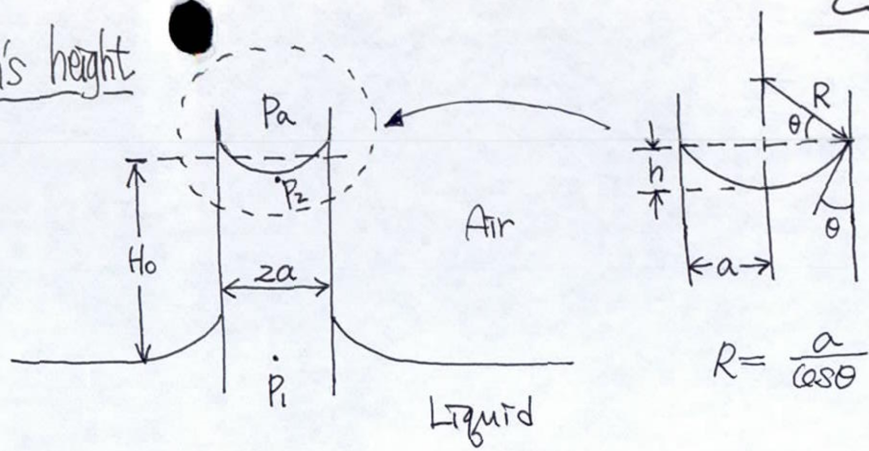
CAPILLARY

Jurin's height



$$R = \frac{a}{\cos\theta}$$

- At equilibrium (maximum column height)

$$p_1 = p_a = p_2 + \rho g H_0$$

at interface $\quad p_a - p_2 = \frac{2\sigma}{R} = \frac{2\sigma\cos\theta}{a}$

$$\therefore \rho g H_0 = \frac{2\sigma\cos\theta}{a}$$

$$\boxed{H_0 = 2\left(\frac{\sigma}{\rho g}\right)\frac{\cos\theta}{a} = 2\frac{\Delta_c^2}{a}\cos\theta}$$

$\Delta_c = \left(\frac{\sigma}{\rho g}\right)^{1/2}$: capillary length

* Dynamics of capillary rise (Washburn)

- Velocity profile $\Leftarrow$ Poiseuille profile

$$U = \frac{dH}{dt} = \frac{a^2}{8\mu}\frac{\Delta p}{H}$$

$$\Delta p = \frac{2\sigma\cos\theta}{a} - \rho g H$$

Then $\quad \frac{dH}{dt} = \frac{a^2}{8\mu}\left(\frac{2\sigma\cos\theta}{aH} - \rho g\right)$

$$\frac{\mu}{\sigma}\frac{dH}{dt} = \frac{1}{8}\left[2\left(\frac{a}{H}\right)\cos\theta - \frac{\rho g a^2}{\sigma}\right]$$

Recalling that $\quad H_0 = 2\left(\frac{\sigma}{\rho g}\right)\frac{\cos\theta}{a}$,

we get

$$dt = \frac{8\mu H_0}{\rho g a^2}\left(\frac{H/H_0}{1 - H/H_0}\right) d\left(\frac{H}{H_0}\right)$$

Upon integrating,

$$t = \tau\left(\ln\frac{1}{1 - H/H_0} - \frac{H}{H_0}\right) \quad \cdots ①$$

where $\tau = 8\frac{\mu}{\sigma}\frac{H_0}{B_o} = 8\frac{\mu H_0}{\rho g a^2}$

(Bond numb. $B_o = \frac{\rho g a^2}{\sigma}$)

$$H = \left(\frac{\rho g a^2 H_0}{4\mu}\right)^{1/2}\sqrt{t} \quad \leftarrow \ln\frac{1}{1-z} = z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots \quad (|z|<1)$$

