



12 Fourier Integrals

12.1 Fourier Integrals

- Extension of the method of Fourier series to nonperiodic functions
- We consider the Fourier series of an arbitrary function f_L of period $2L$ and again let $L \rightarrow \infty$

Example 1. Square wave

$$f_L(x) = \begin{cases} 0 & \text{if } -L < x < 1 \\ 1 & \text{if } -1 < x < 1 \\ 0 & \text{if } 1 < x < L \end{cases}$$

Solution.

- If we let $L \rightarrow \infty$,

$$f(x) = \lim_{L \rightarrow \infty} f_L(x) = \begin{cases} 1 & \text{if } -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

- Since $f_L(x)$ is an even function, $b_n = 0$ for all n .

$$\begin{aligned} a_0 &= \frac{1}{2L} \int_{-1}^1 dx = \frac{1}{L} \\ a_n &= \frac{1}{L} \int_{-1}^1 \cos \frac{n\pi x}{L} dx = \frac{2}{L} \int_0^1 \cos \frac{n\pi x}{L} dx = \frac{2}{L} \cdot \frac{\sin(n\pi/L)}{n\pi/L} \end{aligned}$$

- Amplitude spectrum: $a_n(\omega_n)$, where $\omega_n = n\pi/L$.

From Fourier Series to the Fourier Integral

- Fourier series of any periodic function $f_L(x)$ of period $2L$:

$$f_L(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos \omega_n x + b_n \sin \omega_n x) \quad \omega_n = n\pi/L$$

- $L \rightarrow \infty$

- If we insert a_n and b_n from the Euler formulas

$$\begin{aligned} f_L(x) &= \frac{1}{2L} \int_{-L}^L f_L(v) dv \\ &+ \frac{1}{L} \sum_{n=1}^{\infty} \left[\cos \omega_n x \int_{-L}^L f_L(v) \cos \omega_n v dv + \sin \omega_n x \int_{-L}^L f_L(v) \sin \omega_n v dv \right] \end{aligned}$$

- We now set

$$\begin{aligned} \Delta\omega &= \omega_{n+1} - \omega_n = (n+1)\frac{\pi}{L} - \frac{n\pi}{L} = \frac{\pi}{L} \\ \therefore \frac{1}{L} &= \frac{\Delta\omega}{\pi} \end{aligned}$$

$$\begin{aligned}
f_L(x) &= \frac{1}{2L} \int_{-L}^L f_L(v) dv \\
&+ \frac{1}{\pi} \sum_{n=1}^{\infty} \left[(\cos \omega_n x) \Delta \omega \int_{-L}^L f_L(v) \cos \omega_n v dv + (\sin \omega_n x) \Delta \omega \int_{-L}^L f_L(v) \sin \omega_n v dv \right]
\end{aligned}$$

- We now let $L \rightarrow \infty$ and assume that the resulting nonperiodic function

$$f(x) = \lim_{L \rightarrow \infty} f_L(x).$$

is absolutely integrable on the x -axis; that is, the following limits exist:

$$\int_{-\infty}^{\infty} |f(x)| dx = \lim_{a \rightarrow -\infty} \int_a^0 |f(x)| dx + \lim_{b \rightarrow \infty} \int_b^0 |f(x)| dx \leq M \quad (1)$$

- As $L \rightarrow \infty$, $1/L \rightarrow 0$.

$$\frac{1}{2L} \int_{-L}^L f_L(v) dv \rightarrow 0$$

- Also, $\Delta \omega = \pi/L \rightarrow 0$.

$$f(x) = \frac{1}{\pi} \int_0^{\infty} \left[\cos \omega x \int_{-\infty}^{\infty} f(v) \cos \omega v dv + \sin \omega x \int_{-\infty}^{\infty} f(v) \sin \omega v dv \right] d\omega \quad (2)$$

- If we introduce the notations

$$A(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \cos \omega v dv, \quad B(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \sin \omega v dv \quad (3)$$

- Fourier integral

$$\therefore f(x) = \int_0^{\infty} [A(\omega) \cos \omega x + B(\omega) \sin \omega x] d\omega \quad (4)$$

Theorem 1 [Fourier integral]

If $f(x)$ is piecewise continuous in every finite integral and has a right-hand derivative and a left-hand derivative at every point and if the integral (1) exists, then $f(x)$ can be represented by a Fourier integral (4). At a point where $f(x)$ is discontinuous the value of the Fourier integral equals the average of the left- and right-hand limits of $f(x)$ at that point.

