```
Example: Rigid-End Offsets
4,1,1
:
Boundary
1,4 dof=r,r,f,f,f,r
1,4,3 dof=f,f,f,f,f,r                  : Pinned supports
:
Coordinates
1 x=0+9    y=0 z=0                     : Add depth of column
2 y=18*12-12                           : Subtract half depth of beam
3 x=40*12-9                            : Subtract depth of beam
4 y=0
:
Beam
3,2
1 a=31   i=3100 e=29000                : Beam element
2 a=35   i=2190 e=29000                : Column element
1 1,2,3 m=2 j=0,-12                    : Rigid-end offset at top col. - 12 in.
2 2,3,1 m=1 i=9 j=-9  l=-1.5/12        : Beam - offset both ends 9 in.
3 4,3,1 m=2 j=0,-12 i=0                : Column - offset top - 12 in.
:
Loads
2 l=1 f=3                              : 3-kip load at left joint
:
```

After running SSTAN, we can plot the structure to verify the input. Looking at the plot we can see the rigid-end offsets shown for each member. The maximum moment is found under the beam results. The output for the beam is as follows:

```
                  --------------------- FRAME MEMBER RESULTS ---------------
MEM LOAD NODE              1-2 PLANE                   1-3 PLANE        AXIAL FORCE
 #    #    #         MOMENT        SHEAR        MOMENT        SHEAR
 2    1    2     .00000E+00   .00000E+00   .11444E+04 -.26425E+02   .97755E+01
           3     .00000E+00   .00000E+00  -.17325E+04 -.29075E+02  -.97755E+01
                                                  AXIAL TORQUE =     .00000E+00
** MAXIMUM MIDSPAN MOMENT =    .165E+04 AT DISTANCE    211.40 FROM NODE      2
```

Note that the maximum positive moment is also given for loaded members.

7.6 Geometric Stiffness: P–Δ

Most structures are analyzed assuming small displacements and small strains. The small displacement assumption means that summing equilibrium about the undeformed configuration is sufficiently accurate as not to affect the results. Small strains assumes that the strains are small enough that the second-order terms in the strain tensor can be ignored. We also usually assume that the stress–strain relationship is linear.

It is often important to include the effects of a portion of the large displacement effect into the analysis without including the full effects. One such case is in the analysis of tall buildings, where the lateral displacement in conjunction with the height causes additional displacements that cannot be ignored. This is shown in Figure 7.36. These are called P–Δ *effects* and are due to the additional moment caused by the gravity loads acting through the drift.

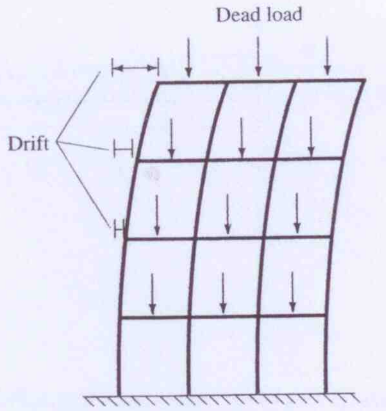

Figure 7.36 Additional P–Δ moments from drift.

P–Δ can be thought of as the additional load due to the axial load of a member not being collinear. This is as a result of considering the deformed system instead of the undeformed system for equilibrium calculations. A simple example will demonstrate this effect. Consider the structure in Figure 7.37, consisting of a rigid bar with a spring attached. Standard analysis says that **R** and **N** are independent (in the undeformed configuration) and the stiffness relationship is

$$\mathbf{Kr} = \mathbf{R} \tag{7.8}$$

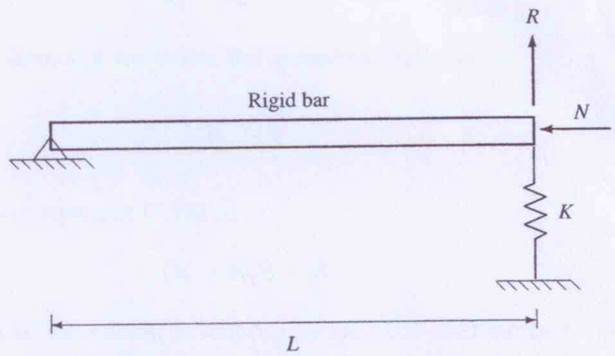

Figure 7.37 Rigid-bar example for geometric stiffness.