as $t \to \infty$

$H \to H_0$

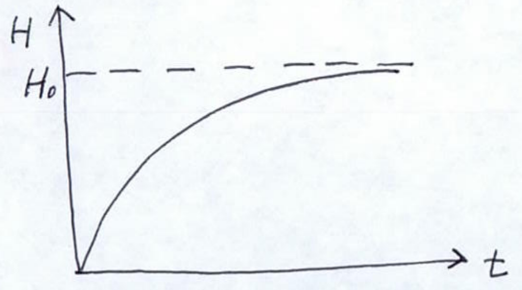
Then Eq. ① becomes

$$\frac{t}{\tau} \approx \ln \frac{1}{1 - H/H_0}$$

$$\exp\left(\frac{t}{\tau}\right) \approx \frac{1}{1 - H/H_0}$$

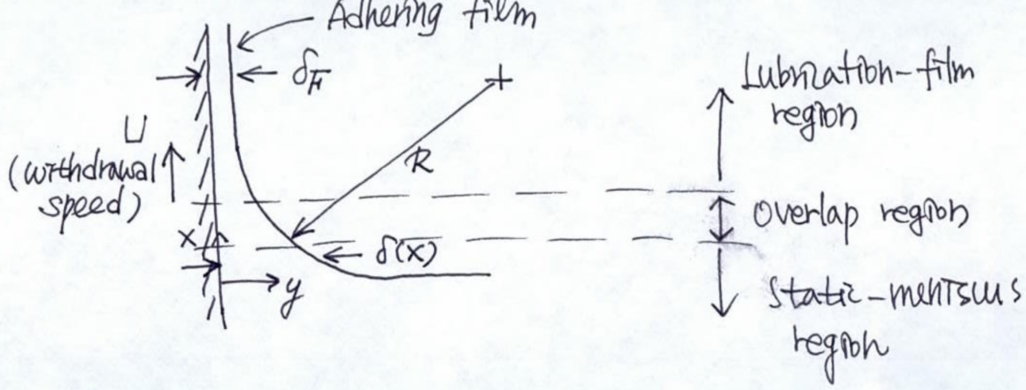
$$1 - \frac{H}{H_0} \approx \exp\left(-\frac{t}{\tau}\right)$$

$$\frac{H}{H_0} \approx 1 - \exp\left(-\frac{t}{\tau}\right)$$



Coating flows Landau-Levich

: Basis of microbubble transport in microchannel.



- In lubrication region

$$-\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2} = 0 \quad \& \quad -\Delta p = \frac{\sigma}{R} \approx \sigma \delta''$$

$$\therefore \sigma \frac{d^3 \delta}{dx^3} + \mu \frac{\partial^2 u}{\partial y^2} = 0$$

$$\rightarrow \eta^3 \frac{d^3 \eta}{d\xi^3} = 1 - \eta$$

$$\eta = \frac{\delta}{\delta_F}, \quad \xi = \frac{x}{\delta_F}\left(\frac{3\mu U}{\sigma}\right)^{1/3}$$

. smooth merging of meniscus curvature.

$$\left(\frac{d^2\eta}{d\xi^2}\right)^{\text{menis}}_{\eta \to 0} = \left(\frac{d^2\eta}{d\xi^2}\right)_{\eta \to \infty} = \alpha$$

$\Leftarrow$

$$\frac{\delta_f}{\Delta_c} = 0.946\, Ca^{2/3}$$

$$\frac{\delta_f}{R} = 0.643\,(3Ca)^{2/3}$$

Contact line dynamics

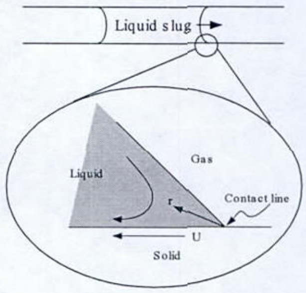
- Singularity at the contact line due to no-slip boundary condition
  - for Stokes flow, the viscous dissipation *f (per volume)*

    $$f = 4\mu \frac{U^2}{r^2}\left(a\cos\varphi - b\sin\varphi\right)^2$$
  - *shear stress* ~ $1/r$
  - Dissipation for the wedge

    $$\Phi_w = \int_{\lambda}^{\Lambda}\int_{0}^{\theta} f r\,dr\,d\theta$$

    λ: cutoff length (molecular scale)