12.2 Applications of the Fourier integral

12.2.1 Main use of the Fourier integral: Solving differential equations

Example 2. Single pulse, sine integral

$$f(x) = \begin{cases} 1 & \text{if } |x| < 1 \\ 0 & \text{if } |x| > 1 \end{cases}$$

Solution.

- From (3)

$$\begin{aligned}
 A(\omega) &= \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \cos \omega v dv = \frac{1}{\pi} \int_{-1}^1 \cos \omega v dv = \frac{\sin \omega v}{\pi \omega} \Big|_{-1}^1 = \frac{\sin \omega}{\pi \omega} + \frac{\sin \omega}{\pi \omega} \\
 &= \frac{2 \sin \omega}{\pi \omega} \\
 B(\omega) &= \frac{1}{\pi} \int_{-1}^1 \sin \omega v dv = -\frac{\cos \omega v}{\pi \omega} \Big|_{-1}^1 = 0 \\
 \therefore f(x) &= \frac{2}{\pi} \int_0^{\infty} \frac{\cos \omega x \sin \omega}{\omega} d\omega
 \end{aligned} \tag{5}$$

- From (5), Dirichlet's discontinuous factor is obtained.

$$\frac{2}{\pi} \int_0^{\infty} \frac{\cos \omega x \sin \omega}{\omega} d\omega = \begin{cases} \pi/2 & \text{if } 0 \leq x \leq 1 \\ \pi/4 & \text{if } x = 1 \\ 0 & \text{if } x > 1 \end{cases}$$

- If $x = 0$,

$$\int_0^{\infty} \frac{\sin \omega}{\omega} d\omega = \frac{\pi}{2}$$

- **Sine integral**

$$\text{Si}(u) = \int_0^u \frac{\sin \omega}{\omega} d\omega$$

- **Gibbs phenomenon:** With increasing a , the oscillations near $x = \pm 1$ are shifted closer to the points $x = \pm 1$.

$$\frac{2}{\pi} \int_0^a \frac{\cos \omega x \sin \omega}{\omega} d\omega = \frac{1}{\pi} \int_0^a \frac{\sin(\omega + \omega x)}{\omega} d\omega + \frac{1}{\pi} \int_0^a \frac{\sin(\omega - \omega x)}{\omega} d\omega$$

- 1st integral: $\omega + \omega x = t \Rightarrow d\omega/\omega = dt/t$ and $0 \leq \omega \leq a \Rightarrow 0 \leq t \leq (x+1)a$.

- 2nd integral: $\omega - \omega x = -t \Rightarrow d\omega/\omega = dt/t$ and $0 \leq \omega \leq a \Rightarrow 0 \leq t \leq (x-1)a$.

$$\begin{aligned}
 \frac{2}{\pi} \int_0^a \frac{\cos \omega x \sin \omega}{\omega} d\omega &= \frac{1}{\pi} \int_0^{(x+1)a} \frac{\sin t}{t} dt - \frac{1}{\pi} \int_0^{(x-1)a} \frac{\sin t}{t} dt \\
 &= \frac{1}{\pi} \text{Si}[a(x+1)] - \frac{1}{\pi} \text{Si}[a(x-1)]
 \end{aligned}$$

12.2.2 Fourier Cosine and Sine Integrals

- If $f(x)$ is an even function,

$$\begin{aligned}
 A(\omega) &= \frac{2}{\pi} \int_0^{\infty} f(v) \cos \omega v dv, \\
 B(\omega) &= \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \sin \omega v dv = 0
 \end{aligned} \tag{6}$$

