



13 Fourier Transform

13.1 Fourier Cosine and Sine Transforms

- An "integral transform" is a transformation that produces from given functions new functions that depend on a different variable and appear in the form of an integral.
→ Laplace transform, Fourier transform

Fourier Cosine Transforms

- For an even function $f(x)$,

$$(a) \quad f(x) = \int_0^\infty A(\omega) \cos \omega x d\omega \quad \text{where} \quad (b) \quad A(\omega) = \frac{2}{\pi} \int_0^\infty f(v) \cos \omega v dv \quad (1)$$

- We now set $A(\omega) = \sqrt{2/\pi} \cdot \hat{f}_c(\omega)$.
- Fourier cosine transform of $f(x)$:

$$\hat{f}_c(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos \omega x dx \quad (2)$$

- Inverse Fourier cosine transform of $\hat{f}_c(\omega)$:

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_c(\omega) \cos \omega x d\omega \quad (3)$$

- Fourier cosine transform : the process of obtaining the transform $\hat{f}_c(\omega)$ from a given f

Fourier Sine Transforms

- For an odd function $f(x)$,

$$(a) \quad f(x) = \int_0^\infty B(\omega) \sin \omega x d\omega \quad \text{where} \quad (b) \quad B(\omega) = \frac{2}{\pi} \int_0^\infty f(v) \sin \omega v dv \quad (4)$$

- We now set $B(\omega) = \sqrt{2/\pi} \cdot \hat{f}_s(\omega)$.
- Fourier sine transform of $f(x)$:

$$\hat{f}_s(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin \omega x dx \quad (5)$$

- Inverse Fourier sine transform of $\hat{f}_s(\omega)$:

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_s(\omega) \sin \omega x d\omega \quad (6)$$

Example 1. Fourier cosine and Fourier sine transforms

$$f(x) = \begin{cases} k & \text{if } 0 < x < a \\ 0 & \text{if } x > a. \end{cases}$$

$$\begin{aligned}\hat{f}_c(\omega) &= \sqrt{\frac{2}{\pi}} k \int_0^a \cos \omega x dx = \sqrt{\frac{2}{\pi}} k \left(\frac{\sin a\omega}{\omega} \right) \\ \hat{f}_s(\omega) &= \sqrt{\frac{2}{\pi}} k \int_0^a \sin \omega x dx = \sqrt{\frac{2}{\pi}} \cdot \left(-\frac{k}{\omega} \right) \cos \omega x |_0^a = \sqrt{\frac{2}{\pi}} k \left(\frac{1 - \cos a\omega}{\omega} \right)\end{aligned}$$

Example 2. $\hat{f}_c(e^{-x})$.

$$\begin{aligned}\hat{f}_c(e^{-x}) &= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x} \cos \omega x dx = \sqrt{\frac{2}{\pi}} \cdot \frac{e^{-x}}{1 + \omega^2} (-\cos \omega x + \omega \sin \omega x) |_0^\infty \\ &= \frac{\sqrt{2/\pi}}{1 + \omega^2}\end{aligned}$$

13.2 Properties of Fourier Cosine/Sine Transforms

- Linear operations

$$\begin{aligned}\hat{f}_c(af + bg) &= \sqrt{\frac{2}{\pi}} \int_0^\infty [af(x) + bg(x)] \cos \omega x dx \\ &= a \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos \omega x dx + b \sqrt{\frac{2}{\pi}} \int_0^\infty g(x) \cos \omega x dx \\ &= a \hat{f}_c(f) + b \hat{f}_c(g)\end{aligned}$$

$$\begin{aligned}(a) \quad \hat{f}_c(af + bg) &= a \hat{f}_c(f) + b \hat{f}_c(g) \\ (b) \quad \hat{f}_s(af + bg) &= a \hat{f}_s(f) + b \hat{f}_s(g)\end{aligned} \tag{7}$$

Theorem 1 [Cosine and sine transforms of derivatives]

Let $f(x)$ be continuous and absolutely integrable on the x -axis, let $f'(x)$ be piecewise continuous on each finite interval, and let $f(x) \rightarrow 0$ as $x \rightarrow \infty$. Then

$$\begin{aligned}(a) \quad \hat{f}_c[f'(x)] &= \omega \hat{f}_s[f(x)] - \sqrt{\frac{2}{\pi}} f(0) \\ (b) \quad \hat{f}_s[f'(x)] &= -\omega \hat{f}_c[f(x)]\end{aligned} \tag{8}$$

