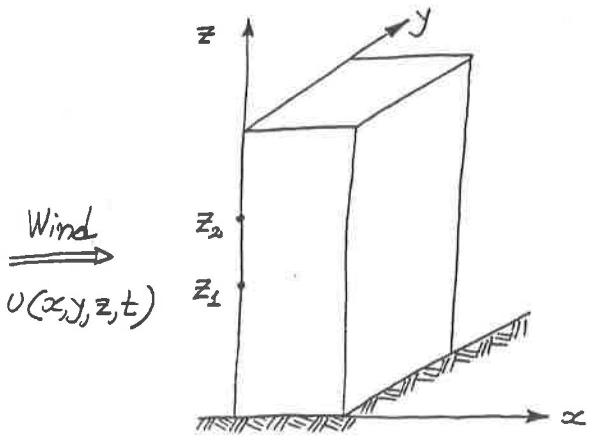
$$\left\{ \begin{aligned} S_{pp}(\xi,\eta;\omega) &= \iint \Psi_{pp}(k_1,k_2;\omega)\, e^{i(k_1\xi + k_2\eta)}\, dk_1 dk_2 \\ \Psi_{pp}(k_1,k_2;\omega) &= \frac{1}{(2\pi)^2}\iint S_{pp}(\xi,\eta;\omega)\, e^{-i(k_1\xi + k_2\eta)}\, d\xi\, d\eta \end{aligned} \right\} \quad \begin{aligned} &(23) \\ &(24) \end{aligned}$$

1-D Case:

$$\left\{ \begin{aligned} S_{pp}(\xi;\omega) &= \int \Psi_{pp}(k;\omega)\, e^{ik\xi}\, dk \\ \Psi_{pp}(k;\omega) &= \frac{1}{2\pi}\int S_{pp}(\xi;\omega)\, e^{-ik\xi}\, d\xi \end{aligned} \right\} \quad \begin{aligned} &(25) \\ &(26) \end{aligned}$$

• Typical Examples of Frequency- Wave Number Spectra

1) Wind Velocity Spectrum:



Assume wind distribution is uniform in the y-direction.

$$S_U(\omega,\xi) = \frac{\tilde{k}\cdot\Phi^2\cdot|\omega|}{2\pi^2\left[1+\left(\frac{\Phi\omega}{2\bar{u}}\right)^2\right]^{4/3}}\cdot e^{-\alpha\cdot|\omega|\cdot|\xi|}$$

where: $\xi = z_1 - z_2$

$\Phi \approx 4{,}000$ ft scale of turbulence

$\tilde{k} \approx 0.01 - 0.04$ surface drag coefficient

$\alpha \approx 0.02$ ft/sec

$\bar{u}$ = mean value of wind

Taking the Fourier transform of $S_U(\omega,\xi)$ with respect to ξ, we compute the frequency - wave number spectrum as:

$$\tilde{S}_U(\omega,k) = \frac{\tilde{k}\cdot\Phi^2\cdot|\omega|}{2\pi^2\left[1+\left(\frac{\Phi\cdot\omega}{2\bar{u}}\right)^2\right]^{4/3}}\cdot\frac{\alpha\cdot|\omega|}{\pi\cdot[\alpha^2\omega^2+k^2]}$$

$S_U(\omega,\xi)$ and $\tilde{S}_U(\omega,k)$ obtained above are good for infinite or semi-infinite problems.

2) <u>Turbulent Boundary Layer Pressure:</u>

$$S_{pp}(\xi,\eta;\omega) = R_x(\xi,\omega)\cdot R_Y(\eta,\omega)\cdot \bar{S}_{pp}(\omega)$$

<u>For subsonic flow:</u>

$$\left\{ \begin{array}{l} R_x(\xi,\omega) = e^{-0.1\cdot\omega\cdot|\xi|/u_c}\cdot\cos\left(\frac{\omega\cdot\xi}{u_c}\right) \\ R_Y(\eta,\omega) = e^{-0.715\cdot\omega\cdot|\eta|/u_c} \end{array} \right\}$$

where: u_c = convection velocity $\approx 0.6\cdot u_\infty$ (u_∞ = free-stream velocity)

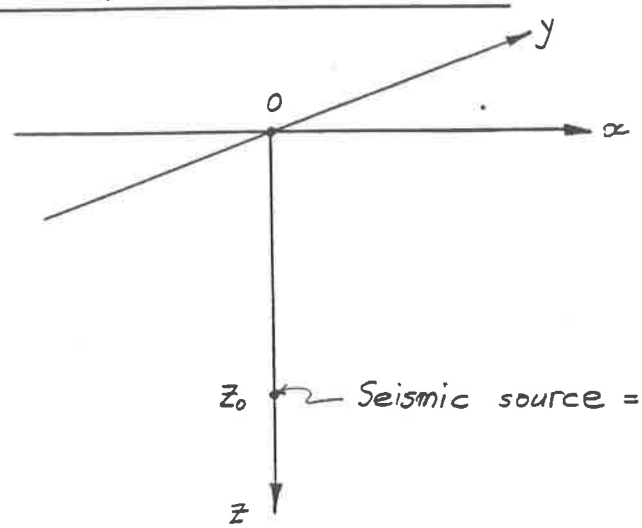
$$\bar{S}_{pp}(\omega) = \frac{q^2\cdot y}{u_\infty}\left[3.7\cdot e^{-2\bar{\omega}} + 0.8\cdot e^{-0.47\cdot\bar{\omega}} - 3.4\cdot e^{-8\cdot\bar{\omega}}\right]\cdot 10^{-5}$$

where: $\bar{\omega} = \frac{\omega\cdot y}{u_\infty}$, $q = \frac{1}{2y}\cdot u_\infty^2$; $y = 0.15$

Taking the Fourier transform of $S_{pp}(\xi,\eta;\omega)$ with respect to ξ and η, we compute the frequency-wave number spectrum as:

$$\tilde{S}_{pp}(k_1,k_2,\omega) = \frac{\left(\frac{\omega}{u_c}\right)^2\cdot\bar{S}_{pp}(\omega)}{\pi^2\cdot\left[\left(0.1\cdot\frac{\omega}{u_c}\right)^2+\left(\frac{\omega}{u_c}+k_1\right)^2\right]\cdot\left[\left(0.715\cdot\frac{\omega}{u_c}\right)^2+k_2^2\right]}$$

3) Seismic Ground Motion:



(x,y) plane = ground surface

Seismic source = point source described by a double couple

Displacements are measured at the ground surface.

$$S_{u_x u_x}(k_x, k_y, \omega) = \left| M_0 \cdot \bar{m}(\omega) \cdot \tilde{G}_x(k_x, k_y, \omega) \right|^2$$

$$\bar{m}(\omega) = \frac{2 \cdot \sin^2 \frac{\omega t^*}{2}}{t^* \omega^2} - i \cdot \frac{\sin \omega t^*}{t^* \cdot \omega^2}$$

$$\tilde{G}_x(\kappa_x, \kappa_y, \omega) = -\frac{1}{8\mu\pi^3} \cdot \frac{\left[4\kappa_x^2\, \gamma\, \nu\, e^{-i\nu z_0} + 4\kappa_y^2\, \gamma\, \nu\, e^{-i\gamma z_0} - 4\kappa^2\gamma^2 e^{-i\gamma z_0} + \kappa_\beta^4\, e^{-i\gamma z_0}\right]}{\left[4\kappa^2\gamma\, \nu + (\kappa_\beta^2 - 2\kappa^2)^2\right]}$$

where: M_0 = seismic moment
t^* = rise time
μ = rigidity of ground
z_0 = depth of seismic source

$$k_\alpha = \frac{\omega}{\alpha} \; ; \; k_\beta = \frac{\omega}{\beta} \; ; \; \nu = \sqrt{k_\alpha^2 - k^2} \; ; \; \gamma = \sqrt{k_\beta^2 - k^2} \; ; \; k = \sqrt{k_x^2 + k_y^2}$$

α, β = complex P-wave and S-wave velocities

- <u>Normal Mode Approach</u>

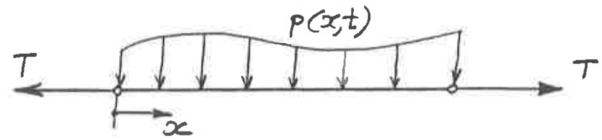
The impulse influence function $h(\vec{r},\vec{\rho},t)$ or the frequency influenc function $H(\vec{r},\vec{\rho},\omega)$ of the linear continuous structure have to be determined to calculate the cross-spectral density of the response

$$S_{WW}(\vec{r}_1,\vec{r}_2;\omega) = \iint_{R\,R} S_{PP}(\vec{\rho}_1,\vec{\rho}_2;\omega)\cdot H(\vec{r}_1,\vec{\rho}_1;\omega)\cdot H^*(\vec{r}_2,\vec{\rho}_2;\omega)\, d\vec{\rho}_1\, d\vec{\rho}_2$$

<u>Let the equation of motion of the linear system be:</u>

$$\mathcal{L}[W] + c\cdot\dot{W} + m_s\,\ddot{W} = p(\vec{\rho},t) \qquad (1)$$

where: $\mathcal{L}$ = linear differential operator (structural operator)
m_s, c = functions of spatial coordinates in general
$p(\vec{\rho},t)$ = random dynamic loading



$$\mathcal{L} = \begin{cases} -T\cdot\dfrac{\partial^2}{\partial x^2} & \text{(taut string)} \\[2ex] EI\cdot\dfrac{\partial^4}{\partial x^4} & \text{(beam)} \quad EI = \text{bending rigidity} \\[2ex] D\left(\dfrac{\partial^4}{\partial x^4} + \dfrac{2\partial^4}{\partial x^2\partial y^2} + \dfrac{\partial^4}{\partial y^4}\right) & \text{(plate)} \quad D = \dfrac{E\,t^3}{12\cdot(1-\nu^2)} \end{cases}$$

- <u>The impulse influence function</u> $h(\vec{r},\vec{\rho},t)$ <u>satisfies:</u>

$$\mathcal{L}[h] + c\cdot\dot{h} + m_s\cdot\ddot{h} = \underbrace{\delta(\vec{r}-\vec{\rho})\cdot\delta(t)}_{\substack{\text{unit impulse at } \vec{r}=\vec{\rho} \\ \text{and at } t=0}} \qquad (2)$$

- Try a solution of Eq. 2 of the form:

$$h(\vec{r},\vec{\rho},t) = \sum_{m=1}^{\infty} a_m(\vec{\rho},t)\cdot Y_m(\vec{r}) \qquad (3)$$

- Substituting Eq. 3 into Eq. 2, we obtain:

$$\sum_{m=1}^{\infty} a_m \cdot \mathcal{L}[Y_m] + c\cdot \sum_{m=1}^{\infty} \dot{a}_m \cdot Y_m + m_s \cdot \sum_{m=1}^{\infty} \ddot{a}_m \cdot Y_m = \delta(\vec{r}-\vec{\rho})\cdot\delta(t) \qquad (4)$$

- If Y_m are the normal modes of undamped free vibrations, then they satisfy the equation:

$$m_s \cdot \ddot{b}_m \cdot Y_m + b_m \cdot \mathcal{L}[Y_m] = 0 \qquad (5)$$

- But we know that in undamped free vibrations, each mode is simple harmonic:

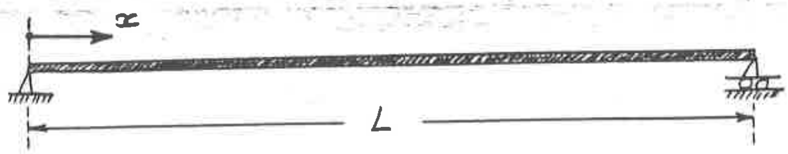
$$\ddot{b}_m = -\omega_m^2 \cdot b_m \qquad (6)$$

- Combining now Eqs. 5 and 6 we obtain:

$$m_s \cdot \omega_m^2 \cdot Y_m = \mathcal{L}[Y_m] \qquad (7)$$

- The natural frequencies of the system can be calculated from Eq. 7.

- Example: simply-supported beam



$\mathcal{L} = E \cdot I \cdot \frac{\partial^4}{\partial x^4}$ <u>Natural Modes:</u> $Y_m(x) = \sin \frac{m\pi x}{L}$

Using Eq. 7, the natural frequencies are obtained as:

$$m_s \cdot \omega_m^2 \cdot Y_m = EI \cdot \left(\frac{m\pi}{L}\right)^4 \cdot Y_m \Rightarrow \omega_m = \sqrt{\frac{EI \cdot \lambda_m}{m_s}}$$

$$\text{where: } \lambda_m = \left(\frac{m\pi}{L}\right)^4$$

- <u>Orthogonality of Normal Modes:</u>

$$\int_R m_s \cdot Y_m(\vec{r})\, Y_n(\vec{r})\, d\vec{r} = \begin{cases} M_m & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases} \tag{8}$$

where M_m = <u>generalized mass</u>

$$\int_R c \cdot Y_m(\vec{r}) \cdot Y_n(\vec{r})\, d\vec{r} = \begin{cases} C_m & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases} \tag{9}$$

where C_m = <u>generalized damping</u>

- <u>Combining now Eqs. 4, 5, 8 and 9, the following expression is obtained:</u>

$$M_m \cdot \ddot{a}_m + C_m \dot{a}_m + M_m \omega_m^2 a_m = Y_m(\vec{\rho}) \cdot \delta(t) \tag{10}$$

In deriving Eq. 10, the following property of Dirac's delta function was used:

$$\int_R \delta(\vec{r}-\vec{\rho})\, F(\vec{r})\, d\vec{r} = F(\vec{\rho})$$

- Assuming now that $a_m = 0$ for $t<0$, the solution of Eq. 10 is obtained as:

$$a_m = \begin{cases} \dfrac{Y_m(\vec{\rho})}{M_m \cdot \omega_m^d} \cdot \exp(-\zeta_m \omega_m \cdot t) \cdot \sin(\omega_m^d \cdot t)\ ; & \vec{\rho} \in R,\ t \geq 0 \\ 0 & \vec{\rho} \notin R \text{ or } t<0 \end{cases} \qquad (11)$$

where:

$$\zeta_m = \frac{C_m}{2 M_m \cdot \omega_m} = \frac{C_m}{C_{cr.}} \quad ; \quad \omega_d^m = \omega_m \cdot \sqrt{1-\zeta_m^2}$$

ζ_m = <u>modal damping ratio</u>

- <u>Eq. 11 is usually written as:</u>

$$a_m = \begin{cases} Y_m(\vec{\rho}) \cdot h_m(t) & \vec{\rho} \in R \\ 0 & \vec{\rho} \notin R \end{cases} \qquad (12)$$

where:

$$h_m(t) = \frac{1}{M_m \cdot \omega_m^d} \cdot \exp(-\zeta_m \cdot \omega_m \cdot t) \cdot \sin(\omega_m^d \cdot t) \qquad (13)$$

$h_m(t)$ is the <u>impulse response function of the m^{th} mode</u>

- Note that $h_m(t)$ has the same form as the impulse response function of the single-degree-of-freedom system.