Example



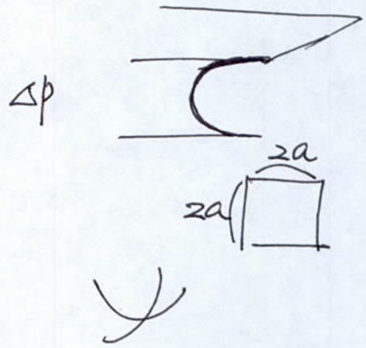
* Contact line singularity

C. Huh and L.E. Scriven, JCIS, vol. 35. 85-101 (1971)

Whitesides. JPC, B

$$fRe = 2\left(-\frac{dp}{dx}\right)\frac{D_h}{\rho U^2}\cdot\frac{\cancel{\rho}UD_h}{\mu} = C$$

$$2\left(\frac{\Delta p}{L}\right)\frac{D_h^2}{\mu U} = C$$



$$U = \frac{2}{C}\left(\frac{\Delta p}{L}\right)\frac{D_h^2}{\mu}$$

$$= \frac{2}{C}\,\frac{D_h^2}{\mu L}\cdot\frac{2\sigma\cos\theta}{a}$$

$$\Delta p = 2\frac{\sigma\cos\theta}{R}$$

$$= \frac{2}{C}\,\frac{4a^2}{\mu L}\,\frac{2\sigma\cos\theta}{\cancel{a}}$$

$$D_h = \frac{4A}{P} = \frac{\cancel{4}(2a)(2a)}{(2a)\cdot\cancel{4}} = 2a$$

$$U = \frac{16}{C}\,\frac{a\sigma\cos\theta}{\mu L} \qquad C = 57$$

$$= \frac{16}{57}\,\frac{a\sigma\cos\theta}{\mu L}$$

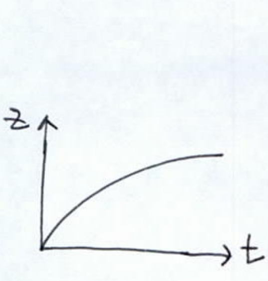
$$U \cong \frac{a\sigma\cos\theta}{3.6\mu L}$$

$$\frac{dz}{dt} = \frac{a\sigma\cos\theta}{3.6\mu z}$$

$$z\,dz = \frac{a\sigma\cos\theta}{3.6\mu}\,dt$$

$$\frac{1}{2}z^2 = \frac{a\sigma\cos\theta}{3.6\mu}\,t$$

$$z = \left(\frac{1.8\,a\sigma\cos\theta}{\mu}\,t\right)^{1/2}$$



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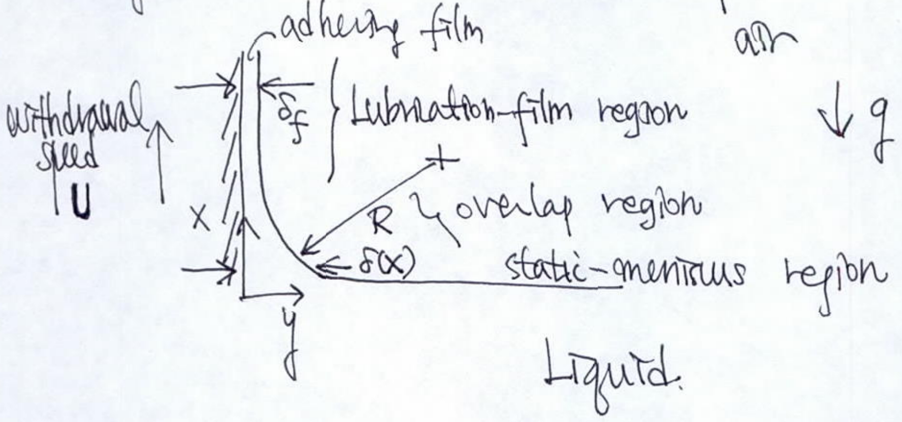
Coating flows (Landau–Levich prob.)



$\boxed{\delta_f \sim U\,?}$

Assumptions

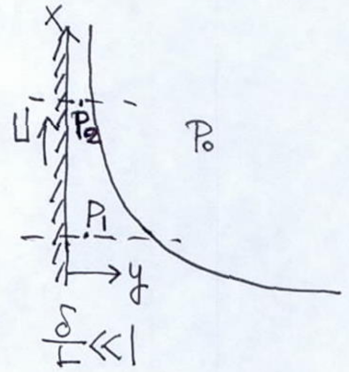
i) steady : $\tau \gg \frac{R^2}{\nu}$

ii) $Re = \frac{\rho U R}{\mu} \ll 1$

iii) gravity drainage neglected

$$\rho g R \ll \frac{\sigma}{R}$$

$$Bo = \frac{\rho g R^2}{\sigma} \ll 1.$$



N.S. $\Rightarrow$ $0 = -\nabla p + \mu \nabla^2 \bar{u}$.