But if the structure is allowed to deform, we can calculate equilibrium in the deformed configuration as shown in Figure 7.38. Summing moments about the pinned end, we get

$$\mathbf{R}L + \mathbf{N}\mathbf{r} = \mathbf{S}L \tag{7.9}$$

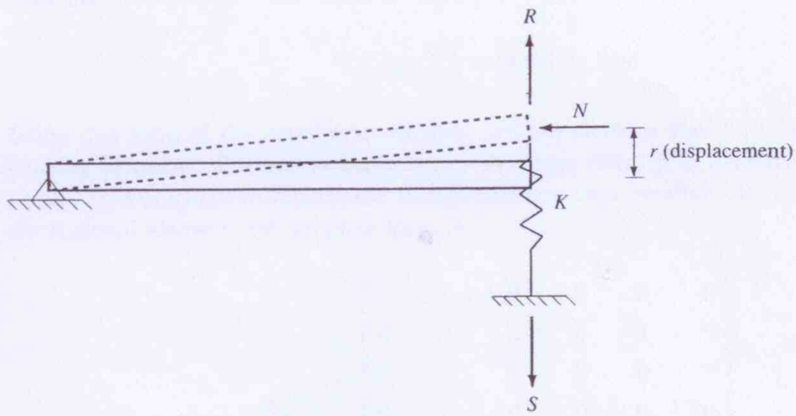

Figure 7.38 Deformed shape for rigid-bar example.

The equation for the response of the spring is

$$\mathbf{K}\mathbf{r} = \mathbf{S} \tag{7.10}$$

Substituting $\mathbf{S}$ from equation (7.10) into (7.9), we get

$$\mathbf{R}L + \mathbf{N}\mathbf{r} = (\mathbf{K}\mathbf{r})L \tag{7.11}$$

Dividing by L and grouping terms, we have

$$\left(\mathbf{K} - \frac{\mathbf{N}}{L}\right)\mathbf{r} = \mathbf{R} \tag{7.12}$$

This can be rewritten further if we define the geometric stiffness as

$$\mathbf{K}_g = \frac{\mathbf{N}}{L} \tag{7.13}$$

giving the final form for equation (7.12) as

$$(\mathbf{K} - \mathbf{K}_g)\mathbf{r} = \mathbf{R} \tag{7.14}$$

Note that the stiffness of the system is reduced as the axial load increases. A reduction in stiffness causes increased displacements. Hence the larger the axial load, the greater the displacements.

At some point, the axial load can become so large that the stiffness becomes zero or even negative. The point at which the stiffness becomes zero defines an instability point. At this point, there are an infinite number of solutions to the set of equations. When the stiffness becomes zero due to the axial load, we have reached the critical load or buckling load:

$$\mathbf{N}_{\text{critical}} = \mathbf{K}L = \text{buckling load} \tag{7.15}$$

Using this form of the geometric stiffness, we can create a matrix version for use with bending elements. The matrix form causes the same effect of a reduction in stiffness. It causes an axial load to reduce the lateral stiffness of a bending element. For the two-dimensional element, the simplest form is

$$\mathbf{K}_g = \frac{\mathbf{N}}{L}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{7.16}$$

This form assumes that the element is rigid for the definition of the geometric stiffness matrix, as in the example above.

7.6.1 Consistent Geometric Stiffness

A more complete form of the geometric stiffness assuming a flexible element can be derived using virtual work. We assume that a general bending element is in its deformed equilibrium configuration, as shown in Figure 7.39. The four end displacements (v_1, v_2, v_3, v_4) are used to define its position. There is an axial force in the element that can vary along its length, $N(X)$. As usual, we apply a small, compatible virtual displacement to the element, as shown in Figure 7.40.

The integration of the strain energy over the length of the element gives the internal

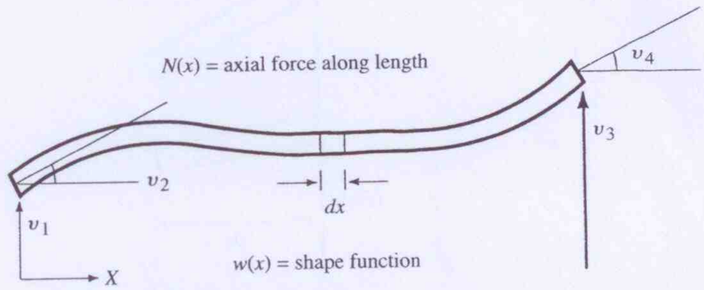

Figure 7.39 Deformed shape for consistent geometric stiffness.