- Combining now Eqs. 3 and 12, the impulse influence function is obtained as:

$$h(\vec{r},\vec{\rho},t)=\begin{cases}\sum_{m=1}^{\infty} Y_m(\vec{r})\,Y_m(\vec{\rho})\cdot h_m(t) & \vec{\rho},\vec{r}\in R,\ t\geq 0\\ 0 & \vec{\rho},\vec{r}\notin R,\ t<0\end{cases} \qquad (14)$$

- The frequency influence function is obtained by taking the Fourier transform of $h(\vec{r},\vec{\rho},t)$ with respect to time:

$$H(\vec{r},\vec{\rho},\omega)=\int_{-\infty}^{\infty} h(\vec{r},\vec{\rho},t)\cdot e^{-i\omega t}\,dt=\begin{cases}\sum_{m=1}^{\infty} Y_m(\vec{r})\,Y_m(\vec{\rho})\cdot H_m(\omega) & \vec{\rho},\vec{r}\in R,\ t\geq 0\\ 0 & \vec{\rho},\vec{r}\notin R,\ t<0\end{cases} \qquad (15)$$

where:

$$\boxed{H_m(\omega)=\frac{1}{M_m\left[\omega_m^2-\omega^2+2i\,\omega_m\cdot\omega\cdot\zeta_m\right]}} \qquad (16)$$

$H_m(\omega)$ = frequency response function of the m^{th} mode

- Consider now that the excitation $P(\vec{\rho},t)$ is a stationary random process. The correlation function of the response has been calculated as:

$$R_{WW}(\vec{r}_1, \vec{r}_2 ; t_1 - t_2) = \int_{-\infty}^{\infty} \iint_{R\,R} S_{PP}(\vec{\rho}_1, \vec{\rho}_2 ; \omega) \cdot H(\vec{r}_1, \vec{\rho}_1, \omega) H^*(\vec{r}_2, \vec{\rho}_2, \omega) \cdot$$

$$\cdot e^{i\omega(t_1 - t_2)} \, d\vec{\rho}_1 \, d\vec{\rho}_2 \, d\omega \qquad (17)$$

- Combining now Eqs. 15 and 17, $R_{WW}(\vec{r}_1, \vec{r}_2 ; \tau)$ is calculated as:

$$R_{WW}(\vec{r}_1, \vec{r}_2, \tau) = \int_{-\infty}^{\infty} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} Y_m(\vec{r}_1) \cdot Y_n(\vec{r}_2) \cdot H_m(\omega) \cdot H_n^*(\omega) \cdot I_{mn}(\omega) \cdot e^{i\omega\tau} d\omega \qquad (18)$$

where:

$$\boxed{I_{mn}(\omega) = \iint_{R\,R} S_{PP}(\vec{\rho}_1, \vec{\rho}_2, \omega) \cdot Y_m(\vec{\rho}_1) \, Y_n(\vec{\rho}_2) \, d\vec{\rho}_1 \, d\vec{\rho}_2} \qquad (19)$$

$I_{mn}(\omega)$ is the cross-spectral density of the generalized forces in the modes m and n

- Since: $R_{WW}(\vec{r}_1, \vec{r}_2, \tau) = \int_{-\infty}^{\infty} S_{WW}(\vec{r}_1, \vec{r}_2, \omega) \cdot e^{i\omega\tau} d\omega$, the response cross-spectral density $S_{WW}(\vec{r}_1, \vec{r}_2, \omega)$ is given by:

$$S_{WW}(\vec{r}_1, \vec{r}_2 ; \omega) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} Y_m(\vec{r}_1) \, Y_n(\vec{r}_2) \, H_m(\omega) \, H_n^*(\omega) \cdot I_{mn}(\omega) \qquad (20)$$

- The response spectral density is obtained by setting: $\vec{r}_1 = \vec{r}_2 = \vec{r}$

$$\boxed{S_{WW}(\vec{r}, \omega) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} Y_m(\vec{r}) \cdot Y_n(\vec{r}) \cdot H_m(\omega) \cdot H_n^*(\omega) \cdot I_{mn}(\omega)} \qquad (21)$$

- Note that $S_{ww}(\vec{r},\omega)$ is <u>real</u> since $I_{mn}(\omega)$ is <u>Hermitian:</u>

$$I_{mn}(\omega) = I^{*}_{nm}(\omega) \tag{22}$$

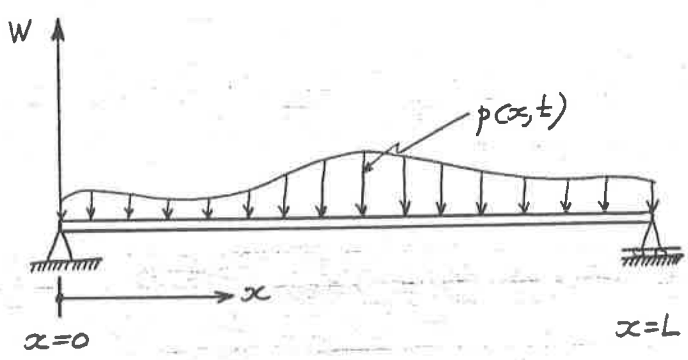
- <u>The double integration appearing in Eq. 21 is really:</u>

<u>3-D problems:</u> $\sum_{m_1}\sum_{m_2}\sum_{m_3}\sum_{n_1}\sum_{n_2}\sum_{n_3}$

<u>2-D problems:</u> $\sum_{m_1}\sum_{m_2}\sum_{n_1}\sum_{n_2}$

<u>1-D problems:</u> $\sum_{m}\sum_{n}$

- <u>Example</u> : <u>simply-supported beam</u>



<u>Normal Modes:</u>

$$Y_m(x) = \sin\frac{m\pi x}{L}$$

<u>The equation of motion is :</u>

$$EI\cdot\frac{\partial^4 w}{\partial x^4} + c\cdot\dot{w} + m_s\ddot{w} = p(x,t)$$

with : $\mathcal{L} = EI\cdot\frac{\partial^4}{\partial x^4}$

1) Calculate the natural frequencies using Eq. 7:

$$m_s \cdot \omega_m^2 \cdot Y_m = \mathcal{L}[Y_m] \Rightarrow m_s \cdot \omega_m^2 \cdot \cancel{Y_m} = EI \cdot \left(\frac{m\pi}{L}\right)^4 \cdot \cancel{Y_m} \Rightarrow$$

$$\Rightarrow \omega_m = \left(\frac{m\pi}{L}\right)^2 \sqrt{\frac{EI}{m_s}}$$

2) Calculate generalized mass:

$$M_m = \int_0^L m_s \cdot Y_m(x) \cdot Y_n(x)\, dx = m_s \int_0^L Y_m^2(x)\, dx = m_s \cdot \frac{L}{2}$$

3) Calculate generalized damping:

$$C_m = \int_0^L c \cdot Y_m(x) \cdot Y_n(x)\, dx = c \cdot \int_0^L Y_m^2(x)\, dx = c \cdot \frac{L}{2}$$

4) Frequency influence function:

$$H(\underset{\substack{\downarrow \\ F}}{x}, \underset{\substack{\downarrow \\ \xi}}{\xi}, \omega) = \begin{cases} \sum_{m=1}^{\infty} \sin\frac{m\pi x}{L} \cdot \sin\frac{m\pi\xi}{L} \cdot H_m(\omega) & \text{for } 0 \leq x, \xi \leq L \\ 0 & \text{otherwise} \end{cases}$$

where:

$$H_m(\omega) = \frac{1}{M_m \cdot [\omega_m^2 - \omega^2 + 2i\zeta_m \cdot \omega_m \cdot \omega]} = \frac{2/L}{EI\left(\frac{m\pi}{L}\right)^4 - \omega^2 m_s + ic\omega} \; ; \; \zeta_m = \frac{C_m}{2 \cdot M_m}$$

- Assume now that the excitation $P(x,t)$ is white-noise with respect to time separation and exponentially decaying with respect to spatial separation:

$$R_{pp}(\xi, \tau) = \sigma^2 \cdot e^{-a \cdot |\xi|} \cdot \delta(\tau) \qquad \tau = t_1 - t_2$$
$$\xi = x_1 - x_2$$

Therefore, the cross-spectral density of $P(x,t)$ is:

$$S_{pp}(\xi, \omega) = \frac{\sigma^2}{2\pi} \cdot e^{-a \cdot |x_1 - x_2|} = \frac{\sigma^2}{2\pi} \cdot e^{-a \cdot |\xi|}$$

Note that $S_{pp}(\xi, \omega)$ is obtained by taking the Fourier transform of $R_{pp}(\xi, \tau)$ with respect to τ.

5) Cross-spectral density of generalized forces:

$$I_{mn}(\omega) = \int_0^L \int_0^L S_{pp}(x_1 - x_2, \omega) \cdot Y_m(x_1) \cdot Y_n(x_2)\, dx_1\, dx_2 =$$

$$= \int_0^L \int_0^L \frac{\sigma^2}{2\pi} \cdot e^{-a|x_1 - x_2|} \cdot \sin\frac{m\pi x_1}{L} \cdot \sin\frac{n\pi x_2}{L}\, dx_1\, dx_2 =$$

$$= \frac{\sigma^2}{2\pi} \cdot \left\{ \frac{a \cdot L}{2} \left[\frac{1}{a^2 + \left(\frac{m\pi}{L}\right)^2} + \frac{1}{a^2 + \left(\frac{n\pi}{L}\right)^2} \right] \cdot \delta_{mn} + \frac{m\pi}{L} \cdot \frac{n\pi}{L} \cdot \frac{2 + e^{-a \cdot L} \cdot \left[(-1)^{m+1} + (-1)^{n+1}\right]}{\left[a^2 + \left(\frac{m\pi}{L}\right)^2\right] \cdot \left[a^2 + \left(\frac{n\pi}{L}\right)^2\right]} \right\}$$ independent of ω!

6) Response spectral density:

$$S_{ww}(x, \omega) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin\frac{m\pi x}{L} \cdot \sin\frac{n\pi x}{L} \cdot H_m(\omega) \cdot H_n^*(\omega) \cdot I_{mn}(\omega) =$$

$$= \sum_{m=1}^{\infty} \left(\sin \frac{m\pi x}{L}\right)^2 \cdot |H_m(\omega)|^2 \cdot I_{mm}(\omega) + \sum_{\substack{m=1 \\ m \neq n}}^{\infty} \sum_{n=1}^{\infty} \sin \frac{m\pi x}{L} \cdot \sin \frac{n\pi x}{L} \cdot$$

$$\cdot H_m(\omega) \cdot H_n^*(\omega) \cdot I_{mn}(\omega)$$

Usually the cross terms are neglected to obtain:

$$S_{WW}(x,\omega) \cong \sum_{m=1}^{\infty} \sin^2 \frac{m\pi x}{L} \cdot |H_m(\omega)|^2 \cdot I_{mm}(\omega)$$

with:

$$|H_m(\omega)|^2 = \frac{4}{m_s^2 \cdot L^2 \cdot [(\omega_m^2 - \omega^2)^2 + 4 \cdot \zeta_m^2 \cdot \omega_m^2 \cdot \omega^2]}$$

$$I_{mm}(\omega) = \frac{\sigma^2}{2\pi} \cdot \left[\frac{a \cdot L}{2} \cdot \frac{2}{a^2 + \left(\frac{m\pi}{L}\right)^2} + \left(\frac{m\pi}{L}\right)^2 \cdot \frac{2 + 2 \cdot (-1)^{m+1} \cdot e^{-a \cdot L}}{\left[a^2 + \left(\frac{m\pi}{L}\right)^2\right]^2}\right]$$

7) Mean-square of the response deflection:

$$E[W^2(x)] = \int_{-\infty}^{\infty} S_{WW}(x,\omega)\, d\omega$$

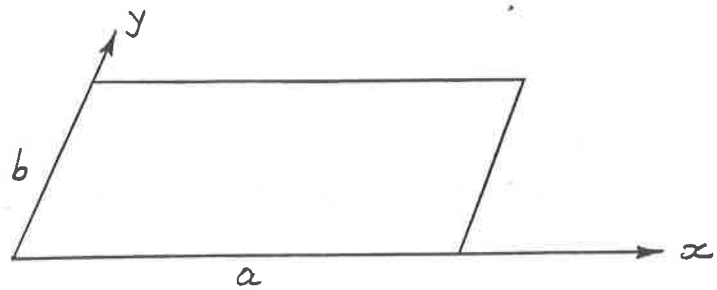
Substituting $S_{WW}(x,\omega)$ into the above expression:

$$E[W^2(x)] = \frac{\sigma^2}{2\pi} \cdot \sum_{m=1}^{\infty} \sin^2 \frac{m\pi x}{L} \cdot \left\{a \cdot L \cdot \left[\frac{1}{a^2 + \left(\frac{m\pi}{L}\right)^2}\right] + \left(\frac{m\pi}{L}\right)^2 \cdot \frac{2 + 2 \cdot (-1)^{m+1} \cdot e^{-a \cdot L}}{\left[a^2 + \left(\frac{m\pi}{L}\right)^2\right]^2}\right\} \cdot$$

$$\cdot \int_{-\infty}^{\infty} \frac{4/L^2}{m_s^2 \cdot [(\omega_m^2 - \omega^2)^2 + 4\zeta_m^2 \cdot \omega_m^2 \cdot \omega^2]}\, d\omega$$

- **Plates**

Rectangular



Normal Modes for the simply-supported case:

$$Y_{mn}(x,y) = \sin\frac{m\pi x}{a} \cdot \sin\frac{n\pi y}{b}$$

The corresponding natural frequencies are:

$$\omega_{mn} = \frac{\pi}{2} \cdot \sqrt{\frac{D}{\rho h}} \cdot \left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2 \right] \quad ; \quad D = \frac{E h^3}{12(1-\nu^2)}$$

$$S_{WW}(x,y,\omega) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{r=1}^{\infty}\sum_{s=1}^{\infty} Y_{mr}(x_1,y_1) \cdot Y_{ns}(x_2,y_2) \cdot H_{mr}(\omega) \cdot H^*_{ns}(\omega) \cdot I_{mnrs}(\omega)$$

Neglecting the cross-terms the above fourth-order summation becomes a double summation

7. MONTE CARLO SIMULATION

- The problems solved in the previous sections were linear and elastic. The solution was given in the frequency domain:

1-D.O.F. : $S_{xx}(\omega) = |H(\omega)|^2 \, S_{FF}(\omega)$

n-D.O.F. : $[S_{xx}(\omega)] = [H(\omega)] \cdot [S_{FF}(\omega)] \cdot [H^*(\omega)]^T$

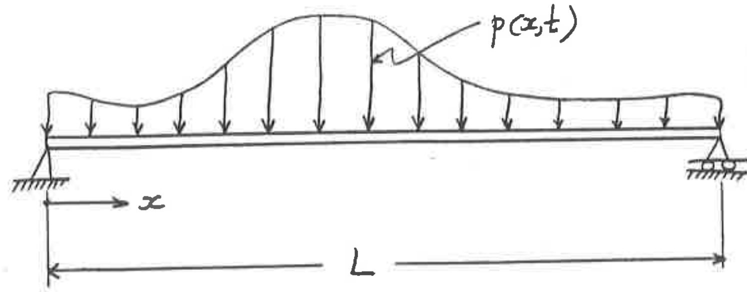
Continuous: $S_{ww}(\vec{r},\omega) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} Y_m(\vec{r}) \cdot Y_n(\vec{r}) \cdot H_m(\omega) \cdot H_n^*(\omega) \cdot I_{mn}(\omega)$

- For nonlinear and nonelastic problems solutions can be always obtained using the Monte Carlo technique. These solutions are obtained in the time-space domain, provided that the solution of the corresponding deterministic problem exists.

- Monte Carlo Simulation :

Consider a structure with random material and geometric properties subjected to random loads. A large number (N) of realizations of all random parameters are generated. The resulting N problems are solved deterministically to obtain N responses. Various statistical properties of the response can be determined from the calculated N responses.

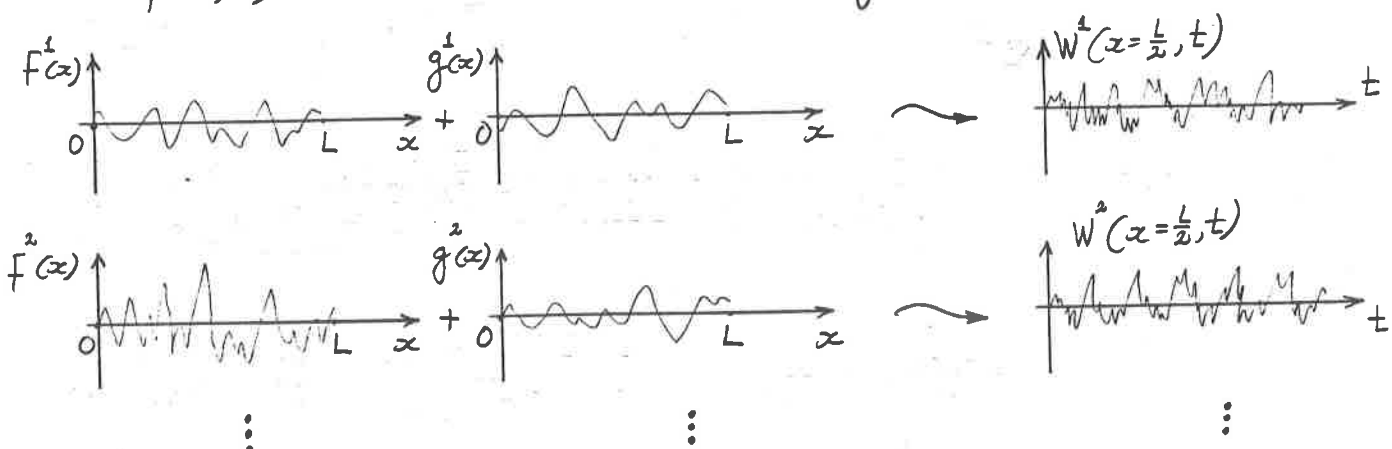
Example: Consider the beam bending problem:



$$\frac{\partial^2}{\partial x^2}\left[EI\frac{\partial^2 w}{\partial x^2}\right]+c\dot{w}+m_s\ddot{w}= = p(x,t)$$

where : $\begin{cases} E = E_0\,[1+f(x)] \\ m_s = m_{s0}\,[1+g(x)] \end{cases}$ ∴ $f(x)$ and $g(x)$ are zero-mean homogeneous stochastic fields with given p.s.d's $S_{ff}(k)$ and $S_{gg}(k)$.