$\delta/L \ll 1$: $\quad 0 = -\frac{dp}{dx} + \mu \frac{\partial^2 u}{\partial y^2}$

Young-Laplace Eq : $p_1 - p_0 = -\frac{\sigma}{R} \approx -\sigma \delta''$

$$-\frac{dp}{dx} = - \frac{d}{dx}(p - p_0)$$

$$= -\frac{d}{dx}\left(-\sigma \frac{d^2\delta}{dx^2}\right) = \sigma \frac{d^3\delta}{dx^3}$$

$\therefore$ N.S. $\Rightarrow$ $\sigma \frac{d^3\delta}{dx^3} + \mu \frac{\partial^2 u}{\partial y^2} = 0$.

B.C. $\quad y = 0 :\ u = U, \qquad y = \delta(x) :\ \frac{\partial u}{\partial y} = 0$.

$$u = U - \frac{\sigma}{\mu}\frac{d^3\delta}{dx^3}\left(\frac{y^2}{2} - \delta y\right)$$

- volume flow rate per width Q

$$Q = \int_0^{\delta(x)} u\,dy = U\delta + \frac{\sigma}{\mu}\frac{d^3\delta}{dx^3}\frac{\delta^3}{3}.$$

but we know that $Q = U\delta_f$ far above the static-meniscus region. Thus eliminating Q, we get

$$\delta^3\frac{d^3\delta}{dx^3} + \left(\frac{3\mu U}{\sigma}\right)\delta = \left(\frac{3\mu U}{\sigma}\right)\delta_f$$

introducing $\eta = \frac{\delta}{\delta_f}$, $\xi = \frac{x}{\delta_f}\left(\frac{3\mu U}{\sigma}\right)^{1/3}$

$$\eta^3\frac{d^3\eta}{d\xi^3} = 1-\eta \quad \cdots (*)$$

- $\therefore$ 3 integration constants, δ_f $\rightarrow$ 4 conditions.

i) $\xi \to \infty$: $\eta \to 1$

$(*) \rightarrow \frac{d^3\eta}{d\xi^3} = 1-\eta$. $\quad \eta''' + \eta = 1$.

solving: $\eta_h = e^{\lambda\xi}$. $\quad \lambda^3 + 1 = 0$.

$(\lambda+1)(\lambda^2 - \lambda + 1) = 0$

$\lambda = -1$, $\lambda = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1}{2} \pm \frac{\sqrt{3}}{2}i$

$$\left[\begin{array}{l}\eta_h = c_1 e^{-\xi} + e^{\frac{1}{2}\xi}\left(c_2\cos\frac{\sqrt{3}}{2}\xi + c_3\sin\frac{\sqrt{3}}{2}\xi\right)\\ \eta_p = 1\end{array}\right.$$

- ii) $\therefore \eta = 1 + Ae^{-\xi}$. $\quad c_2 = c_3 = 0$ to prevent exponential growth.

iii) 'A' may be arbitrarily chosen because (*) is invariant to a shift in the origin of ξ. "A=1"

$$\eta = 1 + e^{-\xi}$$

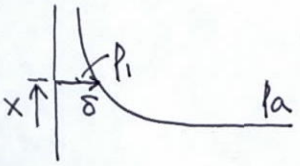
iv) · Smoothly merging the solution of (*) valid in the lubrication-film region into that for the static-meniscus region.

· Matched asymptotics

$$\text{curvature}_{\text{lubricated film}} = \text{curvature}_{\text{static-meniscus}}.$$

$$\left(\frac{d^2\eta}{d\xi^2}\right)^{\text{menis}}_{\eta\to 1} = \left(\frac{d^2\eta}{d\xi^2}\right)^{\text{lubrication}}_{\eta\to\infty} = \alpha$$