- Fourier cosine integral

$$f(x) = \int_0^\infty A(\omega) \cos \omega x d\omega \quad (f \text{ even}) \quad (7)$$

- If $f(x)$ is odd,

$$\begin{aligned} A(\omega) &= \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \cos \omega v dv = 0, \\ B(\omega) &= \frac{2}{\pi} \int_0^{\infty} f(v) \sin \omega v dv \end{aligned} \quad (8)$$

- Fourier sine integral

$$f(x) = \int_0^\infty B(\omega) \sin \omega x d\omega \quad (9)$$

12.2.3 Evaluation of Integrals

Example 3. Laplace integrals

Find the Fourier cosine and sine integrals of

$$f(x) = e^{-kx} \quad (x > 0, k > 0)$$

Solution.

- (a) From (6),

$$A(\omega) = \frac{2}{\pi} \int_0^\infty e^{-kv} \cos \omega v dv$$

- Evaluating the above integrals by integration by parts,

$$\int e^{-kv} \cos \omega v dv = \frac{1}{\omega} e^{-kv} \sin \omega v + \frac{k}{\omega} \int e^{-kv} \sin \omega v dv$$

- By integration by parts,

$$\int e^{-kv} \sin \omega v dv = -\frac{1}{\omega} e^{-kv} \cos \omega v - \frac{k}{\omega} \int e^{-kv} \cos \omega v dv$$

$$\int e^{-kv} \cos \omega v dv = \frac{1}{\omega} e^{-kv} \sin \omega v - \frac{k}{\omega^2} e^{-kv} \cos \omega v - \frac{k^2}{\omega^2} \int e^{-kv} \cos \omega v dv$$

- By multiplying ω^2 on both sides,

$$(\omega^2 + k^2) \int e^{-kv} \cos \omega v dv = e^{-kv} (\omega \sin \omega v - k \cos \omega v)$$

$$\therefore \int e^{-kv} \cos \omega v dv = \frac{e^{-kv}}{\omega^2 + k^2} (\omega \sin \omega v - k \cos \omega v)$$

$$A(\omega) = \frac{2}{\pi} \cdot \frac{e^{-kv}}{\omega^2 + k^2} (\omega \sin \omega v - k \cos \omega v) \Big|_0^\infty$$

$$\begin{aligned}
\therefore A(\omega) &= \frac{2k/\pi}{\omega^2 + k^2} \\
\therefore f(x) = e^{-kx} &= \frac{2k}{\pi} \int_0^\infty \frac{\cos \omega x}{k^2 + \omega^2} d\omega \quad (x > 0, k > 0) \\
\int_0^\infty \frac{\cos \omega x}{k^2 + \omega^2} d\omega &= \frac{\pi}{2k} e^{-kx} \quad (x > 0, k > 0)
\end{aligned} \tag{10}$$

- (b) From (8),

$$B(\omega) = \frac{2}{\pi} \int_0^\infty e^{-kv} \sin \omega v dv$$

- By integration by parts,

$$\begin{aligned}
\int e^{-kv} \sin \omega v dv &= -\frac{\omega}{k^2 + \omega^2} e^{-kv} \left(\frac{k}{\omega} \sin \omega v + \cos \omega v \right) \\
B(\omega) &= -\frac{2}{\pi} \cdot \frac{\omega}{k^2 + \omega^2} e^{-kv} \left(\frac{k}{\omega} \sin \omega v + \cos \omega v \right) |_0^\infty = \frac{2\omega/\pi}{k^2 + \omega^2} \\
f(x) = e^{-kv} &= \frac{2}{\pi} \int_0^\infty \frac{\omega \sin \omega x}{k^2 + \omega^2} d\omega \\
\int_0^\infty \frac{\omega \sin \omega x}{k^2 + \omega^2} d\omega &= \frac{\pi}{2} e^{-kx} \quad (x > 0, k > 0)
\end{aligned} \tag{11}$$

- (10) and (11) Laplace integrals!