$p(x,t)$ is a deterministic loading



The statistical characteristics of the response can be easily obtained from $W^1, W^2, \dots$

- The most important part of the Monte Carlo simulation technique is to generate sample functions (realizations) of a random process with given power spectral density or autocorrelation function. This generation has to be computationally efficient.

SIMULATION OF ONE-DIMENSIONAL STATIONARY STOCHASTIC PROCESSES USING SPECTRAL REPRESENTATION (Shinozuka and Jan, 1972)-(Shinozuka and Deodatis, 1988-1991)

Consider a 1D stationary stochastic process $f_0(t)$:

(1) $E[f_0(t)] = 0$ mean value

(2) $E[f_0(t+\tau) f_0(t)] = R_{f_0f_0}(\tau)$ autocorrelation function

Power spectral density function $S_{f_0f_0}(\omega)$ and $R_{f_0f_0}(\tau)$ are related through the Weiner-Khintchine transform pair:

(3) $$S_{f_0f_0}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_{f_0f_0}(\tau)e^{-i\omega\tau}d\tau$$

(4) $$R_{f_0f_0}(\tau) = \int_{-\infty}^{\infty} S_{f_0f_0}(\omega)e^{i\omega\tau}d\omega$$

PROBLEM: Given the power spectral density function $S_{f_0f_0}(\omega)$ of a random process with zero mean, generate sample functions of this random process.

Example: Given the power spectral density function of an earthquake, generate sample earthquakes corresponding to the same power spectral density function.

It will be shown that the stochastic process $f_0(t)$ can be simulated by the following series, as $N \to \infty$

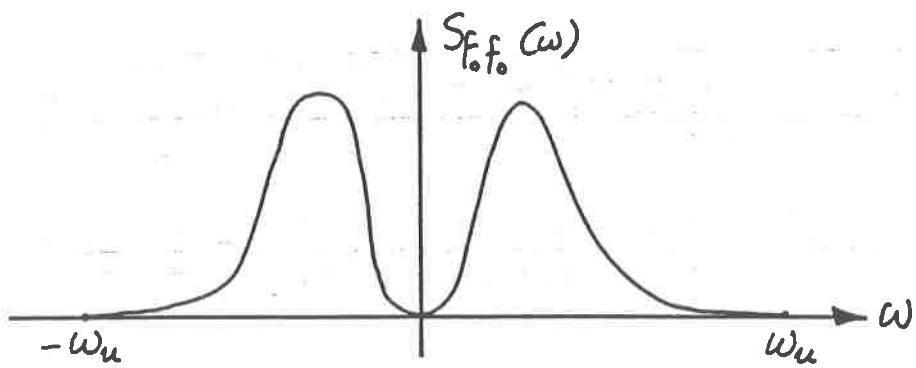
(5) $$f(t) = \sqrt{2} \sum_{j=1}^{N} \sqrt{2S_{f_0 f_0}(\omega_j)\Delta\omega} \cdot \cos(\omega_j t + \phi_j)$$

where

(6) $$\omega_j = j\Delta\omega \qquad j=1,2,\ldots,N$$

and

(7) $$\Delta\omega = \frac{\omega_u}{N} \quad ; \quad \omega_u = \text{upper cutoff frequency}$$



ϕ_j = independent random phase angles uniformly distributed over the range $(0, 2\pi)$

Note that the simulated process is asymptotically Gaussian as N becomes large due to the central limit theorem.

PROOF:

It will be shown now that the expected value and auto-correlation function of the simulated process f(t) are identical to the corresponding targets, $E[f_0(t)] = 0$ and $R_{f_0 f_0}(\tau)$, respectively.

(a) $E[f(t)] = E[f_0(t)] = 0$ has to be shown

(8) $$E[f(t)] = E\left\{\sqrt{2}\sum_{j=1}^{N} A(\omega_j)\cdot\cos(\omega_j t + \varphi_j)\right\}$$

(9) where: $$A(\omega_j) = \sqrt{2 S_{f_0 f_0}(\omega_j)\,\Delta\omega}$$

(10) $$E[f(t)] = \sqrt{2}\sum_{j=1}^{N} A(\omega_j)\, E\{\cos(\omega_j t + \varphi_j)\}$$

But since the φ_j's are uniformly distributed between 0 and 2π:

(11) $$E\{\cos(\omega_j t + \varphi_j)\} = \int_0^{2\pi} p_\varphi \cdot \cos(\omega_j t + \varphi)\, d\varphi =$$
$$= \int_0^{2\pi} \frac{1}{2\pi}\cdot\cos(\omega_j t + \varphi)\, d\varphi = 0$$

Therefore using (10) and (11):

$$E[f(t)] = 0$$

(b) $R_{ff}(\tau) = R_{f_0 f_0}(\tau)$ has to be shown

(12) $$R_{ff}(\tau) = E\{f(t)\,f(t+\tau)\} =$$

$$= 2\sum_{i=1}^{N}\sum_{j=1}^{N} A(\omega_i)\,A(\omega_j)\cdot E\{\underbrace{\cos(\omega_i t+\varphi_i)\cdot\cos(\omega_j t+\omega_j\tau+\varphi_j)}_{\alpha}\}$$

<u>For $i \neq j$:</u> φ_i and φ_j are independent and therefore:

(13) $$E\{\alpha\} = \underbrace{E\{\cos(\omega_i t+\varphi_i)\}}_{=0}\cdot\underbrace{E\{\cos(\omega_j t+\omega_j\tau+\varphi_j)\}}_{=0} = 0$$

<u>For $i=j$:</u>

(14) $$R_{ff}(\tau) = 2\sum_{i=1}^{N} A^2(\omega_i)\,E\{\cos(\omega_i t+\varphi_i)\cdot\cos(\omega_i t+\omega_i\tau+\varphi_i)\}$$

Since: $\cos A\cdot\cos B = \frac{1}{2}\cos(A+B) + \frac{1}{2}\cos(A-B)$

Eq. 14 can be written as:

(15) $$R_{ff}(\tau) = 2\sum_{i=1}^{N} A^2(\omega_i)\cdot\frac{1}{2}\,E\{\cos(2\omega_i t+\omega_i\tau+2\varphi_i)+\cos(\omega_i\tau)\}$$

$$= \sum_{i=1}^{N} A^2(\omega_i)\left[\underbrace{E\{\cos(2\omega_i t+\omega_i\tau+2\varphi_i)\}}_{=0}+E\{\cos(\omega_i\tau)\}\right]$$

$$= \sum_{i=1}^{N} A^2(\omega_i)\cdot\cos(\omega_i\tau)$$

$$= \sum_{i=1}^{N} 2\, S_{f_o f_o}(\omega_i)\, \Delta\omega \cos(\omega_i \tau)$$

As $N \to \infty$, Eq. 15 can be written as:

(16) $$R_{ff}(\tau) = \int_{-\infty}^{\infty} S_{f_o f_o}(\omega) \cos\omega\tau \, d\omega$$

The Wiener-Khintchine theorem states:

(17) $$R_{f_o f_o}(\tau) = \int_{-\infty}^{\infty} S_{f_o f_o}(\omega) \cos\omega\tau \, d\omega$$

Finally, comparing Eqs. 16 and 17:

$$R_{ff}(\tau) = R_{f_o f_o}(\tau)$$

- It has been shown that the stochastic process $f_o(t)$ can be simulated by the following series, as $N \to \infty$:

(18) $$f(t) = \sqrt{2} \sum_{j=1}^{N} \sqrt{2\, S_{f_o f_o}(\omega_j)\, \Delta\omega} \cdot \cos(\omega_j t + \varphi_j)$$

where:

(19) $$\omega_j = j\, \Delta\omega \qquad j = 1, 2, \ldots, N$$

(20) $$\Delta\omega = \frac{\omega_u}{N} \quad ; \quad \omega_u = \text{upper cutoff frequency}$$

Use of Simulation Formula:

φ_j = random numbers uniformly distributed in the range $(0, 2\pi)$ → use uniform $(0,1)$ random number generator and multiply the results by 2π.

ω_u = this value is chosen using engineering judgement or the criterion:

$$2\int_0^{\omega_u} S_{f_0 f_0}(\omega)\, d\omega = 0.99 \cdot \sigma_F^2$$

N = a large number is used, usually $N > 100$

- Usually, the summation of cosines is time-consuming, specially when a large number of sample functions has to be digitally generated.

The speed of simulation can be dramatically increased by using the FFT technique.

In order to take advantage of the FFT algorithm, Eq. 1E has to be written in the following form:

(21) $$f(p\cdot\Delta t) = Re\left\{\sum_{n=0}^{M-1} \sqrt{2}\sqrt{2S_{f_0f_0}(n\Delta\omega)\cdot\Delta\omega}\cdot e^{i\varphi_n}\cdot e^{\frac{inp2\pi}{M}}\right.$$
$$= Re\left\{\sum_{n=0}^{M-1} B_n\cdot e^{\frac{inp2\pi}{M}}\right\} \quad ; \quad p=0,1,\dots,M-1$$

- $\Delta\omega = \dfrac{2\pi}{M\cdot\Delta t}$
- M has to be a power of 2 : $M = 2^m$
- Restriction: $\Delta t < \frac{1}{2}\cdot\frac{2\pi}{\omega_u}$

Note that the number of points where $f(t)$ is calculate is equal to M and equal to the number of points where $S_{f_0f_0}(\omega)$ is also calculated.

- $S_{f_0f_0}(0) = 0$ to have mean zero

SIMULATION OF TWO-DIMENSIONAL HOMOGENEOUS STOCHASTI FIELDS USING SPECTRAL REPRESENTATION (Shinozuka and Jan, 1972) - (Shinozuka and Deodatis, 1988-1991)

Consider a 2D homogeneous stochastic field $f_0(x_1, x_2)$

e.g. see surface elevation, turbulence on wing of aircraft, ground motion over a surface

(22) $$E[f_0(x_1,x_2)] = 0 \qquad \text{mean value}$$

(23) $$E[f_0(x_1+\xi_1, x_2+\xi_2)\, f_0(x_1,x_2)] = R_{f_0f_0}(\xi_1,\xi_2)$$

autocorrelation function

For a homogeneous field, $R_{f_0f_0}(\xi_1,\xi_2)$ is symmetric with respect to the separation vector $\underline{\xi} = [\xi_1, \xi_2]^T$:

(24) $$R_{f_0f_0}(\xi_1,\xi_2) = R_{f_0f_0}(-\xi_1,-\xi_2)$$

or equivalently:

(25) $$R_{f_0f_0}(\underline{\xi}) = R_{f_0f_0}(-\underline{\xi})$$

For some 2D homogeneous fields, the following equation is valid:

(26) $$R_{f_0f_0}(\xi_1,\xi_2) = R_{f_0f_0}(\xi_1,-\xi_2) = R_{f_0f_0}(-\xi_1,\xi_2) = R_{f_0f_0}(-\xi_1,-\xi_2)$$

When Eq. 26 is valid, the 2D homogeneous stochastic field is referred to as a "quadrant field."

The power spectral density function of $f_o(x_1, x_2)$ is defined as:

(27) $$S_{f_of_o}(k_1, k_2) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{f_of_o}(\xi_1, \xi_2)\, e^{-i(k_1\xi_1 + k_2\xi_2)}\, d\xi_1\, d\xi_2$$

and its inverse transform is given by:

(28) $$R_{f_of_o}(\xi_1, \xi_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S_{f_of_o}(k_1, k_2)\, e^{+i(k_1\xi_1 + k_2\xi_2)}\, dk_1\, dk_2$$

where k_1 and k_2 are the wavenumbers in the x_1 and x_2 directions respectively.

Eqs. 27 and 28 represent the 2D version of the Wiener-Khintchine transform pair.

It can be easily shown that the power spectral density function is real and that:

(29) $$S_{f_of_o}(k_1, k_2) = S_{f_of_o}(-k_1, -k_2)$$

If the stochastic field is quadrant, then:

(30) $$S_{f_0f_0}(\kappa_1, \kappa_2) = S_{f_0f_0}(-\kappa_1, \kappa_2) = S_{f_0f_0}(\kappa_1, -\kappa_2) = S_{f_0f_0}(-\kappa_1, -\kappa_2)$$

This equation indicates that the value of the p.s.d. is identical at a corresponding point in each quadrant, hence the name quadrant field.

Finally, it can be shown that:

(31) $$S_{f_0f_0}(\kappa_1, \kappa_2) \geqq 0$$

<u>PROBLEM</u>: Given the power spectral density function $S_{f_0f_0}(\kappa_1, \kappa_2)$ of a 2D homogeneous stochastic field with zero mean, generate sample functions of this stochastic field.

It can be shown that the stochastic field $f_0(x_1, x_2)$ can be digitally simulated by the following series as $N_1 \to \infty$ and $N_2 \to \infty$:

(32) $$f(\underline{x}) = \sqrt{2} \sum_{k_1=1}^{N_1} \sum_{k_2=1}^{N_2} \{[2S_{f_0f_0}(\kappa_{1k_1}, \kappa_{2k_2})\Delta\kappa_1 \Delta\kappa_2]^{\frac{1}{2}} \times \cos(\kappa_{1k_1}x_1 + \kappa_{2k_2}x_2 + \phi^{(1)}_{k_1k_2}) + [2S_{f_0f_0}(\kappa_{1k_1}, -\kappa_{2k_2})\Delta\kappa_1 \Delta\kappa_2]^{\frac{1}{2}} \times \cos(\kappa_{1k_1}x_1 - \kappa_{2k_2}x_2 + \phi^{(2)}_{k_1k_2})\}$$

where:

(33) $$\kappa_{1k_1} = k_1 \Delta\kappa_1 \qquad k_1 = 1, 2, \ldots, N_1$$
(34) $$\kappa_{2k_2} = k_2 \Delta\kappa_2 \qquad k_2 = 1, 2, \ldots, N_2$$

and:

(35) $$\Delta\kappa_1 = \frac{\kappa_{1u}}{N_1} \quad ; \quad \Delta\kappa_2 = \frac{\kappa_{2u}}{N_2}$$

κ_{1u} = upper cutoff wavenumber in the x_1-direction
κ_{2u} = upper cutoff wavenumber in the x_2-direction

Note that $S_{f_0f_0}(\kappa_1, \kappa_2)$ is of insignificant magnitude outside the region defined by:

(36) $$\left\{ \begin{array}{l} -\kappa_{1u} \leq \kappa_1 \leq \kappa_{1u} \\ -\kappa_{2u} \leq \kappa_2 \leq \kappa_{2u} \end{array} \right\}$$

Finally:

$\varphi^{(1)}_{k_1k_2}$ and $\varphi^{(2)}_{k_1k_2}$ are independent random phase angles uniformly distributed between 0 and 2π.

The simulated stochastic field is Gaussian due to the central limit theorem.

