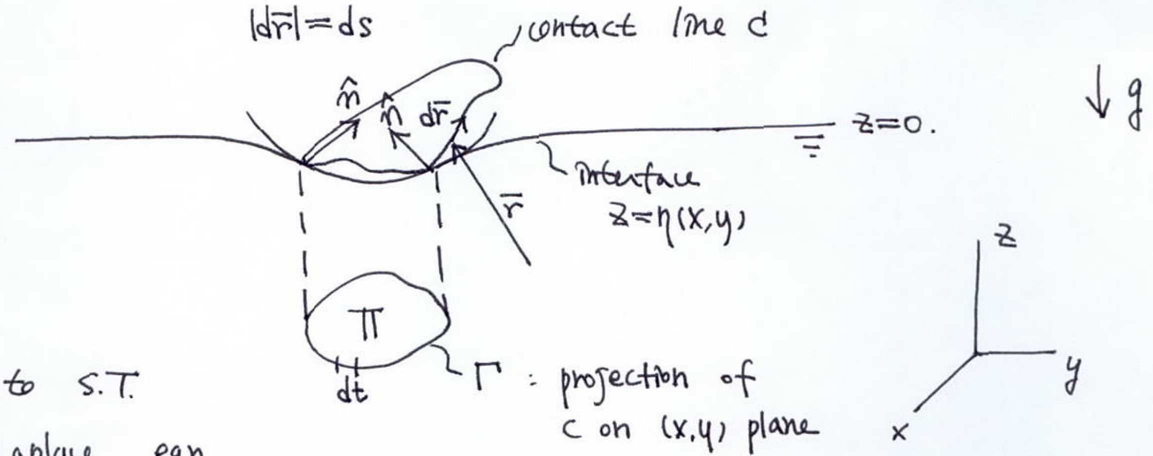
Force on a partly submerged body



* Force due to S.T.

Young-Laplace eqn.

$$p_i - p_o = -\sigma\kappa$$

$$p_i = p_o - \rho g \eta$$

$$\rho g \eta(x,y) = -\sigma \nabla\cdot\hat{n} \qquad \cdots \quad (1)$$

curvature $\kappa = -\nabla\cdot\hat{n}$

$\hat{n}$: unit normal vector to the interface pointing out of the liquid.

$$\hat{n}(x,y) = n_1\hat{i} + n_2\hat{j} + n_3\hat{k}$$

$$= \frac{1}{\sqrt{1+\eta_x^2+\eta_y^2}}\left(-\eta_x\hat{i} - \eta_y\hat{j} + \hat{k}\right)$$

$$\nabla\cdot\hat{n} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\hat{n}$$

$$= \frac{\partial n_1}{\partial x} + \frac{\partial n_2}{\partial y}$$

integrating (1) over (x,y) plane outside Γ,

$$-\int_{R^2/\Pi} \sigma\nabla\cdot\hat{n}\, dA = \int_{R^2/\Pi} \rho g \eta\, dA$$

$$\text{LHS} = \sigma \int_{\Gamma} \hat{\nu}\cdot\hat{n}\, dt \qquad \text{by divergence theorem}$$

$$\text{RHS} = \rho g \cdot (\text{volume between the interface } z=\eta \text{ and the surface } z=0)$$

$$= W_M \quad (\text{weight of the liquid displaced by the meniscus})$$

We will show that $\text{LHS} = \hat{k}\cdot\bar{F}_{st}$: vertical component of surface tension force

· surface tension force on ds $\perp d\bar{r}$, $\perp \hat{n}$

$$d\bar{F}_{st} = \sigma\, d\bar{r}\times\hat{n}$$

$$\bar{F}_{st} = \sigma \int_C \frac{d\bar{r}}{ds}\times\hat{n}\, ds$$

$$= \sigma \int_C \dot{\bar{r}}(s)\times\hat{n}\, ds. \qquad \dot{\bar{r}} = \frac{d\bar{r}}{ds}$$

$$\hat{k}\cdot\bar{F}_{st} = \sigma\int_C \hat{k}\cdot\dot{\bar{r}}\times\hat{n}\, ds \underset{\text{vector identity}}{=} \sigma\int_C \hat{n}\cdot\underbrace{\hat{k}\times\dot{\bar{r}}\, ds}$$