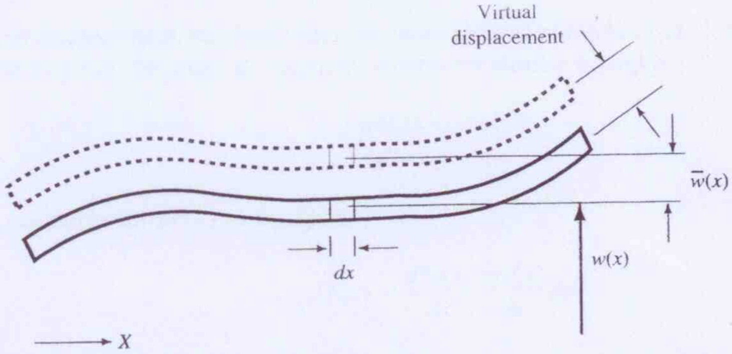

Figure 7.40 Virtual displacement for consistent geometric stiffness.

work in its normal form:

$$\delta W_i = \bar{\mathbf{v}}^{\mathrm{T}}\mathbf{K}_e\mathbf{v} \tag{7.17}$$

The external work is the sum of the work of the end forces through their respective displacements plus the work of the axial load $N(X)$. The end forces through their displacements again are in the standard form:

$$\delta W_e = \bar{\mathbf{v}}^{\mathrm{T}}\mathbf{S} + \int_0^L \mathbf{N}(X) \times \text{displacements}\, dx \tag{7.18}$$

The only new portion is the contribution of the axial load to the external virtual work. To calculate this, we need to look more closely at the deformed shape of the element. If we take a small section of the element dx, we can show the equilibrium and virtual displacement in Figure 7.41. From this we can see that the axial force $N(X)$ goes through a small displacement $de(x)$. The vertical component of the virtual displacement for this section is shown as $dw(x)$. Since the section we are looking at is very small, we can assume that it is essentially straight. In addition, for a small enough length and small

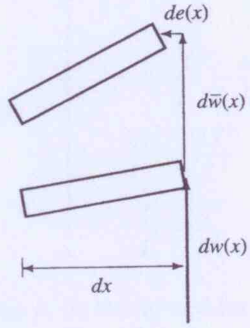

Figure 7.41 Displacements for dx segment.

virtual displacement, the displacements $de(x)$ and $dw(x)$ can be related by similar triangles to the slope of the small dx segment. Hence, by similar triangles,

$$\frac{d\overline{e}(x)}{d\overline{w}(x)} = \frac{dw(x)}{dx} \tag{7.19}$$

We can solve for $de(x)$ in the form

$$d\overline{e}(x) = \frac{d\overline{w}(x)}{dx}\frac{dw(x)}{dx}dx \tag{7.20}$$

We can substitute $\overline{de}(x)$ for the displacement associated with the axial force in equation (7.18) and integrate over the length of the element. This gives us

$$\delta W_e = \overline{\mathbf{v}}^{\mathrm{T}}\mathbf{S} + \int_0^L \mathbf{N}(X)\, d\overline{e}(x)\, dx \tag{7.21}$$

Substituting for $\overline{de}(x)$ using equation (7.20), we get

$$\delta W_e = \overline{\mathbf{v}}^{\mathrm{T}}\mathbf{S} + \int_0^L \mathbf{N}(X)\frac{d\overline{w}(x)}{dx}\frac{d\overline{w}(x)}{dx}dx \tag{7.22}$$

Now we need to define the displaced shape $w(x)$ in order to get $dw(x)/dx$.