Static meniscus curvature



$$\begin{cases} P_a - P_1 = \frac{\sigma}{R} = \sigma\frac{\delta''}{(1+\delta'^2)^{3/2}} \\ P_1 + \rho g x = P_a \end{cases}$$

$$\sigma\frac{\delta''}{(1+\delta'^2)^{3/2}} - \rho g x = 0 \quad \cdots \quad (12)$$

integrating once,

$$\frac{\delta'}{(1+\delta'^2)^{1/2}} = \frac{\rho g x^2}{2\sigma} \underbrace{-1}_{\text{as } x\to 0,\ \delta'\to -\infty}.$$

transition to lubrication regime :

$$\delta' \to 0 \quad \text{or} \quad \frac{\rho g x^2}{2\sigma} \Rightarrow 1. \quad x \to \left(\frac{2\sigma}{\rho g}\right)^{1/2}$$

$$(12) \to \sigma\frac{\delta''}{(1+0)^{3/2}} = \rho g\left(\frac{2\sigma}{\rho g}\right)^{1/2} : \quad \delta'' \to \left(\frac{2\rho g}{\sigma}\right)^{1/2} = \frac{\sqrt{2}}{\ell_c}$$

$$l_c = \sqrt{\frac{\sigma}{\rho g}}$$

In terms of reduced variables

$$\left(\frac{d^2\eta}{d\xi^2}\right)^{\text{menis}}_{\eta\to 1} = \sqrt{2}\,\frac{\delta_f}{l_c}\,(3Ca)^{-2/3} = \alpha \quad \cdots (15)$$

How to obtain α : numerical integration of

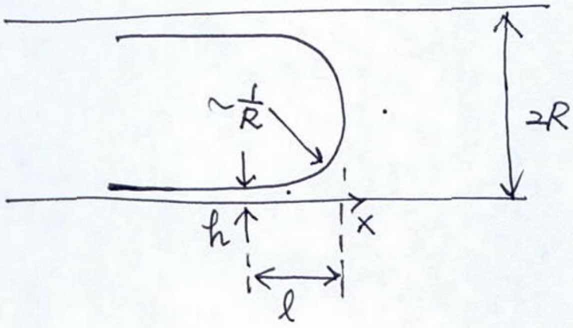
$$\eta^3 \frac{d^3\eta}{d\xi^3} = 1-\eta$$

$$\Rightarrow \alpha = 0.643$$

$$\therefore (15) \rightarrow \frac{\delta_f}{l_c} = 0.946\, Ca^{2/3}$$

$$\frac{\delta_f}{R} = 0.643\,(3Ca)^{2/3}$$

Pressure drop.



$$\frac{d^2h}{dx^2} \sim \frac{1}{R}$$

$$\frac{\delta}{l^2} \sim \frac{1}{R}. \qquad l^2 \sim R\delta. \qquad l \sim \sqrt{R\delta}$$

we know that $\delta \sim R\,Ca^{2/3}$

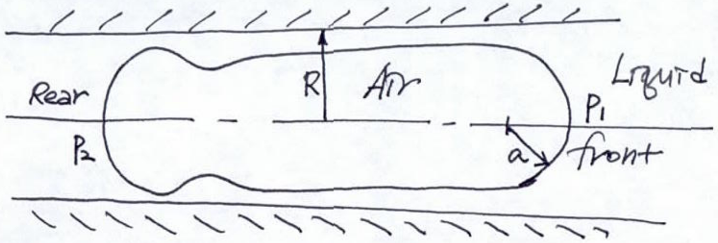
$$l \sim R\,Ca^{1/3}.$$

Pressure drop by viscous dissipation:

$$\Delta P \sim \frac{\mu U}{l} \sim \frac{\mu U}{R} Ca^{-1/3} \sim \frac{\sigma}{R} Ca^{2/3}$$

* Bubble moving in microcapillary

(Bretherton Problem, JFM vol. 10, 166 - 188, 1961)



Similar approach to "coating flow" problem.

- total pressure drop to drive the bubble

$$p_2 - p_1 = 9.40\, Ca^{2/3} \frac{\sigma}{a} \sim \sigma^{1/3} \qquad (a \sim R)$$

$$\sigma \downarrow - \Delta p \downarrow$$

Equivalent single-phase (liquid) pressure drop $\Delta p \sim \mu \frac{U \tilde{l}}{a^2} \sim \frac{\sigma}{a} Ca^{2/3}$

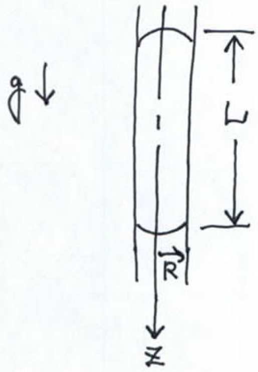
$$\tilde{l} \sim \frac{\sigma}{\mu U} Ca^{2/3} = a\, Ca^{-1/3}$$

$$Ca \sim 10^{-2} \sim 10^{-4}$$

$\tilde{l} \gg a$ $\therefore$ very high pressure required.