For the special case where the stochastic field is quadrant, the simulation formula reduces to:

(37) $$f(x_1,x_2) = \sqrt{2}\sum_{k_1=1}^{N_1}\sum_{k_2=1}^{N_2}[2S_{f_0f_0}(\kappa_{1k_1},\kappa_{2k_2})\Delta\kappa_1\Delta\kappa_2]^{\frac{1}{2}}\cdot$$
$$[\cos(\kappa_{1k_1}x_1+\kappa_{2k_2}x_2+\phi^{(1)}_{k_1k_2})+\cos(\kappa_{1k_1}x_1-\kappa_{2k_2}x_2+\phi^{(2)}_{k_1k_2})]$$

The speed of simulation can be dramatically increased by using the FFT technique. For that reason Eq. 37 has to be written in the following form:

(38) $$f(p\Delta x_1, q\Delta x_2) =$$
$$= Re\left[\sqrt{2}\sum_{k=0}^{M_1-1}\left[\sum_{\ell=0}^{M_2-1} A_{k\ell}\, e^{i\varphi_{k\ell}}\cdot e^{\frac{i\ell q 2\pi}{M_2}}\right]\cdot e^{\frac{ikp2\pi}{M_1}}\right]+$$
$$+ Re\left[\sqrt{2}\sum_{k=0}^{M_1-1}\left[\sum_{\ell=0}^{M_2-1} A_{k\ell}\, e^{i\psi_{k\ell}}\cdot e^{-\frac{i\ell q 2\pi}{M_2}}\right]\cdot e^{\frac{ikp2\pi}{M_1}}\right]$$

where $Re[\cdot]$ represents the real part and:

$$p = 0,1,\dots,M_1-1 \quad ; \quad q = 0,1,\dots,M_2-1$$

$$A_{k\ell} = \sqrt{2S_{f_0f_0}(K_{1k},K_{2\ell})\,\Delta K_1\,\Delta K_2}$$

$$K_{1k} = k\,\Delta K_1 \qquad k = 0, 1, \dots, M_1 - 1$$
$$K_{2\ell} = \ell\,\Delta K_2 \qquad \ell = 0, 1, \dots, M_2 - 1$$

$$\Delta K_1 = \frac{2\pi}{M_1\,\Delta x_1} \quad ; \quad \Delta K_2 = \frac{2\pi}{M_2\,\Delta x_2}$$

- M_1 and M_2 have to be powers of 2 :

$$M_1 = 2^{m_1} \quad \text{and} \quad M_2 = 2^{m_2}$$

- Restrictions : $\left\{ \begin{array}{l} \Delta x_1 < \frac{1}{2}\,\frac{2\pi}{K_{1u}} \\ \Delta x_2 < \frac{1}{2}\,\frac{2\pi}{K_{2u}} \end{array} \right\}$

- $A_{k\ell} = 0$ (if $k=0$ or $\ell=0$) to have mean zero

- The simulated stochastic field is periodic with periods L_{x_1} and L_{x_2} in the two directions x_1 and x_2 respectively :

$$L_{x_1} = M_1\,\Delta x_1 = \frac{2\pi}{\Delta K_1} \quad ; \quad L_{x_2} = M_2\,\Delta x_2 = \frac{2\pi}{\Delta K_2}$$

SIMULATION OF ONE-DIMENSIONAL NON-STATIONARY STOCHASTIC PROCESSES USING SPECTRAL REPRESENTATION

(Shinozuka and Jan, 1972) - (Shinozuka and Deodatis, 1988-1991)

Theory of 1D non-stationary stochastic processes with evolutionary power

Reference: Priestley, M.B., "Evolutionary Spectra and Non-Stationary Processes," Journal of the Royal Statistical Society, Series B, Vol. 27, 1965, pp. 204-237.

If a stochastic process (stationary or non-stationary) can be represented as:

(39) $$y_0(t) = \int_{-\infty}^{\infty} A(t,\omega)\, e^{i\omega t}\, dZ(\omega)$$

where: $A(t,\omega)$ = modulating function
$dZ(\omega)$ = orthogonal increment

then the process $y_0(t)$ is said to be oscillatory.

Note that if $A(t,\omega)$ is constant, $y_0(t)$ is a stationary stochastic process.
The mean square of the oscillatory process is found to be:

(40) $$E\left[y_0^2(t)\right] = \int_{-\infty}^{\infty} |A(t,\omega)|^2\, dF(\omega)$$

The evolutionary power spectrum is introduced as:

(41) $$dF^{\circ}(t,\omega) = |A(t,\omega)|^{2}\, dF(\omega)$$

or:

(42) $$f^{\circ}(t,\omega)\, d\omega = |A(t,\omega)|^{2} f(\omega)\, d\omega$$

if $f(\omega)$ exists such that $dF(\omega) = f(\omega)\,d\omega$, where $dF^{\circ}(t,\omega) = f^{\circ}(t,\omega)\,d\omega$.

In that way the non-stationary spectral contents are defined.

The autocorrelation function is computed from:

(43) $$R_{y_o y_o}(t+\tau, t) = \int_{-\infty}^{\infty} A(t+\tau,\omega)\,\bar{A}(t,\omega)\, e^{i\omega\tau} f(\omega)\, d\omega$$

Note that $R_{y_o y_o}(t+\tau, t)$ is now a function of both t and τ, since the process is non-stationary

A non-stationary stochastic process with evolutionary power can be simulated by the following expression as $N \to \infty$:

(44) $$y(t) = \sqrt{2} \sum_{j=1}^{N} \sqrt{2A^2(t,\omega_j) f(\omega_j) \Delta\omega} \cos(\omega_j t + \varphi_j)$$

where: $\omega_j = j\,\Delta\omega \qquad j = 1, 2, \dots, N$

$\Delta\omega = \dfrac{\omega_u}{N}$

φ_j = random phase angles uniformly distributed over the range $(0, 2\pi)$

Note again that the simulated process is Gaussian.

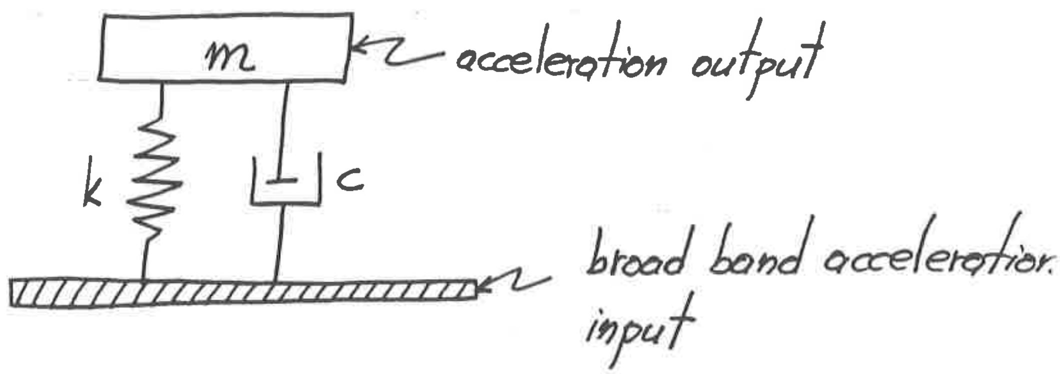
EXAMPLES OF POWER SPECTRAL DENSITY FUNCTIONS

1) The Kanai - Tajimi Spectrum: (1D - stationary)

(45) $$S^{K-T}_{f_0 f_0}(\omega) = \frac{\omega_g^4 + 4\zeta_g^2 \omega_g^2 \omega^2}{(\omega_g^2 - \omega^2)^2 + 4\zeta_g^2 \omega_g^2 \omega^2} \cdot S_0$$

Ground acceleration is idealized as a stationary random process having the above power spectral density function.

This model corresponds to the acceleration of a mass, supported by linear spring and dashpot in parallel as:



The base of the model is undergoing a broad band acceleration.

S_0 = spectrum level (normalized to unit mass) of the broad-band excitation at the base ≡ earthquake magnitude

ω_g = natural frequency of the system ≡ ground resonance frequency

ζ_g = ratio of damping to the critical damping ≡ attenuation of seismic waves in the ground

The main drawback of this model is that the ground motion is considered to be stationary.

2) <u>Non-Stationary Kanai-Tajimi Spectra:</u> (1D)

Kanai-Tajimi Spectra with Evolutionary power

Reference: Deodatis, G and Shinozuka, M. "Auto-Regressive Model for Nonstationary Stochastic Processes", ASCE Journal of Engineering Mechanics, Vol. 114, No. 11, November 1988.

(i) $$f(t,\omega) = |A(t,\omega)|^2 f(\omega)$$

where:
$$\left\{ \begin{aligned} A(t,\omega) &= \frac{\exp(-at)-\exp(-bt)}{\max[\text{Nominator}]} \cdot \sqrt{S^{K-T}_{f_0 f_0}(\omega)} \\ f(\omega) &= 1 \end{aligned} \right\}$$

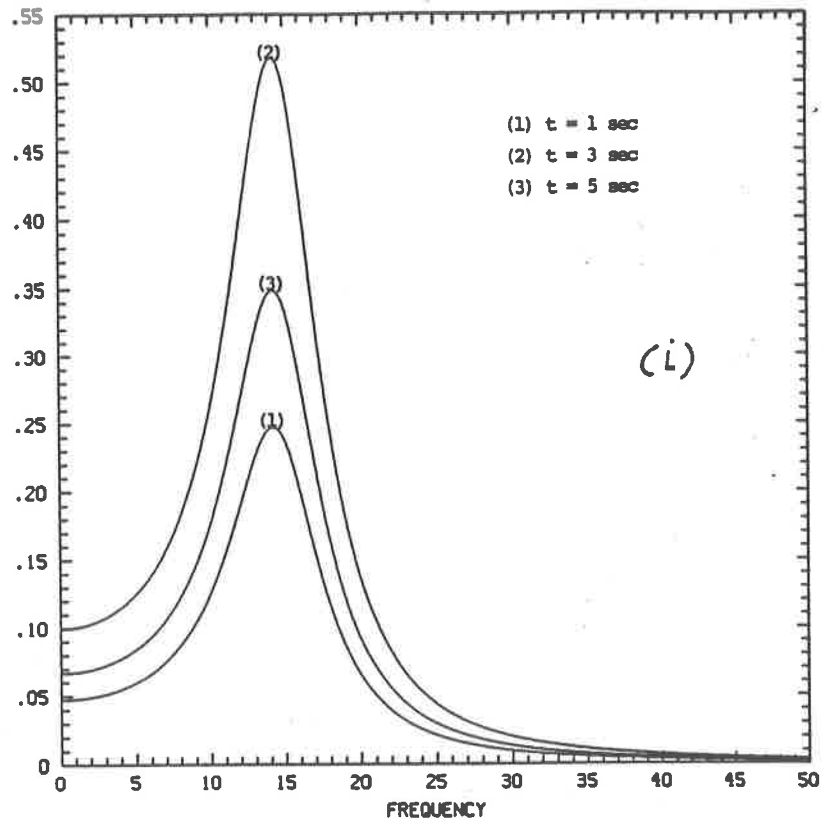
(ii) $$f(t,\omega) = |A(t,\omega)|^2 f(\omega)$$

where:

$$A(t,\omega) = \frac{\exp(-at)-\exp(-bt)}{\max[\text{Nominator}]} \cdot \left\{ S_{01} \cdot \frac{\omega_{g1}^4 + 4\zeta_g^2 \omega_{g1}^2 \omega^2}{(\omega^2-\omega_{g1}^2)^2 + 4\zeta_g^2 \omega_{g1}^2 \omega^2} \right\}^{1/2} +$$

$$+ \exp\left[-\frac{(t-m)^2}{2\sigma^2}\right] \cdot \left\{ S_{02} \cdot \frac{\omega_{g2}^4 + 4\zeta_g^2 \omega_{g2}^2 \omega^2}{(\omega^2-\omega_{g2}^2)^2 + 4\zeta_g^2 \omega_{g2}^2 \omega^2} \right\}^{1/2}$$

$$f(\omega) = 1$$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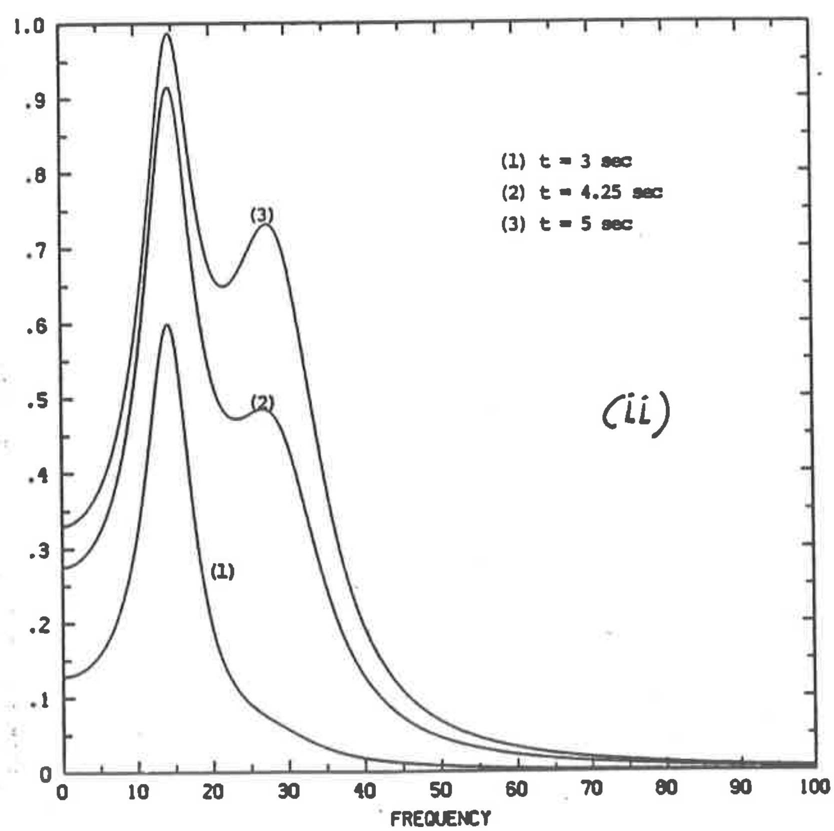
FREQUENCY
(1) t = 1 sec
(2) t = 3 sec
(3) t = 5 sec
(1)
(2)
(3)
(i)


1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
0
0 10 20 30 40 50 60 70 80 90 100
FREQUENCY
(1) t = 3 sec
(2) t = 4.25 sec
(3) t = 5 sec
(1)
(2)
(3)
(ii)

8. RESPONSE OF LINEAR STRUCTURES TO NONSTATIONARY EXCITATIONS (Shinozuka, 1970)

- Consider a linear, SDOF system with impulse response and frequency response functions denoted by $h(t)$ and $H(\omega)$, respectively.
- Assume that the system is excited by a nonstationary random process $X(t)$ with evolutionary power equal to:

$$S_X^n(t,\omega) = |A(t,\omega)|^2 \cdot S_X(\omega)$$

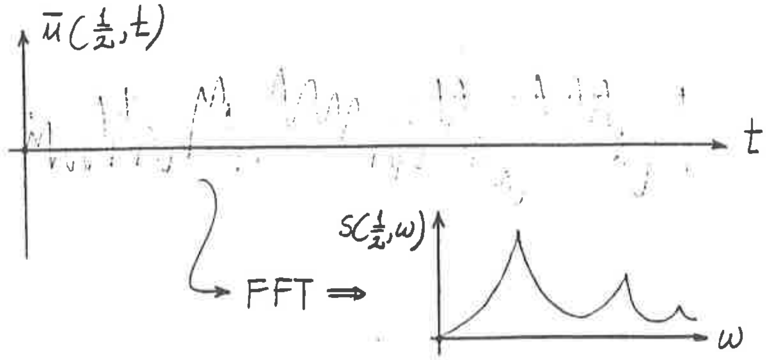
- It can be shown that the response of the system $Y(t)$ will be a nonstationary random process with evolutionary power:

$$S_Y^n(t,\omega) = \left| \int_0^t A(t-\tau,\omega) \cdot e^{-i\omega\tau} h(\tau)\, d\tau \right|^2 \cdot S_X(\omega)$$