Proof.

$$\begin{aligned}
\hat{f}_c[f'(x)] &= \sqrt{\frac{2}{\pi}} \int_0^\infty f'(x) \cos \omega x dx \\
&= \sqrt{\frac{2}{\pi}} \left[f(x) \cos \omega x \Big|_0^\infty + \omega \int_0^\infty f(x) \cdot \sin \omega x dx \right] \\
&= \omega \hat{f}_s[f(x)] - \sqrt{\frac{2}{\pi}} f(0) \\
\hat{f}_s[f'(x)] &= \sqrt{\frac{2}{\pi}} \int_0^\infty f'(x) \sin \omega x dx \\
&= \sqrt{\frac{2}{\pi}} \left[f(x) \sin \omega x \Big|_0^\infty - \omega \int_0^\infty f(x) \cos \omega x dx \right] \\
&= -\omega \hat{f}_c[f(x)]
\end{aligned}$$

$$\begin{aligned}
(a) \quad \hat{f}_c[f''(x)] &= \omega \hat{f}_s[f'(x)] - \sqrt{\frac{2}{\pi}} f'(0) = -\omega^2 \hat{f}_c[f(x)] - \sqrt{\frac{2}{\pi}} f'(0) \\
(b) \quad \hat{f}_s[f''(x)] &= -\omega \hat{f}_s[f'(x)] = -\omega^2 \hat{f}_s[f(x)] + \sqrt{\frac{2}{\pi}} \omega f(0)
\end{aligned} \tag{9}$$

Example 3. An application of the operational formula (9)

Find the Fourier cosine transform of $f(x) = e^{-ax}$, where $a > 0$. solution)

$$\begin{aligned}
(e^{-ax})'' &= a^2 \cdot e^{-ax} \implies a^2 f(x) = f''(x) \\
a^2 \hat{f}_c(f) &= \hat{f}_c(f'') = -\omega^2 \hat{f}_c(f) - \sqrt{\frac{2}{\pi}} f'(0) = -\omega^2 \hat{f}_c(f) + a \sqrt{\frac{2}{\pi}} \\
(a^2 + \omega^2) \hat{f}_c(f) &= a \sqrt{\frac{2}{\pi}} \\
\therefore \hat{f}_c(e^{-ax}) &= \sqrt{\frac{2}{\pi}} \left(\frac{2}{a^2 + \omega^2} \right) \quad (a > 0)
\end{aligned}$$

13.3 Fourier Transform

Complex Form of the Fourier Integral

- The real Fourier integral is

$$f(x) = \int_0^\infty [A(\omega) \cos \omega x + B(\omega) \sin \omega x] d\omega$$

where

$$A(\omega) = \frac{1}{\pi} \int_{-\infty}^\infty f(v) \cos \omega v dv, \quad B(\omega) = \frac{1}{\pi} \int_{-\infty}^\infty f(v) \sin \omega v dv.$$

- Substituting $A(\omega)$ and $B(\omega)$ into the integral for f , we have

$$\begin{aligned} f(x) &= \frac{1}{\pi} \int_0^\infty \int_{-\infty}^\infty f(v)[\cos \omega v \cos \omega x + \sin \omega v \sin \omega x] dv d\omega \\ &= \frac{1}{\pi} \int_0^\infty \left[\int_{-\infty}^\infty f(v) \cos(\omega x - \omega v) dv \right] d\omega \\ F(\omega) &= \int_{-\infty}^\infty f(v) \cos(\omega x - \omega v) dv \end{aligned}$$

- $F(\omega)$ is an even function, since $\cos \omega(x - v)$ is an even function of ω . For example,

$$\begin{aligned} f(x) &= \int_{-c}^c \cos xt dt = \frac{1}{x} \sin xt \Big|_{-c}^c = \frac{2 \sin cx}{x} \\ f(-x) &= \frac{2 \sin(-cx)}{-x} = \frac{2 \sin cx}{x} = f(x) \\ \therefore f(x) &\text{ is an even function.} \end{aligned}$$

- $f(v)$ does not depend on ω , and we integrate with respect to v .

$$f(x) = \frac{1}{\pi} \int_0^\infty F(\omega) d\omega = \frac{1}{2\pi} \int_{-\infty}^\infty F(\omega) dx$$