The displaced shape must have a form that is consistent with the boundary conditions, that is, the end displacements v_1, v_2, v_3, and v_4. Since there are four boundary conditions, we can define a cubic function for the displaced shape. We can put these cubic functions into a matrix equation of the form

$$\mathbf{w(x)} = \mathbf{H(x)} \cdot \mathbf{v} \tag{7.23}$$

where $\mathbf{v}$ is the vector of end displacements for the element. $\mathbf{H(x)}$ is the matrix that converts these end displacements into a continuous displacement along the length. These functions are usually referred to as *shape functions*. The cubic form is

$$\mathbf{w(x)} = \begin{bmatrix} 1 - \frac{3X^2}{L^2} + \frac{2X^3}{L^3} \\ -X + \frac{2X^2}{L} - \frac{X^3}{L^2} \\ \frac{3X^2}{L^2} - \frac{2X^3}{L^3} \\ \frac{X^2}{L} - \frac{X^3}{L^2} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} \tag{7.24}$$

Note that the matrix is given in its transposed form. These are called the *cubic Hermitian polynomials* or *beam functions*. These functions can be used along with the finite-element method to derive the bending element stiffness matrix we have used in this book.

We take the derivative of these functions to form the matrix $dw(x)/dx$. We can then substitute into equation (7.22) and integrate to get the consistent form of the geometric stiffness. Note that when using the matrix form of virtual work, the virtual displacement term is transposed so that the work becomes a scaler value. As a result, the $d\overline{w}(x)/dx$ is replaced by $d\overline{w}(x)^{\mathrm{T}}/dx$ in equation (7.22). This final matrix has the form

$$\mathbf{K}_g = \frac{\mathbf{N}L}{30}\begin{array}{c} \begin{array}{cccc} v_1 & \theta_1 & v_2 & \theta_2 \end{array} \\ \begin{bmatrix} \dfrac{36}{L^2} & \dfrac{3}{L} & -\dfrac{36}{L^2} & \dfrac{3}{L} \\ \dfrac{3}{L} & 4 & -\dfrac{3}{L} & -1 \\ -\dfrac{36}{L^2} & -\dfrac{3}{L} & \dfrac{36}{L^2} & -\dfrac{3}{L} \\ \dfrac{3}{L} & -1 & -\dfrac{3}{L} & 4 \end{bmatrix} \end{array} \tag{7.25}$$

You could have derived the rigid member form using the same method and the linear shape functions:

$$\mathbf{w(x)} = \left\langle 1 - \frac{X}{L}, \frac{X}{L} \right\rangle \begin{bmatrix} v_1 \\ v_3 \end{bmatrix} \tag{7.26}$$

7.6.2 Geometric Stiffness Example

As an example, let's look at a simple cantilever beam. We will consider two cases: (1) no geometric stiffness and (2) geometric stiffness included. For this problem we will use two DOFs at the tip, lateral displacement, w, and rotation, θ. The loads on the structure will be a moment, R_0, and in the geometric case will include the axial load. The structure is shown in Figure 7.42.

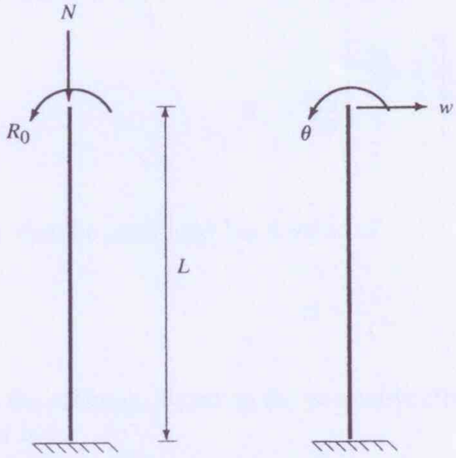

Figure 7.42 Geometric stiffness example for a cantilever beam.

Following standard procedure, the global stiffness will be a 2×2 matrix. This matrix can be assembled directly from the bending element stiffness matrix correcting for the 90° rotation. Since the only load is the applied moment, we generate the equilibrium equations in matrix form:

$$\frac{2EI}{L}\begin{bmatrix} \frac{6}{L^2} & \frac{3}{L} \\ \frac{3}{L} & 2 \end{bmatrix}\begin{bmatrix} w \\ \theta \end{bmatrix} = \begin{bmatrix} 0 \\ R_0 \end{bmatrix} \tag{7.27}$$

If we solve the set of equations symbolically, we get

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$$\begin{bmatrix} w \\ \theta \end{bmatrix} = R_0 \begin{bmatrix} \frac{L}{EI} \\ -\frac{L^2}{2EI} \end{bmatrix} \rightarrow R_0 \begin{bmatrix} -\frac{L^2}{2EI} \\ \frac{L}{EI} \end{bmatrix} \tag{7.28}$$