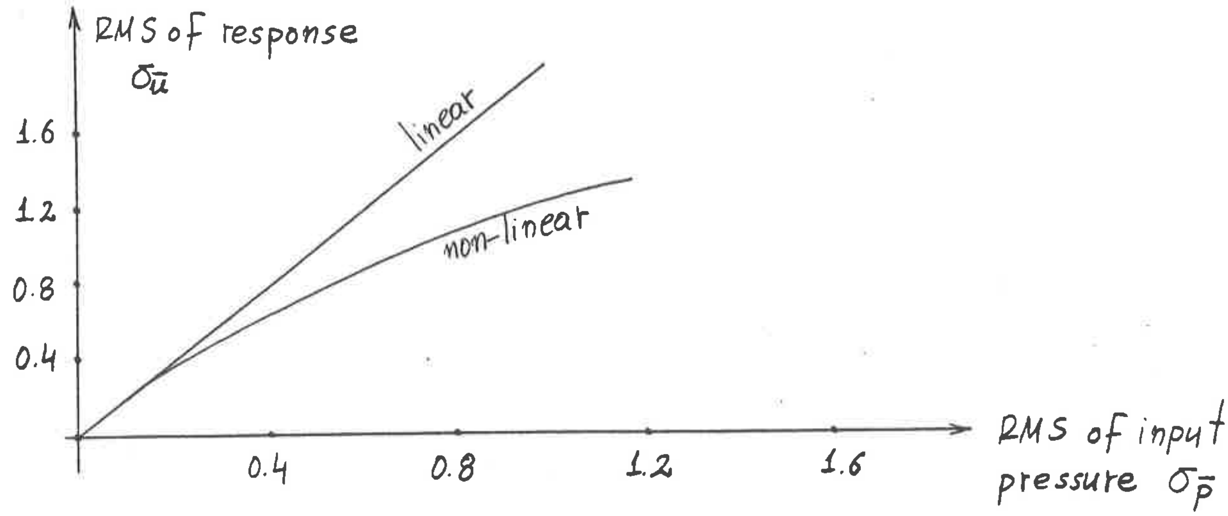
- Note that when $A(t,\omega) = 1$ and $t \to \infty$, the integral in the above equation reduces to $H(\omega)$, which is the well-known transfer function in the stationary case. Then:

$$S_Y^n(t,\omega) = S_Y(\omega) = |H(\omega)|^2 \cdot S_X(\omega)$$

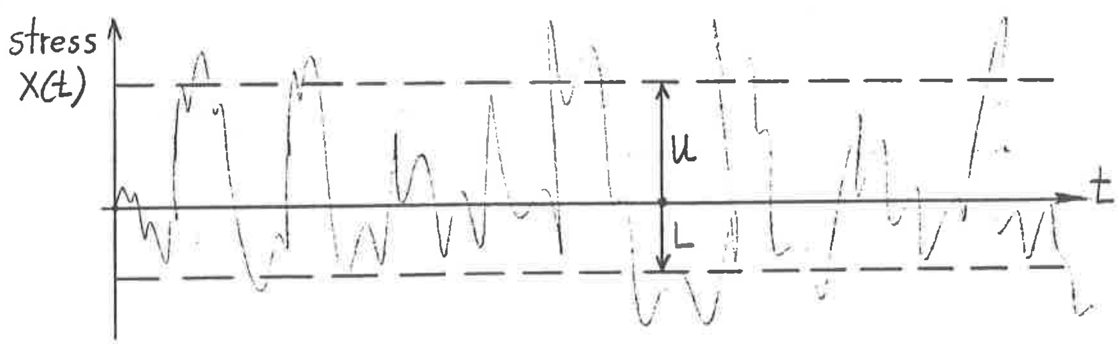
For $\bar{x} = \frac{1}{2} \longrightarrow$



• And: $\sigma_u^2 = \int_0^\infty S\, d\omega$ or by spatial averaging



9.0 STRUCTURAL FAILURES RESULTING FROM DYNAMIC RESPONSE



Structure fails if one of the events occur:

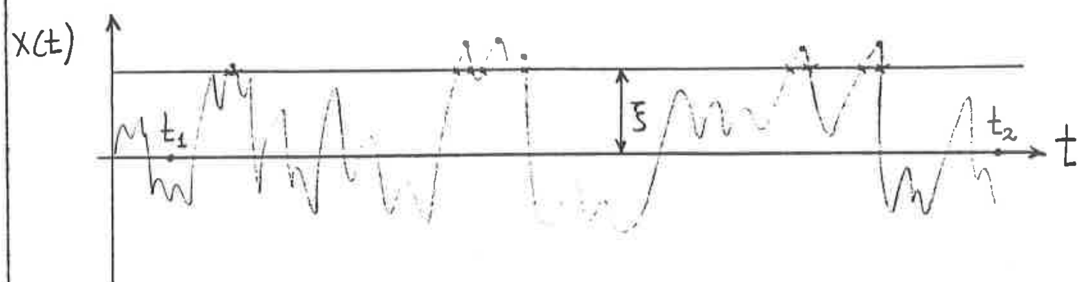
1./ X(t) reaches for the first time either the upper bound U or the lower bound L. (First excursion fails).

2./ Failure occurs when the accumulated damage reaches a fixed total (Fatigue failure).

For these studies it is necessary to know:

(1) Within given time interval, the number of times that $X(t)$ crosses a specified threshold level (threshold crossing).

(2) Within given time interval, the number of peaks (or troughs) in $X(t)$ above or below a specified level (peak distribution).



9.1 Threshold Crossing

$X(t)$ is a continuous random process.

Crossing problem is a counting process

- $\eta(\xi, t_1, t_2)$ = counting process which counts the random number of times $X(t)$ crosses a threshold level ξ from either above or below in time interval (t_1, t_2).

- We shall now try to estimate the statistics of η.

Construct a model for η:

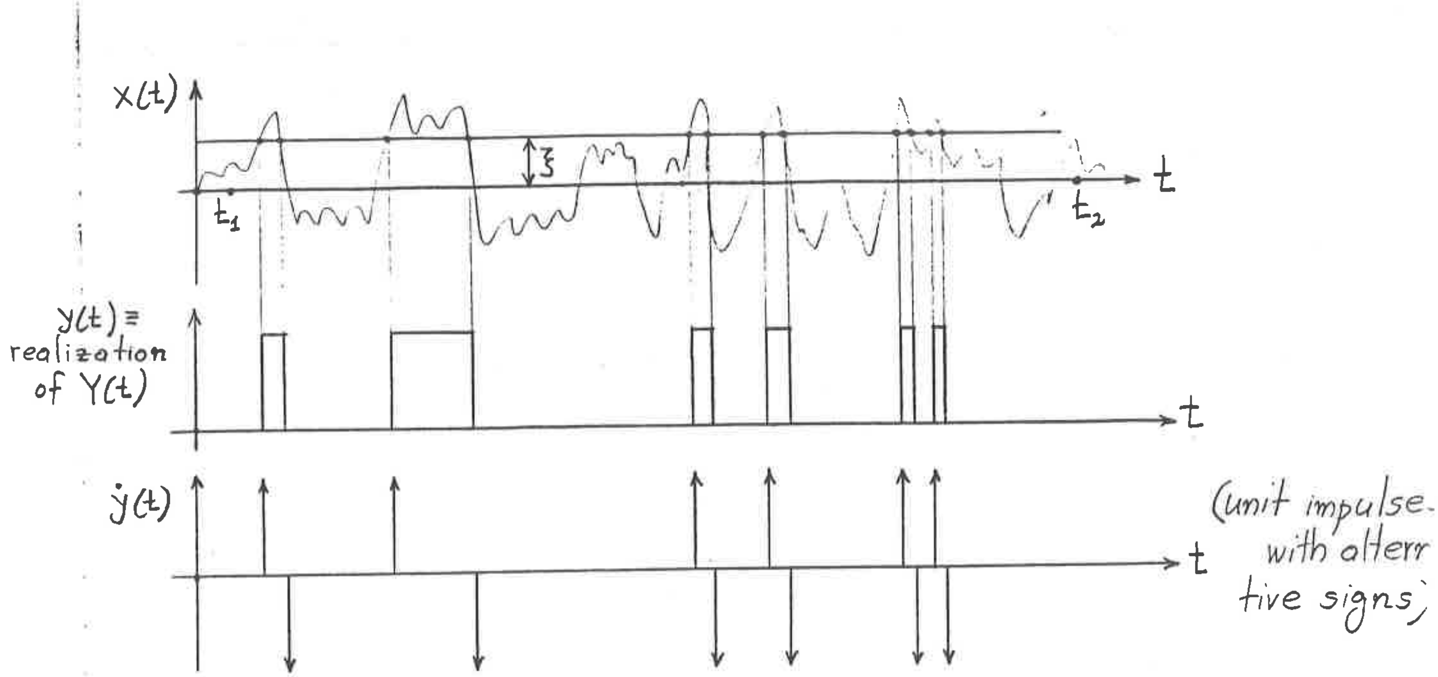
$$Y(t) = H[X(t) - \xi] \qquad (9\text{-}1)$$

$H(t)$ = Heavisides step function

$$Y(t) = \begin{cases} 1 & X(t) > \xi \\ 1/2 & X(t) = \xi \\ 0 & X(t) < \xi \end{cases} \qquad (9\text{-}2)$$

$$\dot{Y}(t) = \dot{X}(t) \cdot \delta[X(t) - \xi] \qquad (9\text{-}3)$$

$\delta(t)$ = Dirac's delta function



Now: $$\eta(\xi, t_1, t_2) = \int_{t_1}^{t_2} |\dot{x}(t)|\, \delta[x(t)-\xi]\, dt \qquad (9\text{-}4)$$

Expected value:

$$E[\eta(\xi, t_1, t_2)] = \int_{t_1}^{t_2} E\{|\dot{x}(t)|\, \delta[x(t)-\xi]\}\, dt = \qquad (9\text{-}5)$$

$$= \int_{t_1}^{t_2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\dot{x}|\, \delta(x-\xi)\, P_{x\dot{x}}(x, \dot{x}, t)\, dx\, d\dot{x}\, dt = \qquad (9\text{-}6)$$

$$= \int_{t_1}^{t_2}\int_{-\infty}^{\infty} |\dot{x}|\, P_{x\dot{x}}(\xi, \dot{x}, t)\, d\dot{x}\, dt \qquad (9\text{-}7)$$

The rate of threshold crossing per unit time $N(\xi,t)$ is:

$$N(\xi,t) = |\dot{x}(t)| \cdot \delta[x(t)-\xi] \qquad (9\text{-}8)$$

Therefore: $$\boxed{E[N(\xi,t)] = \int_{-\infty}^{\infty} |\dot{x}|\, P_{x\dot{x}}(\xi, \dot{x}, t)\, d\dot{x}} \qquad (9\text{-}9)$$

The correlation function of $N(\xi,t)$ can be computed from:

$$E[N(\xi,t_1)\,N(\xi,t_2)] = R_{NN}(\xi,t_1,t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\dot{x}_1|\cdot|\dot{x}_2|\cdot p_{X\dot{X}}(\xi,\dot{x}_1,t_1;\dot{x}_2,t_2) \cdot d\dot{x}_1\, d\dot{x}_2 \qquad (9\text{-}1$$

When we count upcrossings only:

$$\boxed{E[N_+(\xi,t)] = \int_0^{\infty} \dot{x}\; p_{X\dot{X}}(\xi,\dot{x},t)\, d\dot{x}} \qquad (9\text{-}11)$$

$$R_{N_+N_+}(\xi,t_1,t_2) = \int_0^{\infty}\int_0^{\infty} \dot{x}_1\,\dot{x}_2\; p_{X\dot{X}}(\xi,\dot{x}_1,t_1;\dot{x}_2,t_2)\, d\dot{x}_1\, d\dot{x}_2 \qquad (9\text{-}12)$$

For a weakely stationary case: $\boxed{E[N_+(\xi)] = \frac{1}{2}E[N(\xi)]}$ (9-13a,

$$R_{N_+N_+}(\xi,\tau) = \frac{1}{4}R_{NN}(\xi,\tau) \qquad (9\text{-}13b)$$

- Consider $X(t)$ is stationary Gaussian r.p. with zero mean.

$\therefore$ $X(t)$ and $\dot{X}(t)$ are independent.

$$p_{X\dot{X}}(x,\dot{x}) = p_X(x)\cdot p_{\dot{X}}(\dot{x}) = \frac{1}{\sqrt{2\pi}\cdot\sigma_X}\cdot e^{-\frac{x^2}{2\sigma_X^2}}\cdot\frac{1}{\sqrt{2\pi}\,\sigma_{\dot{X}}}\cdot e^{-\frac{\dot{x}^2}{2\sigma_{\dot{X}}^2}}$$

$$E[N_+(\xi)] = \int_0^{\infty}\dot{x}\cdot p_{X\dot{X}}(\xi,\dot{x})\,d\dot{x} = \frac{1}{2\pi\cdot\sigma_X\cdot\sigma_{\dot{X}}}\cdot e^{-\frac{\xi^2}{2\sigma_X^2}}\int_0^{\infty}\dot{x}\cdot e^{-\frac{\dot{x}^2}{2\sigma_{\dot{X}}^2}}\,d\dot{x}$$

Let now: $\dfrac{\dot{x}^2}{2\sigma_{\dot{X}}^2} = u \longrightarrow du = \dfrac{\dot{x}}{\sigma_{\dot{X}}^2}\,d\dot{x}$

$$E[N_+(\xi)] = \frac{1}{2\pi\,\sigma_X\cdot\sigma_{\dot{X}}}\cdot e^{-\frac{\xi^2}{2\sigma_X^2}}\cdot\sigma_{\dot{X}}^2\;\overbrace{\int_0^{\infty} e^{-u}\,du}^{=1}$$

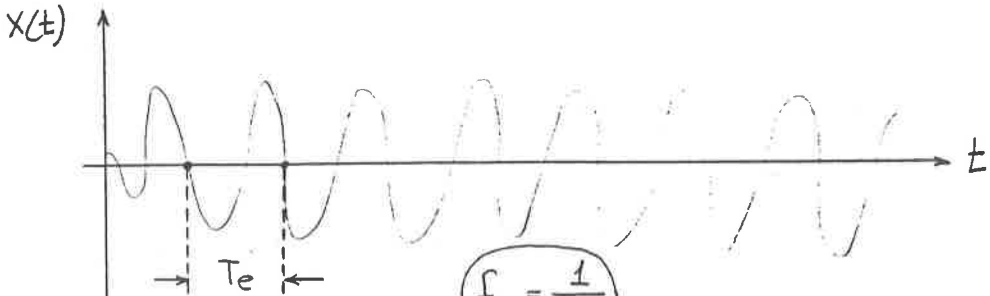
And finally: $\boxed{E[N_+(\xi)] = \frac{1}{2\pi}\cdot\frac{\sigma_{\dot{X}}}{\sigma_X}\cdot e^{-\frac{\xi^2}{2\sigma_X^2}}}$ (9-15)

The above is the upcrossing rate for a stationary Gaussian r.p.

$$\underline{\text{At } \xi=0}: \quad E[N_+(0)] = \frac{1}{2\pi}\cdot\frac{\sigma_{\dot{x}}}{\sigma_x} = \frac{1}{2\pi}\cdot\left[\frac{\int_{-\infty}^{\infty}\omega^2 S_x(\omega)\,d\omega}{\int_{-\infty}^{\infty} S_x(\omega)\,d\omega}\right]^{1/2} \qquad (9\text{-}16)$$

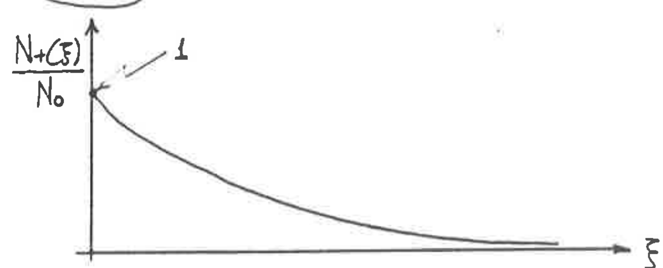
• $E[N_+(0)] = f_e$ = effective frequency for a narrow band process

Set now: $\quad N_0 \equiv E[N_+(0)] \quad ; \quad N_+(\xi) = E[N_+(\xi)]$



$$f_e = \frac{1}{T_e}$$

$$\boxed{\frac{N_+(\xi)}{N_0} = e^{-\frac{\xi^2}{2\sigma_x^2}}}$$



9.2 Peak Distribution

A peak in a sample function $x(t)$ occurs when $\dot{x}(t)=0$ and $\ddot{x}(t)$ is negative. Therefore to obtain distribution of peaks, we need joint information on $x, \dot{x}, \ddot{x}$.