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^\infty \left[\int_{-\infty}^\infty f(v) \cos(\omega x - \omega v) dv \right] d\omega \quad (10)$$

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^\infty \left[\int_{-\infty}^\infty f(v) \sin(\omega x - \omega v) dv \right] d\omega = 0 \quad (11)$$

$$G(\omega) = \int_{-\infty}^\infty f(v) \sin(\omega x - \omega v) dv$$

- $G(\omega)$ is an odd function since $\sin(\omega x - \omega v)$ is an odd function of ω .
- (1) + i(2) with $e^{ix} = \cos x + i \sin x$ (Euler formula)

$$f(v) \cos(\omega x - \omega v) + i f(v) \sin(\omega x - \omega v) = f(v) e^{i(\omega x - \omega v)}$$

- *Complex Fourier Integral*

$$\therefore f(x) = \frac{1}{2\pi} \int_{-\infty}^\infty \int_{-\infty}^\infty f(v) e^{i\omega(x-v)} dv d\omega \quad (12)$$

13.4 Fourier Transform and Its Inverse

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty f(v) e^{-i\omega v} dv \right] e^{i\omega x} d\omega$$

- Fourier transform of f :

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty f(x) \cdot e^{-i\omega x} dx \quad (13)$$

- Inverse Fourier transform of $\hat{f}(\omega)$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x} d\omega \quad (14)$$

- Existence of the Fourier Transform (13)

1. $f(x)$ is piecewise continuous on every finite interval.
2. $f(x)$ is absolutely integrable on the x -axis.

Example 4. Fourier transform

Find the Fourier transform of

$$f(x) = \begin{cases} k & \text{if } 0 < x < a \\ 0 & \text{otherwise} \end{cases}$$

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_0^a k e^{-i\omega x} dx = \frac{k}{\sqrt{2\pi}} \left(\frac{e^{-i\omega a} - 1}{-i\omega} \right) = \frac{k(1 - e^{-ia\omega})}{i\omega \sqrt{2\pi}}$$

Example 5. Fourier transform

Find the Fourier transform of e^{-ax^2} , where $a > 0$.

$$\begin{aligned} F(e^{-ax^2}) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp[-ax^2 - ix\omega] dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp \left[-\left(\sqrt{ax} + \frac{i\omega}{2\sqrt{a}} \right)^2 + \left(\frac{i\omega}{2\sqrt{a}} \right)^2 \right] dx \\ &= \frac{1}{\sqrt{2\pi}} \exp \left(-\frac{w^2}{4a} \right) \int_{-\infty}^{\infty} \exp \left[-\left(\sqrt{ax} + \frac{i\omega}{2\sqrt{a}} \right)^2 \right] dx \end{aligned}$$

Solution.

- Set

$$v = \sqrt{ax} + \frac{i\omega}{2\sqrt{a}} \quad \Rightarrow \quad dx = \frac{dv}{\sqrt{a}}$$

$$\therefore I = \int_{-\infty}^{\infty} \exp \left[-\left(\sqrt{ax} + \frac{i\omega}{2\sqrt{a}} \right)^2 \right] dx = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} e^{-v^2} dv$$

- i) Error function

$$\begin{aligned} \operatorname{erf}(x) &= \frac{2}{\sqrt{\pi}} \frac{1}{\sqrt{a}} \int_0^x e^{-v^2} dv \\ \operatorname{erf}(0) &= 0, \quad \operatorname{erf}(\infty) = 1 \end{aligned}$$

$$\int_0^{\infty} e^{-v^2} dv = \frac{\sqrt{\pi}}{2} \quad \Rightarrow \quad \therefore I = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} e^{-v^2} dv = \sqrt{\frac{\pi}{a}}$$