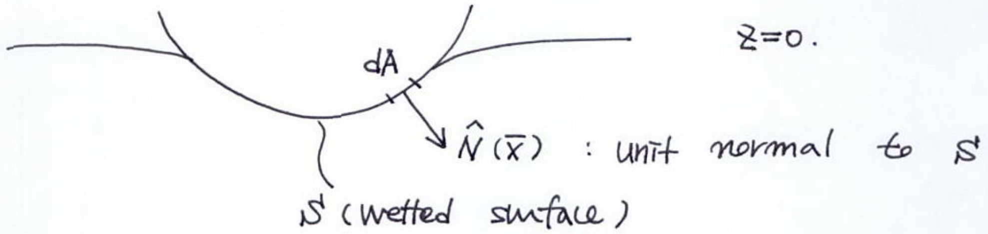
$$= \hat{k}\times d\bar{r}$$

$$= \hat{\nu}\, dt$$

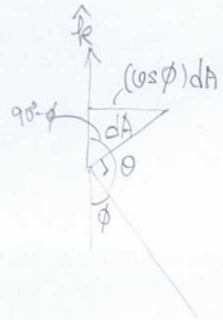
$\hat{\nu}$: inward normal vector of Γ

$$\therefore\ \hat{k}\cdot\bar{F}_{st} = \sigma\int_{\Gamma} \hat{n}\cdot\hat{\nu}\, dt = W_M.$$

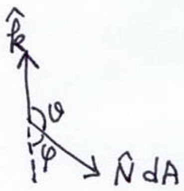
* Force due to hydrostatic pressure



$$\overline{F}_p = \int_S \rho g z \hat{N}(\bar{x})\, dA$$



vertical component of $\overline{F}_p$



$$\hat{k}\cdot\overline{F}_p = \rho g \int_S z \underbrace{\hat{k}\cdot\hat{N}(\bar{x})\, dA}_{= \pm dx\,dy}$$

$$\hat{k}\cdot\hat{N} dA = -\cos\varphi\, dA$$

for depressed interface $\hat{N}(x) \searrow$: $\hat{k}\cdot\hat{N} dA = -dx\,dy$

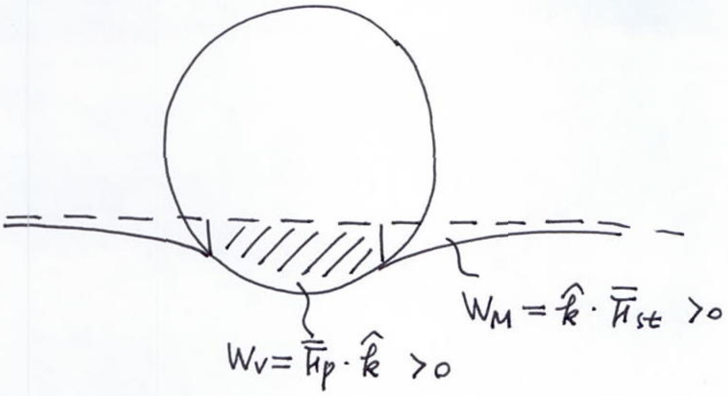
$$\hat{k}\cdot\overline{F}_p = -\rho g \int_\Pi z(x,y)\, dx\,dy$$

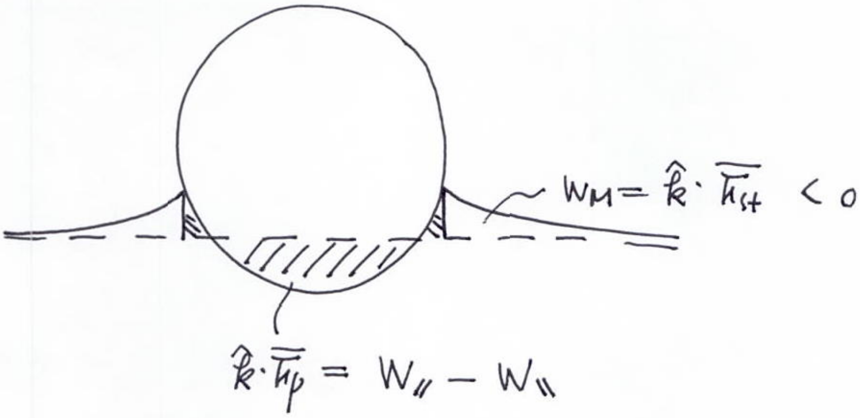
$= W_v$ (weight of liquid in the vertical cylinder through C, which intersects the horizontal plane in the curve Γ)

* Total vertical force on the body

$$\hat{k}\cdot(\overline{F}_{st} + \overline{F}_p) = W_M + W_v$$

: weight of liquid that would fill the region bounded below by the interface and the wetted surface of the body, and above by the undisturbed interface $z=0$.


$W_M = \hat{k} \cdot \overline{H}_{st} > 0$
$W_v = \overline{H}_p \cdot \hat{k} > 0$


$W_M = \hat{k} \cdot \overline{H}_{st} < 0$
$\hat{k} \cdot \overline{H}_p = W_{ll} - W_{ll}$