The element end forces for the two DOFs can be recovered by multiplying the stiffness times the displacements and we get

$$\begin{bmatrix} S_w \\ S_\theta \end{bmatrix} = \begin{bmatrix} 0 \\ R_0 \end{bmatrix} \tag{7.29}$$

This solution clearly satisfies equilibrium. Now, let's include the effects of geometric stiffness. We will add an axial load, N, to the system. We will not include any additional DOFs. As a result, the axial load only has the effect of reducing the lateral stiffness. The global stiffness is identical to the first case. We need to assemble the geometric stiffness in the same manner as the global stiffness. Using the consistent stiffness matrix, we get

$$\mathbf{K}_g = \frac{\mathbf{N}L}{30}\begin{bmatrix} \frac{36}{L^2} & \frac{3}{L} \\ \frac{3}{L} & 4 \end{bmatrix} \tag{7.30}$$

If we assume that the axial load has a value of

$$\mathbf{N} = \frac{2EI}{L^2} \tag{7.31}$$

we can form the stiffness, including the geometric effects. Note that for a cantilever, the buckling load is

$$\mathbf{N}_{\text{buckling}} = \frac{\pi^2 EI}{4L^2} \tag{7.32}$$

Compared to the buckling load, the choice for **N** in equation (7.31) is not unreasonable. Its expression in terms of the EI/L is useful so that the stiffness will be in a form that can easily be inverted. The final equilibrium equation is

$$(\mathbf{K} - \mathbf{K}_g)\mathbf{r} = \mathbf{R} \tag{7.33}$$

If we substitute in for the elastic stiffness and the geometric stiffness, we get the following matrix equations:

$$\frac{2EI}{L}\begin{bmatrix} \dfrac{4.8}{L^2} & \dfrac{2.9}{L} \\ \dfrac{2.9}{L} & 1.866 \end{bmatrix}\begin{bmatrix} w \\ \theta \end{bmatrix} = \begin{bmatrix} 0 \\ R_0 \end{bmatrix} \tag{7.34}$$

[2-34]

Notice how the combined stiffness matrix has changed. All the terms have been reduced. Most notably, the 1,1 term has been reduced to one-fourth of its original value. This term directly reflects the lateral stiffness. Again, if we solve for the displacements, we get

$$\begin{bmatrix} w \\ \theta \end{bmatrix} = R_0\begin{bmatrix} \cancel{\dfrac{4.367L}{EI}} \\ \cancel{-\dfrac{2.644L^2}{2EI}} \end{bmatrix} = R_0\begin{bmatrix} -2.629\dfrac{L^2}{EI} \\ 4.351\dfrac{L}{EI} \end{bmatrix} \tag{7.35}$$

$= R_0 \left[\cancel{-2.629 \frac{L^2}{EI}} \right]$ 2E

Comparing these to the purely elastic case, we see that the displacements are more than four times their elastic value. In this case, the axial load caused a very large P–Δ effect. We can again recover the member end forces. We use the combined elastic and geometric stiffness matrix. The geometric matrix is required for the element to maintain equilibrium. We can look at the contributions to the member forces from each portion of the stiffness (elastic and geometric). The elastic portion of the member forces is

변형 stiffn × matrix → 부재력

$$\begin{bmatrix} S_w \\ S_\theta \end{bmatrix} = \begin{bmatrix} -\dfrac{5.45}{L}R_0 \\ 1.64R_0 \end{bmatrix} \tag{7.36}$$

Here notice the increase in the end moment and that there is now a shear at the end of the column. These are due to the added displacements caused by the inclusion of the geometric stiffness. Notice that the structure is not in equilibrium using the elastic forces alone. Looking at the structure as a whole, we have the free body shown in Figure 7.43.

