Chap. 5. Drag, scale and the Reynolds number

* Dimensional Analysis

§ Variables and constants

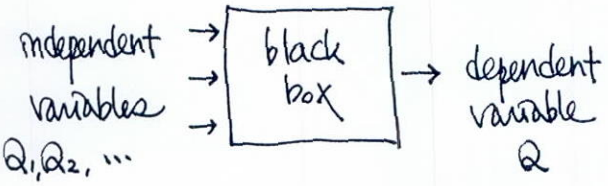
- dimensional variables : s, t ; p, V, t
- dimensional constants : s_0, V_0, g ; ρ, g, c

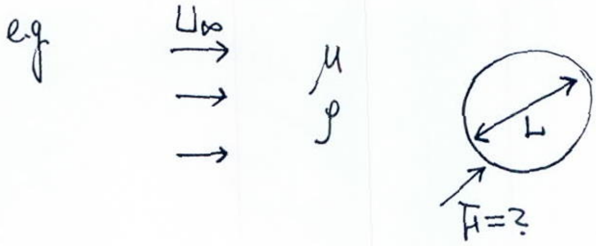
} → need to be nondimensionalized in dimensional analysis

- pure constants : dimensionless. $\frac{1}{2}$, 2, π, e

5.3 The Buckingham Pi theorem

- physical process





<u>step 1</u> : Identify a complete set of independent quantities $Q_1, \cdots, Q_n$ on which Q depends:

$$Q = \text{fn}\,(Q_1, Q_2, \cdots, Q_n) \qquad \cdots (1)$$

* complete : no other quantities affect the value of Q once the values of the set are specified

Step 2 : List the dimensions of Q; $Q_1, Q_2, \cdots, Q_n$

Step 3 : Pick from the indep. variables $Q_1, \cdots, Q_n$ a "complete", "dimensionally independent" subset $Q_1, \cdots, Q_k$ ($k \le n$). Then express the dimensions of the remaining independent variables $Q_{k+1}, \cdots, Q_n$ and the dependent variable Q in terms of products of powers of $Q_1, \cdots, Q_k$.

* dimensionally independent : each of the members of a subset has a dimension that cannot be expressed as a product of powers of the dimensions of the remaining members

* complete : all the remaining quantities $Q_{k+1}, \cdots, Q_n$ have dimensions which can be expressed as products of powers of the dimensions of $Q_1, \cdots, Q_k$

Step 4 :

$$Q^* = \frac{Q}{Q_1^a \cdots Q_k^b}$$: dimensionless (nondimensional)

$$Q_{k+1}^* = \frac{Q_{k+1}}{Q_1^p \cdots Q_k^q} \quad \cdots$$

Define dimensionless forms of the dependent variable Q and the $n-k$ remaining independent variables $Q_{k+1}, \cdots, Q_n$ by dividing each one with the product of powers of $Q_1, \cdots Q_k$ →

which has the same dimension as the quantity it divides.

Step 5 : Final result : $\boxed{Q^* = fn\,(Q_{k+1}^*, \cdots, Q_n^*)}$

why) An alternative form of eq (1) :

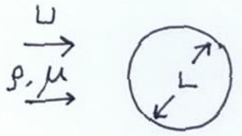
$$\underset{\text{dimensionless}}{Q^*} = fn\,(\underbrace{Q_1, \cdots, Q_k}_{\text{dimensional}} ;\ \underbrace{Q_{k+1}^*, Q_{k+2}^*, \cdots, Q_n^*}_{\text{dimensionless}})$$

· principle of dimensional homogeneity → $Q_1, \cdots, Q_k$: should be absent.

The Buckingham pi theorem

When a complete relationship between dimensional physical quantities is expressed in dimensionless form, the number of independent quantities which appear in it is reduced from the original n to $n-k$, where k is the maximum number of the original n quantities that are dimensionally independent.

~~Ex. 5.2~~ e.g. force on sphere / cylinder



→ drag
↑ (lift = 0) symmetric

Step 1 : $F = fn\,(L, U, \rho, \mu)$ $\quad n = 4$

Step 2 :

F :	MLT^{-2}	$kg \cdot m/s^2$
L :	L	m
U :	LT^{-1}	m/s
ρ :	ML^{-3}	kg/m^3
μ :	$ML^{-1}T^{-1}$	$kg/m \cdot s$

Step 3

fundamental dimensions : L, M, T.

\# of fund. dim. = # dimensionally indep. var. = k

Fund. dim.	Quantity
L	L
T	U
M	ρ

$k = 3$

Step 4

$F: \quad MLT^{-2} = (ML^{-3})(LT^{-1})^2 L^2$

$\sim \rho U^2 L^2$

$F^* = \frac{F}{\rho U^2 L^2}$.

$\mu: \quad ML^{-1}T^{-1} : (ML^{-3})(LT^{-1})L$

$\sim \rho U L$

$\mu^* = \frac{\mu}{\rho U L}$ $\qquad n - k = 4 - 3 = 1.$

Step 5

$$\frac{F}{\rho U^2 L^2} = f\left(\frac{\mu}{\rho U L}\right) \quad \text{OR} \quad f\left(\frac{\rho U L}{\mu}\right) = f(Re_L)$$

Ex 5.3 power input P to a centrifugal pump.