- ii) $(u, v) \Rightarrow (r, \theta)$

$$I = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} e^{-v^2} dv = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} e^{-u^2} du$$

$$I^2 = \frac{1}{a} \int_{-\infty}^{\infty} e^{-u^2} du \int_{-\infty}^{\infty} e^{-v^2} dv = \frac{1}{a} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(u^2+v^2)} dudv$$

$$r^2 = u^2 + v^2, \quad dudv = rdrd\theta$$

$$\begin{aligned} I^2 &= \frac{1}{a} \int_0^{2\pi} \int_0^{\infty} e^{-r^2} r dr d\theta = \frac{1}{a} \int_0^{2\pi} d\theta \int_0^{\infty} e^{-r^2} r dr \\ &= \frac{2\pi}{a} \left(-\frac{1}{2} e^{-r^2} \right) \Big|_0^{\infty} = \frac{\pi}{a} \\ \hat{f}(e^{-ax^2}) &= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\omega^2}{4a}\right) \sqrt{\frac{\pi}{a}} = \frac{1}{\sqrt{2a}} e^{-\omega^2/4a} \end{aligned}$$

13.5 Physical Interpretation: Spectrum

- Spectral representation
- $\hat{f}(\omega)$: the spectral density $\hat{f}(\omega)$ measures the intensity of $f(x)$ int the frequency interval between ω and $\omega + \Delta\omega$.

$$\int_{-\infty}^{\infty} |\hat{f}(\omega)|^2 d\omega \quad \text{total energy}$$

- Harmonic oscillator

$$\begin{aligned} my'' + ky &= 0 \\ my'y'' + ky'y &= 0 \end{aligned}$$

- By integration

$$\frac{1}{2}m(y')^2 + \frac{1}{2}ky^2 = E_0 \quad \text{constant}$$

- General solution is

$$y = a_1 \cos \omega_0 x + b_1 \sin \omega_0 x = c_1 e^{i\omega_0 x} + c_{-1} e^{-i\omega_0 x}, \quad \omega_0^2 = \frac{k}{m}$$

where $c_1 = (a_1 - ib_1)/2$, $c_{-1} = \bar{c}_1 = (a_1 + ib_1)/2$.

- Set $A = c_1 e^{i\omega_0 x}$, $B = c_{-1} e^{-i\omega_0 x}$.

$$y = A + B, \quad \rightarrow \quad v = y' = A' + B' = i\omega_0(A - B)$$

$$E_0 = \frac{1}{2}mv^2 + \frac{1}{2}ky^2 = \frac{1}{2}(i\omega_0)^2(A - B)^2 + \frac{1}{2}k(A + B)^2$$

$$- \omega_0^2 = k/m \quad \rightarrow \quad m\omega_0^2 = k$$

$$E_0 = \frac{1}{2}k[-(A - B)^2 + (A + B)^2] = 2kAB = 2kc_1 e^{i\omega_0 x} c_{-1} e^{-i\omega_0 x} = 2kc_1 c_{-1} = 2kc_1 \bar{c}_1 = 2k|c_1|^2$$

\therefore The energy is proportional to the square of the amplitude $|c_1|$.

13.6 Linearity. Fourier Transformation of Derivatives

Theorem 2 [Linearity of the Fourier transform]

The Fourier transform is a linear operation, that is, for any functions $f(x)$ and $g(x)$ whose Fourier transforms exist and any constants a and b ,

$$\hat{f}(af + bg) = a\hat{f}(f) + b\hat{f}(g)$$

Proof.

$$\begin{aligned}\hat{f}[af(x) + bg(x)] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [af(x) + bg(x)] e^{-i\omega x} dx \\ &= \frac{a}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx + \frac{b}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x) e^{-i\omega x} dx \\ &= a\hat{f}[f(x)] + b\hat{f}[g(x)]\end{aligned}$$

Theorem 3 [Fourier transform of the derivative of $f(x)$]

Let $f(x)$ be continuous on the x -axis and $f(x) \rightarrow 0$ as $|x| \rightarrow \infty$. Furthermore, let $f'(x)$ be absolutely integrable on the x -axis. Then

$$\hat{f}[f'(x)] = i\omega \hat{f}[f(x)] \quad (15)$$

Proof.

$$\begin{aligned}\hat{f}[f'(x)] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f'(x) e^{-i\omega x} dx \\ &= \frac{1}{\sqrt{2\pi}} \left[f(x) e^{-i\omega x} \Big|_{-\infty}^{\infty} + -i\omega \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx \right]\end{aligned}$$

- Since $f(x) \rightarrow 0$ as $|x| \rightarrow \infty$,

$$\begin{aligned}\hat{f}[f'(x)] &= i\omega \hat{f}[f(x)] \\ \hat{f}[f''(x)] &= i\omega \hat{f}[f'(x)] = (i\omega)^2 \hat{f}[f(x)] \\ F[f''(x)] &= -\omega^2 \hat{f}[f(x)]\end{aligned} \quad (16)$$