To check equilibrium, we would need to include the forces from geometric stiffness. These forces must be combined with the elastic portion to get the true forces. The geometric forces are recovered in the same manner as the elastic forces: stiffness times displacements.

$$\begin{bmatrix} S_w \\ S_\theta \end{bmatrix}_{\text{geometric}} = \begin{bmatrix} +\dfrac{5.45}{L}R_0 \\ -0.64R_0 \end{bmatrix} \leftarrow -K_g \begin{bmatrix} w \\ \theta \end{bmatrix} \tag{7.37}$$

* $P-\Delta$

$$B_2 = \frac{1}{1-\frac{P}{P_E}} = \frac{1}{1-\frac{\frac{2EI}{L^2}}{\frac{\pi^2 EI}{4L^2}}} = \frac{1}{1-\frac{8}{\pi^2}} = 5.28$$

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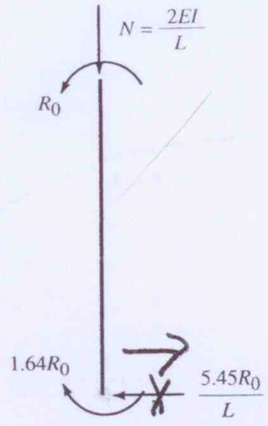


Figure 7.43 Free body of elastic results.

The geometric stiffness matrix can be used on large structures to include the effects of axial loads reducing the lateral stiffness. The process becomes more involved when indeterminate structures are to be analyzed. The difficulty arises because the axial force in a member is not known until the structure is analyzed. As a result, the process becomes iterative in nature.

For statically determinate structures, the axial loads in the members caused by the loading is known and does not depend on the deflection. Therefore, for any value of load, the axial load in all members is constant and the resulting force–displacement plot is always linear. A plot of the stiffness for different axial load values is given in Figure 7.44.

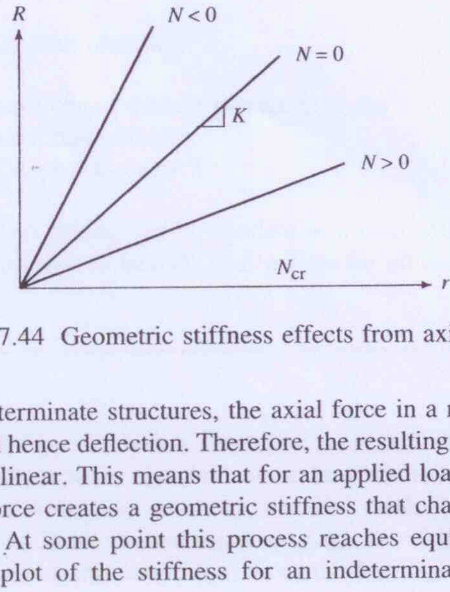


Figure 7.44 Geometric stiffness effects from axial loads.

For statically indeterminate structures, the axial force in a member is dependent on the applied loading and hence deflection. Therefore, the resulting force–displacement plot for the structure is nonlinear. This means that for an applied load you get a certain axial force. But this axial force creates a geometric stiffness that changes the deflections and hence the axial force. At some point this process reaches equilibrium or the structure becomes unstable. A plot of the stiffness for an indeterminate structure is given in Figure 7.45.

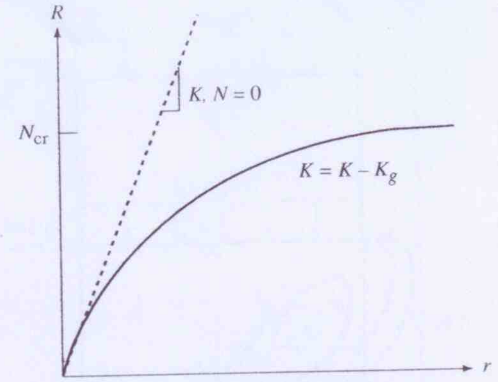


Figure 7.45 Geometric stiffness effects for indeterminate structures.

The process for including geometric effects in indeterminate structures is:

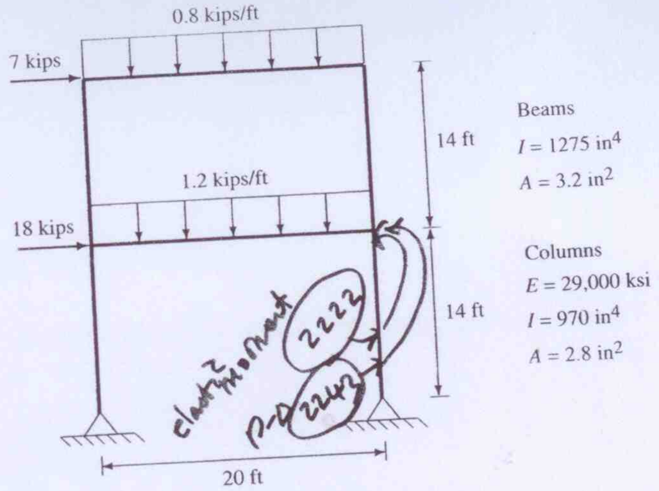
1. Form stiffness with *no* geometric stiffness.
2. Impose loads on the structure.
3. Solve for the axial forces in each member.
4. Use the axial force to form $K - K_g$.
5. Impose loads on the structure.
6. Solve for axial forces in each member.
7. Check if axial forces have changed since preceding cycle:
   (a) If the same, recover forces and quit.
   (b) If different, repeat steps 4 through 7.