• It has been shown that the number of extrema in $X(t)$ above a specified level ξ, can be obtained from: (between t_1^+ and t_2^+)

$$\mathcal{E}(\xi, t_1^+, t_2^+) = \int_{t_1^+}^{t_2^+} |\ddot{x}(t)|\,\delta[\dot{x}(t)]\cdot H[x(t)-\xi]\,dt \qquad (9\text{-}18)$$

The expected value of $\mathcal{E}$ is:

$$E[\mathcal{E}(\xi, t_1^*, t_2^*)] = \int_{t_1^*}^{t_2^*} E\{|\ddot{x}(t)|\,\delta[\dot{x}(t)]\,H[x(t)-\xi]\}\,dt = \qquad (9\text{-}19)$$

$$= \int_{t_1^+}^{t_2^+} dt \iiint_{-\infty}^{\infty} |\ddot{x}|\,\delta(\dot{x})\,H(x-\xi)\,P_{x\dot{x}\ddot{x}}(x,\dot{x},\ddot{x},t)\,dx\,d\dot{x}\,d\ddot{x} \qquad (9\text{-}20)$$

Denote the random number of peaks per unit time as M:

$$E[M(\xi,t)] = -\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} d\dot{x} \int_{-\infty}^{0} P_{x\dot{x}\ddot{x}}(x,\dot{x},\ddot{x},t)\,\ddot{x}\,\delta(\dot{x})\,H(x-\xi)\,d\ddot{x} \qquad (9\text{-}2\text{-}$$

$\llcorner$ $\ddot{x}$ is negative for peaks and positive for trough

$$\boxed{E[M(\xi,t)] = -\int_{\xi}^{\infty}\int_{-\infty}^{0} \ddot{x}\cdot P_{x\dot{x}\ddot{x}}(x,0,\ddot{x},t)\,dx\,d\ddot{x}} \qquad (9\text{-}21b)$$

By letting $\xi \to -\infty$, then: $E[M(\xi,t)] = E[M_T(t)]$

expected total number of peaks per unit time

$$E[M_T(t)] = -\int_{-\infty}^{\infty}\int_{-\infty}^{0} \ddot{x}\,P_{x\dot{x}\ddot{x}}(x,0,\ddot{x},t)\,dx\,d\ddot{x} \qquad (9\text{-}22)$$

Let $X(t)$ be a stationary and Gaussian random process

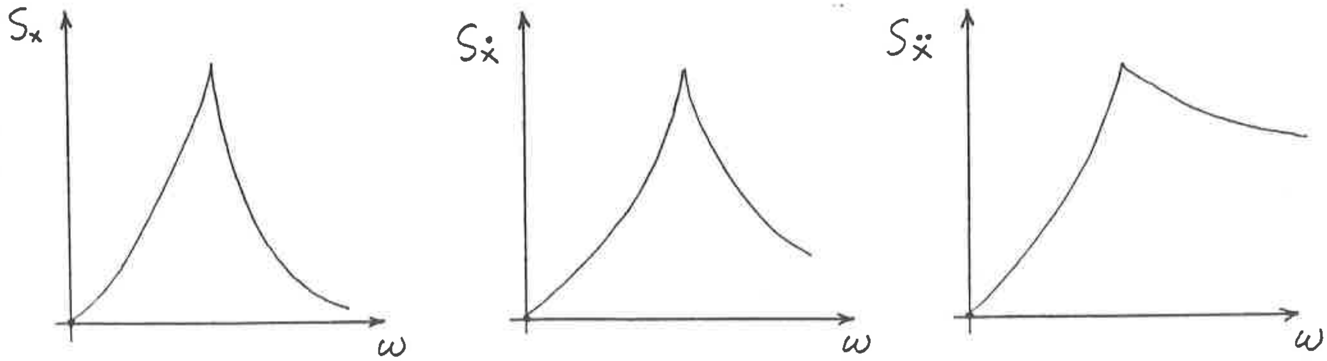
$$P_{x\dot{x}\ddot{x}}(x,0,\ddot{x}) = \frac{1}{(2\pi)^{3/2}\cdot|S|^{1/2}}\cdot\exp\left[-\frac{1}{2\cdot|S|}\cdot\left(\sigma_2^2\cdot\sigma_3^2\cdot x^2 + 2\sigma_2^4\,x\cdot\ddot{x} + \sigma_1^2\cdot\sigma_2^2\cdot\ddot{x}^2\right)\right] \qquad (9\text{-}23)$$

where: $|S| = \sigma_1^2\,\sigma_2^2\,\sigma_3^2 - \sigma_2^6 \qquad (9\text{-}24)$

$$\left\{\begin{aligned} \sigma_1^2 &= \sigma_x^2 = \int_{-\infty}^{\infty} S_x(\omega)\,d\omega \\ \sigma_2^2 &= \sigma_{\dot{x}}^2 = \int_{-\infty}^{\infty} \omega^2\cdot S_x(\omega)\,d\omega \\ \sigma_3^2 &= \sigma_{\ddot{x}}^2 = \int_{-\infty}^{\infty} \omega^4\cdot S_x(\omega)\,d\omega \end{aligned}\right\} \qquad (9\text{-}25)$$

Substituting eq. (9-23) into eq. (9-22) we get:

$$E[M_T] = \frac{1}{2\pi}\cdot\frac{\sigma_3}{\sigma_2} = \frac{1}{2\pi}\left[\frac{\int_{-\infty}^{\infty}\omega^4\cdot S_x(\omega)\cdot d\omega}{\int_{-\infty}^{\infty}\omega^2\cdot S_x(\omega)\cdot d\omega}\right]^{1/2} \qquad (9\text{-}26)$$



$S_{\dot{x}} = \omega^2\cdot S_x$ $\qquad$ $S_{\ddot{x}} = \omega^4\cdot S_x$

- But $S_{\ddot{x}}$ usually blows up!
- Next, we would like to obtain the probability density function of peak distribution.

Consider a conditional probability distribution of peak magnitude

$$\left\{\begin{aligned} F_{\Xi}(\xi,t) &= 1 - \frac{E[M(\xi,t)]}{E[M_T(t)]} \qquad (9\text{-}27)\\ P_{\Xi}(\xi,t) &= -\frac{1}{E[M_T(t)]}\cdot\frac{\partial}{\partial\xi}E[M(\xi,t)]\end{aligned}\right\}$$

probability that a peak will have magnitude less or equa to ξ, at time t

(Huston & Skopinski, 1950

Note: $\frac{\partial}{\partial\xi}E[M(\xi,t)] = -\frac{\partial}{\partial\xi}\int_{\xi}^{\infty}\int_{-\infty}^{0}\ddot{x}\, P_{x\dot{x}\ddot{x}}(x,0,\ddot{x},t)\,dx\,d\ddot{x} = \int_{-\infty}^{0}\ddot{x}\cdot p(\xi,0,\ddot{x},t)\,d\ddot{x}$

Leibnitz rule

probability that a peak will have magnitude between ξ and $\xi+d\xi$: $P_{\Xi}(\xi,t)\cdot d\xi$

Therefore: $$\boxed{P_{\Xi}(\xi,t) = -\frac{1}{E[M_T(t)]}\int_{-\infty}^{0}\ddot{x}\,P_{x\dot{x}\ddot{x}}(\xi,0,\ddot{x},t)\,d\ddot{x}} \qquad (9\text{-}28)$$

Substituting eq. (9-23) into eq. (9-28) and using eq. (9-26) we get:

$$P_{\Xi}(\xi) = \frac{1}{(2\pi)^{1/2}\cdot\sigma_1^2\cdot\sigma_2\cdot\sigma_3}\left[|S|^{1/2}\cdot\exp\left(-\frac{\sigma_2^2\cdot\sigma_3^2\cdot\xi^2}{2|S|}\right) + \sigma_2^4\cdot\xi\cdot\left(\frac{\pi}{2\sigma_1^2\cdot\sigma_2^2}\right)^{1/2}\cdot\left(1+\right.\right.$$

$$+\,\mathrm{erf}\,\frac{\sigma_2^3\cdot\xi}{\sqrt{2}\cdot|s|^{1/2}\cdot\sigma_1}\Big)\cdot\exp\Big(-\frac{\xi^2}{2\sigma_1^2}\Big)\Big] \qquad (9\text{-}29)$$

where: $\quad \mathrm{erf}\,x = \frac{2}{\sqrt{\pi}}\int_0^x e^{-u^2}\,du \qquad \left\{\begin{array}{l}\mathrm{erf}(0)=0\\ \mathrm{erf}(\infty)=1\end{array}\right\}$

Alternative expression for Eq. (9-29):

· Let now: $\quad \alpha = \frac{E[N_+(0)]}{E[M_T]} = \frac{\sigma_2^2}{\sigma_1\sigma_3} \qquad (9\text{-}30)$

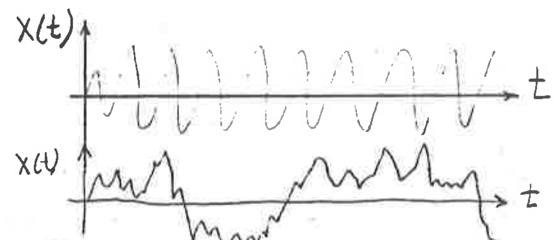
Then: $\quad P_\Xi(\xi) = \frac{(1-\alpha^2)^{1/2}}{\sqrt{2\pi}\cdot\sigma_1}\cdot\exp\left\{-\xi^2\left[2\sigma_1^2(1-\alpha^2)\right]^{-1}\right\} +$

$$+\,\frac{\alpha\cdot\xi}{2\sigma_1^2}\cdot\left\{1+\mathrm{erf}\left[\frac{\xi}{\sigma_1}(2\alpha^{-2}-2)^{-1/2}\right]\right\}\cdot\exp\Big(-\frac{\xi^2}{2\sigma_1^2}\Big) \qquad (9\text{-}31)$$

We have: $\quad 0 \leqq \alpha \leqq 1$

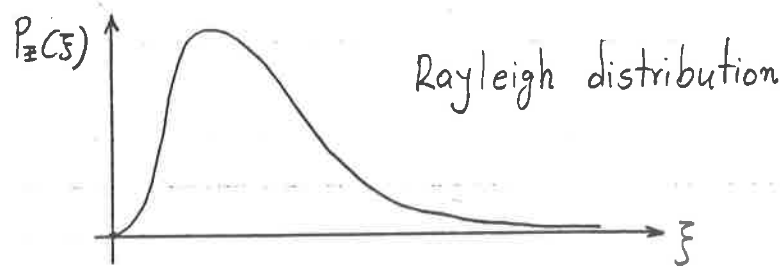
$\alpha = 1 \rightarrow$ narrow band process :

$\alpha \rightarrow 0$ very large number of peaks for each zero upcrossing



· From eq. (9-31) $\longrightarrow$ (for $\alpha=1$)

$$P_\Xi(\xi) = \frac{\xi}{\sigma_1^2}\cdot\exp\Big(-\frac{\xi^2}{2\sigma_1^2}\Big) \qquad \left\{\begin{array}{c}0\leqq\xi<\infty\\ \alpha=1\end{array}\right\} \qquad \text{Rayleigh distribution} \qquad (9\text{-}32)$$



For $\alpha \rightarrow 0$:

$$P_\Xi(\xi) \cong \frac{1}{\sqrt{2\pi}\cdot\sigma_1}\cdot\exp\Big(-\frac{\xi^2}{2\sigma_1^2}\Big) \qquad \left\{\begin{array}{c}-\infty<\xi<\infty\\ \alpha=0\end{array}\right\} \qquad \text{Normal distribution} \qquad (9\text{-}33)$$

Conclusion: For most practical cases, the peak distribution of a stationary r.p. lies between Gaussian and Rayleigh.

Useful Approximations (for a narrow-band stationary random processes)

$$E[N_+(\xi)] \approx E[M(\xi)]$$

$$E[N_+(0)] \approx E[M_T] \qquad \left(= -\frac{1}{E[M_T(t)]} \frac{\partial}{\partial \xi} E[M(\xi, t)] \right)$$

exact expression

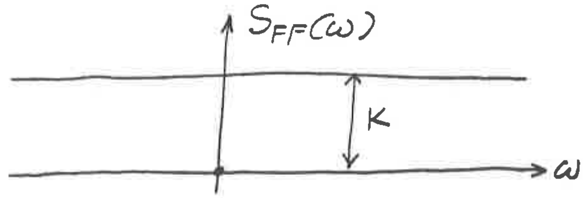
$$P_\Xi(\xi) \doteq -\frac{1}{E[N_+(0)]} \cdot \frac{d}{d\xi} E[N_+(\xi)] \qquad (9\text{-}34)$$

$$P_\Xi(\xi) \doteq -\frac{1}{E[N_+(0)]} \cdot \frac{d}{d\xi} \int_0^\infty \dot{x} \cdot P_{x\dot{x}}(\xi, \dot{x})\, d\dot{x} \qquad (9\text{-}35)$$

Example: consider the following nonlinear Duffing type equation:

$$\ddot{X} + \beta \dot{X} + \omega_0^2 (X + \varepsilon X^3) = F(t)$$

where: $F(t)$ = Gaussian white-noise with intensity K



ε = positive number, meaning that the system has a hardening spring type of nonlinearity

- Although $F(t)$ is Gaussian, the response $X(t)$ is not Gaussian because of the nonlinearity of the system. However $X(t)$ is a stationary random process
- If the system is lightly damped and if the nonlinearity is small ($\varepsilon \ll 1$), then $X(t)$ is expected to be a narrow-band random process.
- It has been found (e.g. Caughey, 1963) that the stationary displacement and velocity are independent random processes with joint probability density:

$$p_{X\dot{X}}(x, \dot{x}) = C \cdot \exp\left\{ -\frac{\beta}{\pi K} \left[\frac{\dot{x}^2}{2} + \omega_0^2 \left(\frac{x^2}{2} + \varepsilon \frac{x^4}{4} \right) \right] \right\}$$

- Constant C can be evaluated from the normalization condition:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p_{X\dot{X}}(x, \dot{x}) \, dx \, d\dot{x} = 1$$

Using normalization condition get:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} C \cdot e^{-\frac{\beta}{\pi K}\cdot\frac{\dot{x}^2}{2}} \cdot e^{-\frac{\beta}{\pi K}\cdot\omega_0^2\cdot\left(\frac{x^2}{2}+\epsilon\cdot\frac{x^4}{4}\right)} \, dx\, d\dot{x} = 1$$

Integrating for $\dot{x}$ we get:

$$C\cdot\sqrt{\frac{2\pi K}{\beta}}\cdot\int_{-\infty}^{\infty} e^{-\frac{\beta\cdot\omega_0^2}{\pi K}\left(\frac{x^2}{2}+\epsilon\cdot\frac{x^4}{4}\right)}\, dx = 1$$

Let: $e^{-\frac{\beta\cdot\omega_0^2}{\pi K}\epsilon\cdot\frac{x^4}{4}} = 1-\beta\cdot\frac{\omega_0^2}{\pi K}\cdot\epsilon\cdot\frac{x^4}{4} + O(\text{higher order})$

$\hookrightarrow$ neglect since $\varepsilon \ll 1$.