Slug motion in capillary

(1) Simplest case : Poiseuille flow



N-S eqn.

$$0 = -\frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right) + \rho g$$

$$\mu\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right) = \frac{\partial}{\partial z}(p - \rho g z)$$

$$\frac{\mu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = \frac{\partial}{\partial z}(p-\rho g z) = \frac{dP}{dz}$$

$$\mu\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = r\frac{dP}{dz}$$

$$\mu r\frac{\partial u}{\partial r} = \frac{r^2}{2}\frac{dP}{dz} + C_1$$

at $r=0$, $\frac{\partial u}{\partial r} = 0 \rightarrow C_1 = 0$.

$$\mu\frac{\partial u}{\partial r} = \frac{r}{2}\frac{dP}{dz}$$

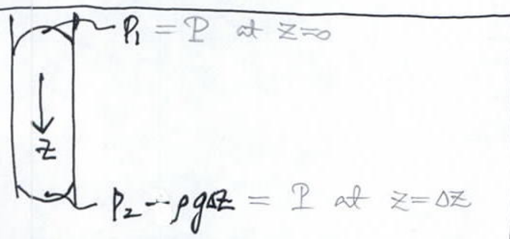
$$\mu u = \frac{r^2}{4}\frac{dP}{dz} + C_2$$

at $r=R$. $u=0$

$$0 = \frac{R^2}{4}\frac{dP}{dz} + C_2 \quad \therefore C_2 = -\frac{R^2}{4}\frac{dP}{dz}$$

$$u = \frac{1}{\mu}\left(-\frac{dP}{dz}\right)\frac{1}{4}(R^2 - r^2)$$

*



$\Delta P = (P_2 - P_1) - \rho g \Delta z$

$\frac{\Delta P}{\Delta z} = -\rho g$ must be very long!

if $|P_2 - P_1| \ll \rho g z$

$(P_2 - P_1)$ is caused by capillary effect.

$$\therefore u(r) = \frac{\rho g}{4\mu}(R^2 - r^2)$$

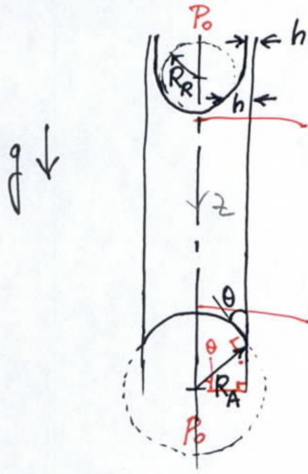
$$U = \frac{1}{\pi R^2}\int_0^R u(2\pi r)\,dr = \frac{1}{R^2}\int_0^{R^2} u\,dr^2$$

$$= \frac{\rho g}{4\mu}\frac{1}{R^2}\left[R^2\eta - \frac{1}{2}\eta^2\right]_0^{R^2} = \frac{\rho g}{4\mu}\frac{1}{R^2}\left(R^4 - \frac{1}{2}R^4\right)$$

$$= \frac{\rho g}{8\mu}R^2 \quad : \text{Average velocity}$$

(2) Fully wetting liquid slug in dry tube

(Driving force = gravity)



P_1 : $P_1 = P_0 - \frac{2\sigma}{R_R}$. $R_R = R - h$

$\therefore P_1 = P_0 - \frac{2\sigma}{R-h}$

$P_2 - \rho g L$: $P_2 = P_0 - \frac{2\sigma}{R_A}$. $R_A \cos\theta = R$

$\therefore P_2 = P_0 - \frac{2\sigma\cos\theta}{R}$

θ: advancing cont. angle

$$u = \frac{1}{4\mu}\left(-\frac{dP}{dz}\right)(R^2 - r^2) \qquad P = p - \rho g z$$

$$\frac{dP}{dz} = \frac{dp}{dz} - \rho g$$

$$\frac{dP}{dz} = \frac{\Delta P}{L} = \frac{1}{L}\left(p_2 - \rho g L - p_1\right)$$

$$= \frac{1}{L}\left(p_0 - \frac{2\sigma\cos\theta}{R} - \rho g L - p_0 + \frac{2\sigma}{R-h}\right)$$

$$= \frac{1}{L}\left(-\frac{2\sigma\cos\theta}{R} - \rho g L + \frac{2\sigma}{R-h}\right)$$

$$U = \frac{R^2}{8\mu}\left(-\frac{dP}{dz}\right)$$

$$= \frac{R^2}{8\mu}\frac{1}{L}\left(\frac{2\sigma\cos\theta}{R} + \rho g L - \frac{2\sigma}{R-h}\right)$$