Using this iterative process, P–Δ effects can be handled in a relatively simple analysis procedure. SSTAN has the capability to include P–Δ effects for all beam elements.

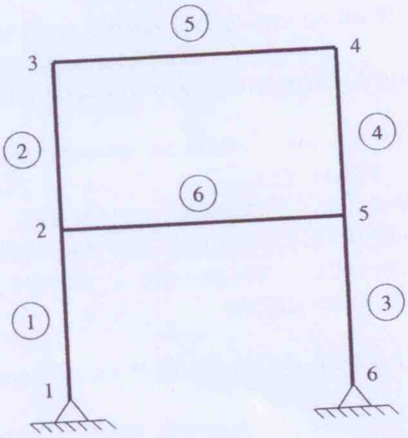
Condensed Example: P–Δ Analysis

Given the two-story single-bay frame shown at the top of page 300, find the maximum and end moments in the first-floor beam member. Show how the results change when P–Δ effects are included. To analyze this structure, we need to number the joints and members. The numbering shown in the second figure on page 300 will be used.

From the given numbering and the dimensions from the first figure, we can create the following SSTAN input file.



P–Δ example frame.



Node and element numbering.

```
Example: P-Delta
6,1,1  p=1      : 6 nodes, 1 type (beams), 1 load
:
Coordinates
1 x=0 y=0 z=0
3 y=28*12     g=1,3
6 x=20*12 y=0
4 y=28*12     g=4,6
:
Boundary
1,6     DOF=r,r,f,f,f,r       : Set all to 2 dimen. (X-Y plane)
1,6,5   DOF=f,f,f,f,f,r       : Pinned supports
:
```

```
Beam
6,2
1 a=3.2 i=1275 e=29000      : Beams
2 a=2.8 i=970               : Columns
C--- Columns
1 1,2,6 m=2
2 2,3,6
3 6,5,1
4 5,4,1
C--- Beams
5 3,4,2 m=1   l=-0.8/12     : Roof, K node below - neg. load
6 2,5,1       l=-1.2/12     : 1st floor
:
Loads
2 l=1  f=18
3      f=7
:
```

Notice that the second line of the input file has P=1 to turn on the P–Δ effects. To check the results without P–Δ, this should be removed.

The results for member 6, the first-floor beam, with no P–Δ effects are as follows:

1층 천장

```
---------------------- FRAME MEMBER RESULTS ----------------
MEM LOAD NODE            1-2 PLANE                  1-3 PLANE        AXIAL FORCE
 #    #    #       MOMENT        SHEAR        MOMENT        SHEAR
 6    1    2    .00000E+00   .00000E+00  -.17340E+04   .57060E+01   .66168E+01
           5    .00000E+00   .00000E+00  -.25154E+04  -.29706E+02  -.66168E+01
                                               AXIAL TORQUE =    .00000E+00
```

The results from the same member when P–Δ effects are included are

```
---------------------- FRAME MEMBER RESULTS ----------------
MEM LOAD NODE            1-2 PLANE                  1-3 PLANE        AXIAL FORCE
 #    #    #       MOMENT        SHEAR        MOMENT        SHEAR
 6    1    2    .00000E+00   .00000E+00  -.17557E+04   .58859E+01   .64798E+01
           5    .00000E+00   .00000E+00  -.25370E+04  -.29886E+02  -.64798E+01
                                               AXIAL TORQUE =    .00000E+00
```

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Notice how when the P–Δ effects are included, the moments and shears increase. This is a result of the added load, which increases deflections.

7.7 Problems

7.1. Use SSTAN to analyze the following structures and find the maximum moment and shear in the structures.