Therefore:

$$C\cdot\frac{\pi K}{\beta}\cdot\int_{-\infty}^{\infty}\left(1-\frac{\beta\cdot\omega_0^2}{\pi K}\epsilon\cdot\frac{x^4}{4}\right) e^{-\frac{\beta\cdot\omega_0^2}{\pi K}\cdot\frac{x^2}{2}}\, dx \cong 1$$

And finally integrating:

$$\boxed{C \cong \frac{\beta\cdot\omega_0}{2\pi\cdot\pi K}\cdot\left(1-\epsilon\cdot\frac{3}{4}\,\frac{\pi K}{\beta\cdot\omega_0^2}\right)^{-1}}$$

- Expected number of upcrossings per unit time (threshold ξ):

$$E[N_+(\xi)] = \int_0^{\infty}\dot{x}\, p_{X\dot{X}}(\xi,\dot{x})\, d\dot{x} = C\cdot e^{-\frac{\beta}{\pi K}\cdot\left(\frac{\xi^2}{2}+\varepsilon\cdot\frac{\xi^4}{4}\right)\cdot\omega_0^2}\cdot\int_0^{\infty}\dot{x}\cdot e^{-\frac{\beta\cdot\dot{x}^2}{2\pi K}}\, d\dot{x}$$

Let now: $u = \frac{\beta\cdot\dot{x}^2}{2\pi K} \Rightarrow du = \frac{\beta}{\pi K}\dot{x}\, d\dot{x} \Rightarrow \dot{x}\, d\dot{x} = \frac{\pi K}{\beta} du$

and taking into account that: $\int_0^{\infty} e^{-u}\, du = 1$

$$E[N_+(\xi)] = G \cdot \frac{\pi K}{\beta} \cdot e^{-\frac{\beta \cdot \omega_0^2}{\pi K} \cdot \left(\frac{\xi^2}{2} + \epsilon \frac{\xi^4}{4}\right)}$$

$$E[N_+(0)] = G \cdot \frac{\pi K}{\beta} = f_e = \text{effective frequency since } X(t) \text{ is a narrow-band random process}$$

$$\omega_e = 2\pi f_e = 2\pi \cdot \frac{\pi \cdot K}{\beta} \cdot \frac{\beta \cdot \omega_0}{2\pi^2 K} \cdot \left(1 - \varepsilon \cdot \frac{3}{4} \cdot \frac{\pi \cdot K}{\beta \cdot \omega_0^2}\right)^{-1} = \frac{\omega_0}{\left(1 - \underbrace{\varepsilon \cdot \frac{3}{4} \cdot \frac{\pi K}{\beta \cdot \omega_0^2}}_{\text{nonlinear effect}}\right)}$$

If $\varepsilon = 0 \longrightarrow$ linear system $\longrightarrow \omega_e = \omega_0$

The nonlinearity increases the expected effective frequency. Although the above conclusion is based on the case of Gaussian white-noise excitation, we expect it to be valid for a broadband excitation provided that the excitation spectral density is slowly varying in the neighborhood of ω_0.

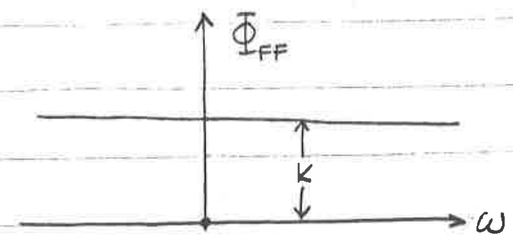
Example: Consider a nonlinear Duffing type equation:

$$\boxed{\ddot{X}+\beta\dot{X}+\omega_0^2(X+\epsilon X^3)=F(t)} \leftarrow \text{Gausion white-noise}$$

We would like to obtain the joint probability function density (of the stationary response and its derivative): $P_{X\dot{X}}(x,\dot{x})$

~~(Using perturbation techniques)~~ It has been found that (Lin) the stationary displacement and velocity are independent with joint probability density:

$$P_{X\dot{X}}(x,\dot{x})=C\cdot\exp\left\{-\frac{\beta}{\pi K}\left[\frac{\dot{x}^2}{2}+\omega_0^2\cdot\left(\frac{x^2}{2}+\epsilon\cdot\frac{x^4}{4}\right)\right]\right\}$$



- constant C can be evaluated from normalization condition:

$$\iint_{-\infty}^{\infty} P_{X\dot{X}}(x,\dot{x})\,dx\,d\dot{x}=1$$

$$E[N_+(\xi)]=\int_0^\infty \dot{x}\,P_{X\dot{X}}(\xi,\dot{x})\,d\dot{x}=$$

$$=C\cdot e^{-\frac{\beta}{\pi K}\cdot\left(\frac{\xi^2}{2}+\epsilon\cdot\frac{\xi^4}{4}\right)\omega_0^2}\cdot\int_0^\infty \dot{x}\cdot e^{-\frac{\beta\cdot\dot{x}^2}{\pi K 2}}\,d\dot{x}$$

Let: $u=\frac{\beta}{2\pi K}\dot{x}^2 \longrightarrow du=\frac{\beta}{\pi K}\dot{x}\,d\dot{x}$, $\dot{x}\,d\dot{x}=\frac{\pi K}{\beta}du$

and since: $\int_0^\infty e^{-u}du=1$

$$E[N_+(\xi)]=C\cdot\frac{\pi K}{\beta}\cdot e^{-\frac{\beta\cdot\omega_0^2}{\pi\cdot K}\left(\frac{\xi^2}{2}+\epsilon\cdot\frac{\xi^4}{4}\right)}$$

$$E[N_+(0)]=C\cdot\frac{\pi K}{\beta}$$

Using normalization condition get:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} C\cdot e^{-\frac{\beta}{\pi K}\cdot\frac{\dot{x}^2}{2}}\cdot e^{-\frac{\beta}{\pi K}\cdot\omega_0^2\cdot\left(\frac{x^2}{2}+\epsilon\cdot\frac{x^4}{4}\right)}\,dx\,d\dot{x} = 1$$

Integrating for $\dot{x}$ we get:

$$C\cdot\frac{\pi K}{\beta}\cdot\int_{-\infty}^{\infty} e^{-\frac{\beta\cdot\omega_0^2}{\pi K}\left(\frac{x^2}{2}+\epsilon\cdot\frac{x^4}{4}\right)}\,dx = 1$$

Let: $e^{-\frac{\beta\cdot\omega_0^2}{\pi K}\epsilon\cdot\frac{x^4}{4}} = 1-\beta\cdot\frac{\omega_0^2}{\pi K}\cdot\epsilon\cdot\frac{x^4}{4} + \underbrace{\mathcal{O}(\text{higher order})}_{\text{neglect since } \varepsilon\ll 1}$

Therefore:

$$\frac{\pi K}{\beta}\cdot\int_{-\infty}^{\infty}\left(1-\frac{\beta\cdot\omega_0^2}{\pi K}\epsilon\cdot\frac{x^4}{4}\right) e^{-\frac{\beta\cdot\omega_0^2}{\pi K}\cdot\frac{x^2}{2}}\,dx \cong 1$$

And finally integrating:

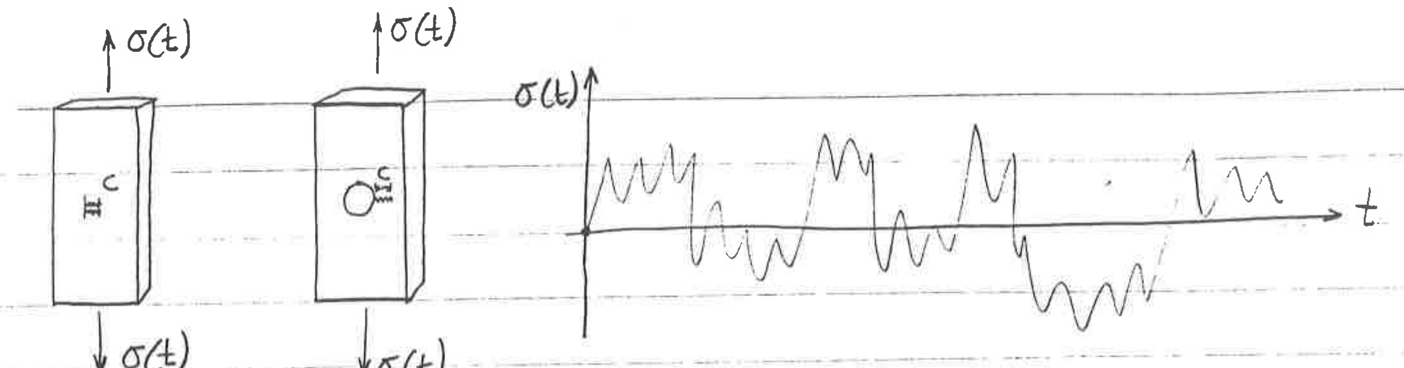
$$\boxed{C \cong \frac{\beta\cdot\omega_0}{2\pi\cdot\pi K}\cdot\left(1-\epsilon\cdot\frac{3}{4}\frac{\pi K}{\beta\cdot\omega_0^2}\right)^{-1}}$$

$$\omega_e = 2\pi \underbrace{f_e}_{=E[N_+(0)]} = 2\pi\cdot C\cdot\frac{\pi K}{\beta} = \omega_0\Big(1-\underbrace{\epsilon\cdot\frac{3}{4}\cdot\frac{\pi}{\beta}\cdot\frac{K}{\omega_0^2}}_{\text{non-linear effect}}\Big)^{-1}$$

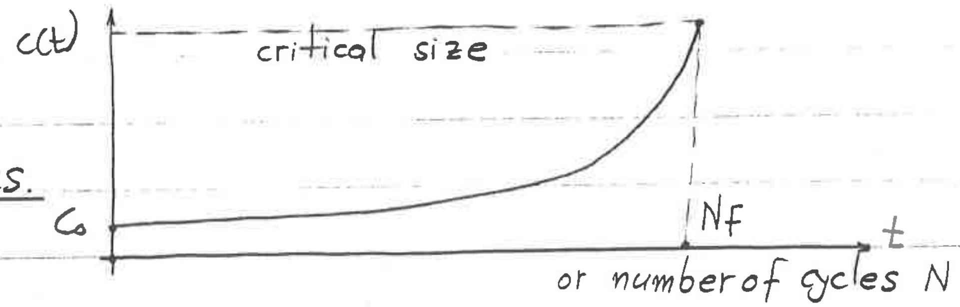
If: $\varepsilon = 0 \Rightarrow$ linear system $\omega_e = \omega_0$

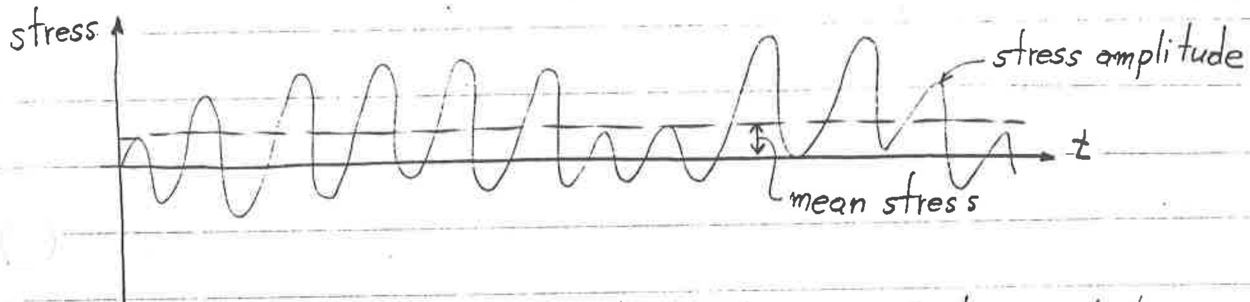
9.3 Fatigue Failure

- When metal tensile specimens are subjected to fluctuating stress, small dislocations (cracks) grow under cyclic stress until failure is reached.

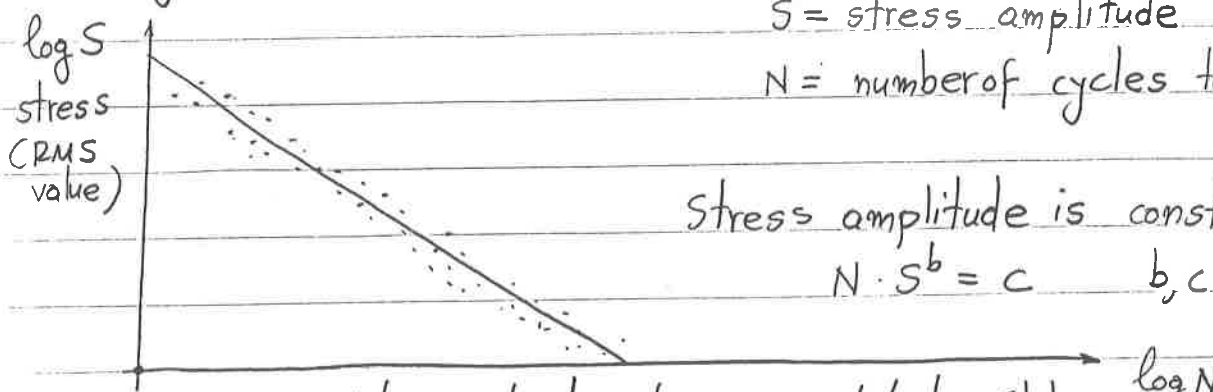


We need fracture mechanics.





- When the mean stress is not very large, it does not have much effect on fatigue life.
- When a large number of specimens are tested under (deterministic cyclic) a constant amplitude loading, the S-N curve is obtained.



S = stress amplitude

N = number of cycles to fatigue failure

Stress amplitude is constant!

$$N \cdot S^b = c \qquad b, c > 0$$

Not constant stress amplitude (deterministic)

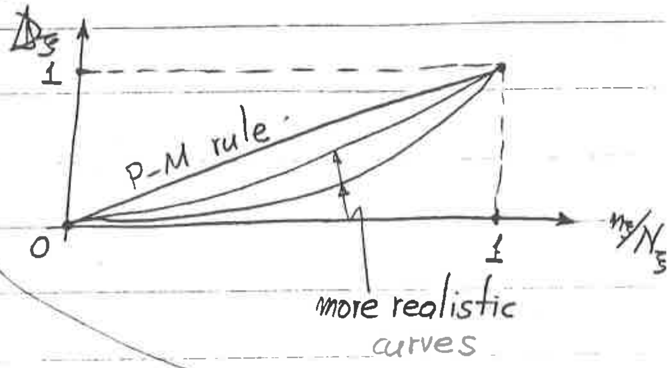
-cording to Palmgren-Miner rule, the accumulated (fatigue) damage ~~D in a structure can be calculated from:~~ is linear:

Damage due to the application of n_ξ cycles at a stress amplitude ξ:

$$\Delta_\xi = \frac{n_\xi}{N_\xi} \qquad n_\xi \leq N_\xi \qquad N_\xi = \text{number of cycles to fatigue failure at stress amplitude } \xi$$

Under variable stress amplitudes, the total damage is:

$$D = \sum \Delta_\xi = \sum \frac{n_\xi}{\Delta_\xi}$$



This is linear and very approximate!

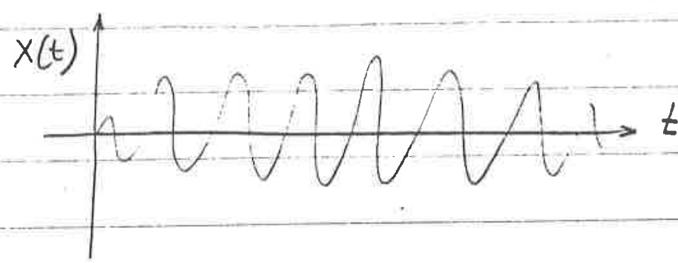
~~N = total number of possible cycles~~
~~n = applied cycles~~

• Thus fatigue failure occurs when $D = 1$.

The P-M theory implies that the order of application of the different stress levels has no effect on the resulting total accumulated damage = very desirable property to use theory with random stress

Random Stress

Let ν_0^+ to be the expected frequency of the narrow band stress response $X(t)$:



$$\nu_0^+ = N_+(0)$$

In given time T, the expected number of cycles is: $T \cdot \nu_0^+$

• The expected fraction of these cycles whose stress amplitude lie between ξ and $\xi + d\xi$ is: $p_\Xi(\xi)\, d\xi$

probability density of peaks

The expected number of such peaks is: (in given time T)

$$n(\xi) = \nu_0^+ \cdot T \cdot p_\Xi(\xi)\, d\xi \qquad (9\text{-}37)$$

A single peak of amplitude ξ causes an incremental damage of $\frac{1}{N(\xi)}$ according to P-M rule. [Note: we have somehow modified the P-M rule by changing n_ξ to number of peaks at level ξ]

$N(\xi)$ = number of cycles to failure in a constant amplitude test at an amplitude ξ.