Example 6. An application of the operational formula (15)

Find the Fourier transform of xe^{-x^2} from Table III, Sec. 10.11

$$\begin{aligned}\hat{f}(xe^{-x^2}) &= \hat{f}\left(-\frac{1}{2}(e^{-x^2})'\right) = -\frac{1}{2}\hat{f}[(e^{-x^2})'] \\ &= -\frac{1}{2}i\omega \hat{f}(e^{-x^2}) = -\frac{i\omega}{2} \cdot \frac{1}{\sqrt{2}}e^{-\omega^2/4} = -\frac{i\omega}{2\sqrt{2}}e^{-\omega^2/4}\end{aligned}$$

13.7 Convolution

- The convolution $f * g$ of functions f and g is defined by

$$h(x) = (f * g)(x) = \int_{-\infty}^{\infty} f(p)g(x-p)dp = \int_{-\infty}^{\infty} f(x-p)g(p)dp \quad (17)$$

- The convolution of functions correspond to the multiplication of their Fourier transforms.

Theorem 4 [Convolution theorem]

Suppose that $f(x)$ and $g(x)$ are piecewise continuous, bounded, and absolutely integrable on the x-axis. Then,

$$\hat{f}(f * g) = \sqrt{2\pi}\hat{f}(f)\hat{f}(g) \quad (18)$$

Proof.

$$\begin{aligned} F(f * g) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(p)g(x-p)e^{-i\omega x}dpdx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(p)g(x-p)e^{-i\omega x}dxdp \end{aligned}$$

- We now take $x - p = q$. $x = p + q$ and $dx = dq$.

$$\begin{aligned} \hat{f}(f * g) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(p)g(q)e^{-i\omega(p+q)}dqdp \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(p)e^{-i\omega p}dp \int_{-\infty}^{\infty} g(q)e^{-i\omega q}dq \\ &= \sqrt{2\pi}\hat{f}(f)\hat{f}(g) \end{aligned}$$

- By taking the inverse Fourier transform in both sides of (18)

$$(f * g)(x) = \int_{-\infty}^{\infty} \hat{f}(\omega)\hat{g}(\omega)e^{i\omega x}d\omega \quad (19)$$

Theorem 5 [Time convolution]

$$\hat{f}^{-1}[F(\omega)G(\omega)] = \int_{-\infty}^{\infty} f(p)g(x-p)dp = f(x) * g(x)$$

where $F(\omega) = \hat{f}(f)$ and $G(\omega) = \hat{f}(g)$.

- If f and g are zeros for $x < 0$,

$$f(x) * g(x) = \int_0^{\infty} f(p)g(x-p)dp.$$

- Moreover, when $x - p < 0$ together with $f = g = 0$ ($x < 0$), $g(x-p) = 0$.

$$f(x) * g(x) = \int_0^x f(p)g(x-p)dp.$$

Theorem 6 [Unilateral time convolution]

If $F(\omega)$ and $G(\omega)$ are the Fourier transforms of two functions $f(x)$ and $g(x)$, which are identically zero for $x < 0$, then,

$$\hat{f}^{-1}[F(\omega)G(\omega)] = \int_0^t f(p)g(x-p)dp$$

Example 7. Find the inverse of the Fourier transform

$$F(\omega) = \frac{1}{2\pi(6 + 5i\omega - \omega^2)}.$$

- Since $-\omega^2 = (i\omega)^2$,

$$F(\omega) = \frac{1}{2\pi\{6 + 5i\omega + (i\omega)^2\}} = \frac{1}{2\pi} \cdot \frac{1}{2+i\omega} \cdot \frac{1}{3+i\omega}.$$

- From Table III. Sec. 10.11,

$$g(x) = \begin{cases} e^{-2x} & x > 0 \\ 0 & x < 0 \end{cases} \quad \text{and} \quad h(x) = \begin{cases} e^{-3x} & x > 0 \\ 0 & x < 0 \end{cases}$$

$$f(x) = \int_0^x e^{-2p} e^{-3(x-p)} dp = e^{-3x} \int_0^x e^p dp = \begin{cases} e^{-2x} - e^{-3x} & x > 0 \\ 0 & x < 0 \end{cases}$$