OR $$\frac{8\mu L}{R^2}U = \rho g L + \frac{2\sigma\cos\theta}{R} - \frac{2\sigma}{R-h}$$: Bico & Quéré (2) JCIS

cf.) h = tube radius − actual meniscus radius

(2-1) Limiting case for (2) : $\theta \ll 1$

$$\cos\theta \simeq 1 - \frac{\theta^2}{2}$$

$$\frac{1}{R-h} = \frac{1}{R}\left(\frac{1}{1-h/R}\right) = \frac{1}{R}\left(1-\frac{h}{R}\right)^{-1} \approx \frac{1}{R}\left(1+\frac{h}{R}\right)$$

Then

$$\frac{8\mu L}{R^2} U = \rho g L + \frac{2\sigma}{R}\left(1-\frac{\theta^2}{2}\right) - 2\sigma \frac{1}{R}\left(1+\frac{h}{R}\right)$$

$$\frac{8\mu L}{R^2} U = \rho g L - \frac{2\sigma}{R}\left(\frac{\theta^2}{2} + \frac{h}{R}\right)$$ Bico eq [3]

For wetting liquids, $\frac{h}{R} = fn(Ca)$ ← Bretherton's law

$(Ca = \frac{\mu U}{\sigma})$. $\theta = fn(Ca)$ ← Hoffman's law

- Bretherton's law ; $\frac{h}{R} = 2.9 \frac{h_\infty}{R} = 3.88\, Ca^{2/3}$.
- Hoffman's law : $\theta = (6\Gamma Ca)^{1/3}$.

Note that Hoffman's law accounts for the viscous dissipation near contact line !

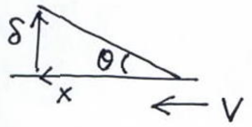
Using Bretherton's and Hoffman's laws,

$$\frac{8\mu L}{R^2} U = \rho g L - \frac{2\sigma}{R}\left[\frac{1}{2}(6\Gamma)^{2/3} Ca^{2/3} + 3.88\, Ca^{2/3}\right]$$

$$\frac{8\mu L}{R^2} U = \rho g L - \frac{2\sigma}{R}\beta\, Ca^{2/3}$$

where $\beta = \frac{1}{2}(6\Gamma)^{2/3} + 3.88$.

* Derivation of Hoffman – Tanner's law
for wetting liquids ($\theta \ll 1$)



$$\tau \sim \mu \frac{V}{\delta} \qquad \tan\theta = \frac{\delta}{x}. \qquad \delta \simeq x\theta$$

$$\sim \mu \frac{V}{x\theta}$$

viscous force per unit depth

$$f \sim \int \tau \, dx \sim \mu \int_{\lambda}^{R} \frac{V}{\theta x} \, dx$$

[λ : cutoff length (molecular)
R : length scale
(tube radius)]

$$\sim \frac{\mu V}{\theta} \ln\left(\frac{R}{\lambda}\right)$$

capillary pressure caused by surface bending
(otherwise, $\theta_{eq} = 0$)

$$\frac{\sigma \cos\theta}{a} \cong \frac{\sigma}{a}\left(1 - \frac{\theta^2}{2}\right)$$

$\theta_{eq} = 0$
(taken care of by Poiseuille flow)

interface bending effect
: reducing driving force (capillary)
~ related to wedge dissipation

capillary force per length : $\sim \sigma\theta^2$

$\therefore$ viscous force $\sim$ capillary force

$$\frac{\mu V}{\theta} \ln\left(\frac{R}{\lambda}\right) \sim \sigma\theta^2$$

$$\theta^3 \sim \frac{\mu V}{\sigma} \ln\left(\frac{R}{\lambda}\right) \sim \Gamma Ca.$$