The expected damage (fractional) due to all cycles having peaks between ξ and $\xi + d\xi$ is:

$$\frac{n(\xi)}{N(\xi)} = \nu_0^+ \cdot T \cdot \frac{P_\Xi(\xi)\,d\xi}{N(\xi)} \qquad (9\text{-}38)$$

The total expected damage is the sum of all contributions:

$$\boxed{E[D] = \nu_0^+ \cdot T \cdot \int_0^\infty \frac{P_\Xi(\xi)}{N(\xi)}\,d\xi} \qquad (9\text{-}39)$$

Thus we need $P_\Xi(\xi)$ and $N(\xi)$ ← can be found only experimentally

Let: $N = \frac{c}{S^b}$ where: c = const. depending on material
b = exponent determined experimentally
S = stress

Assume: $P_\Xi(\xi)$ is Rayleigh distribution (= narrow band process)

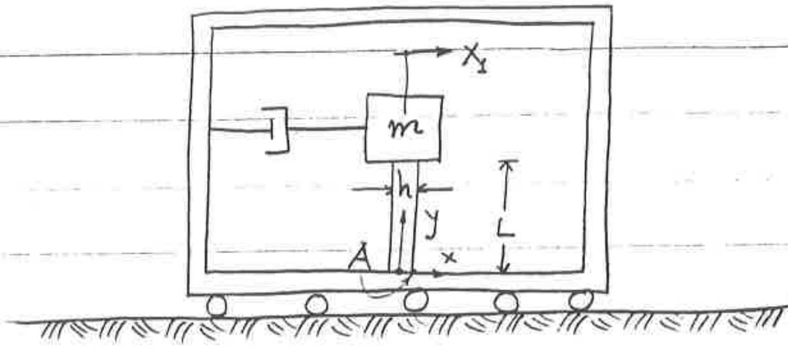
Then: $E[D(T)] \doteq \frac{\nu_0^+ \cdot T}{c \cdot \sigma_1^2} \int_0^\infty \xi^{b+1} \exp\left(-\frac{\xi^2}{2\sigma_1^2}\right) d\xi \qquad (9\text{-}40)$

↑ mean damage

Note: $N(\xi) = \frac{c}{\xi^b}$ and integrating:

$$E[D(T)] = \frac{\nu_0^+ \cdot T}{c} \cdot \left(\sqrt{2} \cdot \sigma_1\right)^b \cdot \Gamma\left(1 + \frac{b}{2}\right)$$

Example:



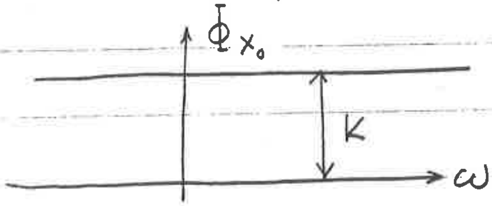
Excitation is accelaration $\ddot{X}_0$.

Stress response: $y = 0$, $x = \frac{h}{2}$

stress

$$X(t) = \frac{3Eh}{2L^2} \cdot \underbrace{[X_1(t) - X_0(t)]}_{\text{displacement of mass } m\text{, relative to car}}$$

Consider the input is stationary Gaussian white-noise.

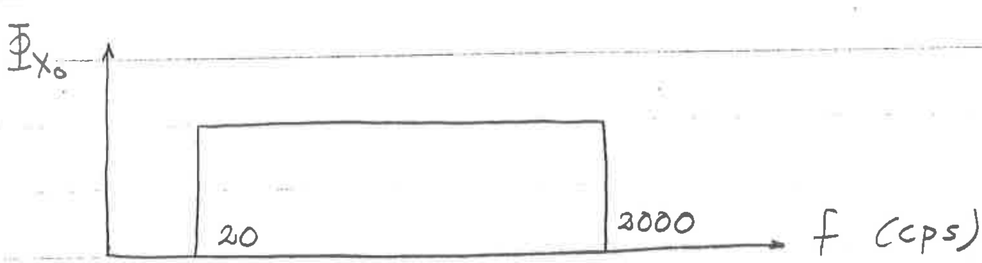


$$E[X^2] = \left(\frac{3Eh}{2L^2}\right)^2 \frac{\pi}{2} \cdot \frac{K}{\zeta \omega_1^3} = \sigma_1^2$$

$$\omega_1 = \frac{k}{m} = \frac{3EI}{mL^3} = \frac{Eh^4}{4me^3}$$

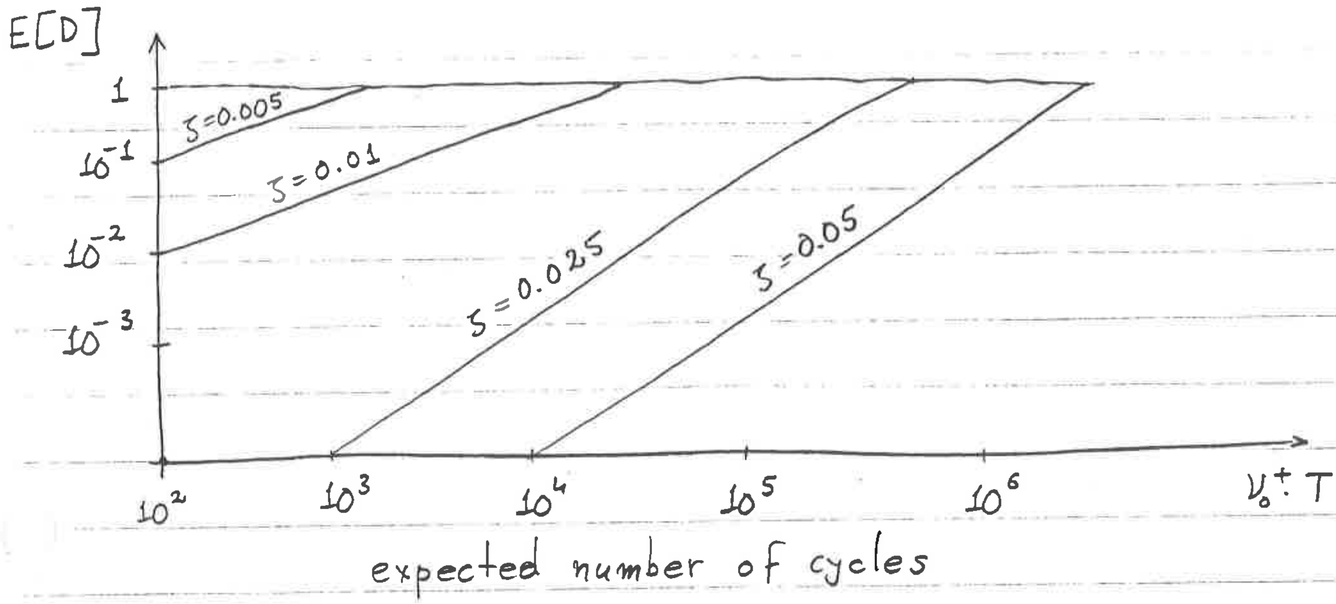
$$\nu_0^+ = \frac{\sigma_{\dot{x}}}{2\pi\sigma_x} \doteq \frac{\omega_1}{2\pi}$$

Take : $\begin{cases} L = 4'', \; h = 0.25'', \; E = 10.3 \times 10^6 \text{ psi}, \; b = 6.09 \\ c = (2 \times 10^5)^{6.09}, \; m = 7.28 \times 10^{-4} \cdot \frac{lb.sec^2}{in} \\ \omega_1 = 465 \; rad/sec, \; K = 5930 \; \frac{(in/sec)^2}{rad/sec} \end{cases}$



$$\sigma_1 = \frac{2320}{\sqrt{5}} \text{ psi}$$

$$E[D(T)] = \frac{\nu_0^+ T}{C} (\sqrt{2} \cdot \sigma_1)^b \cdot \Gamma\left(1 + \frac{b}{2}\right)$$



SUMMARY

1) Single Degree of freedom.

$$\Phi_{xx}(\omega) = |H(\omega)|^2 \cdot \Phi_{FF}(\omega)$$

$$\sigma_x^2 = \int_{-\infty}^{\infty} \Phi_{xx}(\omega)\, d\omega$$

2) Multi Degree of freedom

$$[\Phi_{xx}(\omega)] = [H(\omega)][\Phi_{FF}(\omega)][H^*(\omega)]^T$$

3) Continuous Systems (Modal Analysis)

$$\Phi_{xx}(\vec{r},\omega) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} Y_m(\vec{r})\, Y_n(\vec{r})\, H_m(\omega)\, H_n^*(\omega) \cdot I_{mn}(\omega)$$

modes

cross-spectral densities of generalized random force

4) Simulation, Nonlinear and Non-elastic systems
Time Domain, Monte Carlo Method

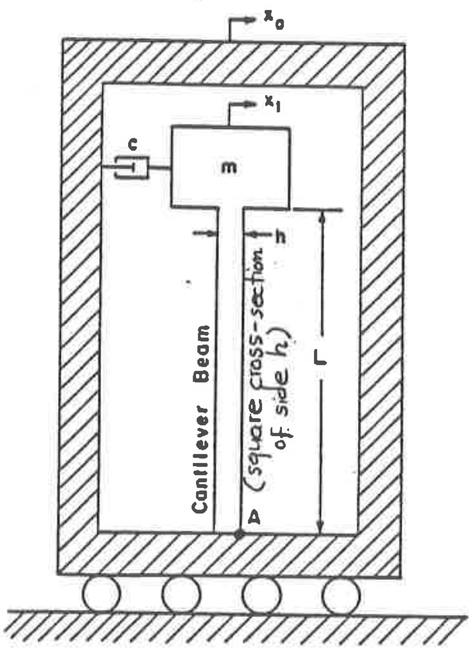
5) Threshold Crossing - Peak distribution

$\{N_+(\xi),\ P_I(\xi)\}$ depend on: $P_{x\dot{x}}$ $(P_{x\dot{x}\ddot{x}})$

Exam:

1. Multi-degree of freedom
2. Continuous structures (modal analysis) ⊛
3. Threshold crossing - peak distribution. (maybe simulation)

Example:



- Excitation of the above system is acceleration $\ddot{X}_0$.
- Examine the fatigue damage at the root of the beam (point A) due to random vibration of the vehicle.

- Equation of motion:

$$m\ddot{X}_1 + c(\dot{X}_1 - \dot{X}_0) + k(X_1 - X_0) = 0$$

or by setting $Y = X_1 - X_0$

$$m\ddot{Y} + c\cdot\dot{Y} + k\cdot Y = -m\ddot{X}_0 \implies \ddot{Y} + 2\zeta\omega_0\dot{Y} + \omega_0^2\cdot Y = -\ddot{X}_0$$

- Stress at point A:

$$S(t) = \frac{3Eh}{2L^2}\cdot Y(t) = \frac{3Eh}{2L^2}\cdot\left[X_1(t) - X_0(t)\right]$$

- Acceleration input: stationary, Gaussian white-noise with intensity S_0

- Mean-square stress:

$$\sigma_s^2 = E[S^2] = \left(\frac{3Eh}{2L^2}\right)^2 \cdot E[Y^2] = \left(\frac{3Eh}{2L^2}\right)^2 \cdot \frac{\pi \cdot S_0}{2\zeta \cdot \omega_0^3}$$

where: $$\omega_0^2 = \frac{k}{m} = \frac{3EI}{L^3 \cdot m} = \frac{E \cdot h^4}{4mL^3}$$

- Expected value of $N_+(0)$ of stress S (Gaussian):

$$E[N_+(0)] = \nu_0^+ = \frac{1}{2\pi} \cdot \frac{\sigma_{\dot{s}}}{\sigma_s} = \frac{\omega_0}{2\pi}$$

- Assuming that the stress $S(t)$ is a narrow-band process:

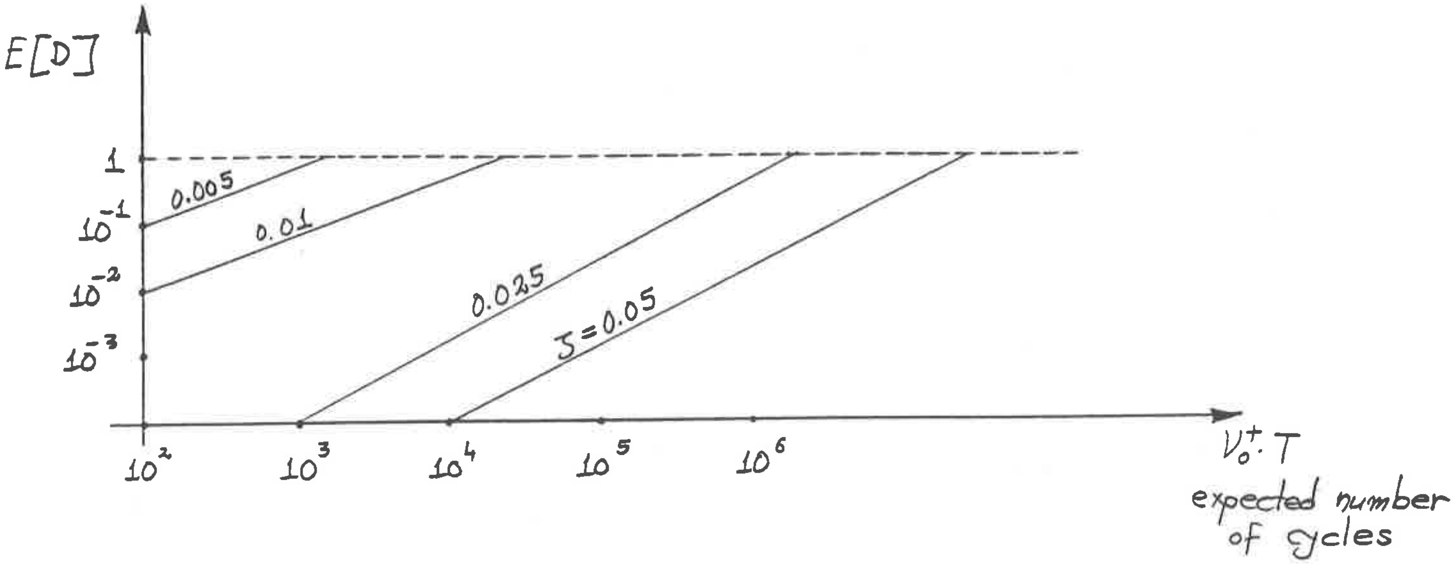
$$E[D(T)] = \frac{\nu_0^+ \cdot T}{c} \cdot (\sqrt{2} \cdot \sigma_s)^b \cdot \Gamma\left(1 + \frac{b}{2}\right)$$

- Numerical values:

$L = 4''$
$h = 0.25''$
$E = 10.3 \cdot 10^6$ psi (aluminum alloy)
S-N curve : $N \cdot S^{6.09} = (2 \cdot 10^5)^{6.09} \rightarrow b = 6.09$ & $c = (2 \cdot 10^5)^{6.09}$
$m = 7.28 \cdot 10^{-4} \ \frac{\text{lb} \cdot \text{sec}^2}{\text{inch}}$
$S_0 = 5930 \ \frac{(\text{inch}/\text{sec}^2)^2}{\text{rad}/\text{sec}}$

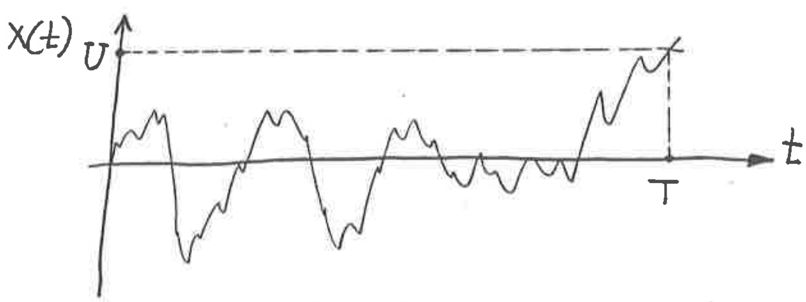
- Results: $\omega_o = 465\ \text{rad/sec}$

  $$\sigma_s = \frac{2320}{\sqrt{5}}\ \text{psi}$$



- First-Passage or First-Excursion Problem

  - $X(t)$ is a stationary random process (it can denote the random stress at some point of a structure)



T: time to failure

  - We would like to have information concerning the probabilistic characteristics of T, e.g. the probability density function of T, $p_T(t)$.
  - This problem is not solved today except for the following very special case.

- Assuming that the upcrossings of level U constitute a Poisson process (not a good assumption most of the times, since differen crossings must be independent), $P_T(t)$ is given by:

$$P_T(t) = \nu_U^+ \cdot e^{-\nu_U^+ \cdot t} \qquad t \geq 0$$

where $\nu_U^+ = E[N_+(U)] = \int_0^\infty \dot{x}\, p_{X\dot{X}}(U, \dot{x})\, d\dot{x}$

Then: $\left\{ \begin{array}{l} E[T] = \dfrac{1}{\nu_U^+} \\ \sigma_T = \dfrac{1}{\nu_U^+} \end{array} \right\}$

Probability of failure in the interval $0 < t < T_0$:

$$P_{T_0} = \int_0^{T_0} P_T(t)\, dt = 1 - \exp[-\nu_U^+ \cdot T_0]$$