Step1 : $P = fn(Q, D, \Omega, \rho, \mu)$ $\qquad n=5$

Step2 : $P : \frac{kg \cdot m^2}{s^2}$, $Q : \frac{m^3}{s}$, $D : m$

$\Omega : s^{-1}$, $\rho : kg/m^3$, $\mu : kg/m \cdot s$

step 3. $k=3$

$m \rightarrow D\ (m)$

$s \rightarrow \Omega\ (s^{-1})$

$kg \rightarrow \rho\ (kg/m^3)$

step 4. $P:\ \dfrac{kg\cdot m^2}{s^3} = \left(\dfrac{kg}{m^3}\right)\left(\dfrac{1}{s^3}\right) m^5 \sim \rho\Omega^3D^5$

$P^* = \dfrac{P}{\rho\Omega^3D^5}.$

$Q:\ \dfrac{m^3}{s} = m^3\cdot\left(\dfrac{1}{s}\right) \sim D^3\Omega$

$Q^* = \dfrac{Q}{D^3\Omega}$ $\qquad n-k=2$

$\mu:\ \dfrac{kg}{m\cdot s} = \left(\dfrac{kg}{m^3}\right)\cdot\left(\dfrac{1}{s}\right)\cdot m^2 \sim \rho\Omega D^2$

$\mu^* = \dfrac{\mu}{\rho\Omega D^2}$

step 5. $\dfrac{P}{\rho\Omega^3D^5} = f\left(\dfrac{Q}{\Omega D^3},\ \dfrac{\mu}{\rho\Omega D^2}\right).$

Ex. 5.4.

step 1 $Q = f\left(R,\ \mu,\ \dfrac{dp}{dx}\right)$ $\qquad n=3$

step 2. $Q:\ \dfrac{m^3}{s}$

$R:\ m$

$\mu:\ \dfrac{kg}{m\cdot s}$

$\dfrac{dp}{dx}:\ \dfrac{kg}{m^2\cdot s^2}$

step 3.

m	R
s	μ
kg	dp/dx

$k=3$

step 4. $Q : \frac{m^3}{s} = \left(\frac{kg}{m^2 \cdot s^2}\right)\left(\frac{m \cdot s}{kg}\right) \cdot m^4 \sim \frac{dp}{dx} \frac{1}{\mu} \cdot R^4$

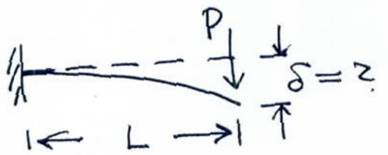
$$Q^* = \frac{Q}{\frac{1}{\mu} R^4 \left(\frac{dp}{dx}\right)}$$

step 5. when $\underline{n - k = 0}$: $\boxed{Q^* = \text{const.}}$

$$\frac{Q}{\frac{1}{\mu} R^4 \left(\frac{dp}{dx}\right)} = \text{const}$$

$$Q = \frac{c}{\mu} R^4 \left(\frac{dp}{dx}\right)$$

Ex. 5.5.



step 1 : $\delta = f(P, L, E, I)$ $\quad n = 4$

step 2 :

δ : m

P : kg·m/s²

L : m

E : kg/m·s²

I : m⁴

step 3

m	L
kg	E
s	P

$k = 3.$

step 4. $\delta : m = m \sim L$

$$\delta^* = \frac{\delta}{L}$$

$I : m^4 \sim L^4$

$$\sim \left(\frac{kg \cdot m}{s^2}\right)^2 \left(\frac{m \cdot s^2}{kg}\right)^2 \sim P^2 E^2$$

Return to step 3

$$m = \left(\frac{kg \cdot m}{s^2}\right)^{1/2} \left(\frac{m \cdot s^2}{kg}\right)^{1/2}$$

$$L \sim \left(\frac{P}{E}\right)^{1/2}$$

$\therefore$ L, P, E : not independent dimensionally.

m	L
kg/s^2	E

$k = 2$

<u>step 4</u> $\quad \delta^* = \frac{\delta}{L}$

P : $\frac{kg \cdot m}{s^2} = \left(\frac{kg}{m\, s^2}\right) \cdot m^2 = EL^2$. $\quad P^* = \frac{P}{EL^2}$

I : $m^4 \sim L^4$. $\quad I^* = \frac{I}{L^4}$

step 5. $\quad \frac{\delta}{L} = f\left(\frac{P}{EL^2}, \frac{I}{L^4}\right)$

If we take $\quad \delta = f(P, L, EI)$,

$$\frac{\delta}{L} = f\left(\frac{PL^2}{EI}\right) \xrightarrow[\text{regime : } \delta \propto P]{\text{"physical argument" in linear}} \frac{\delta}{L} = C\, \frac{PL^2}{EI}.$$

5.4 Nondimensionalization of the basic equations

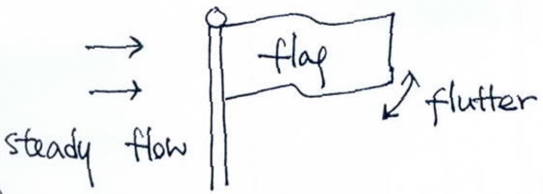
continuity : $\nabla \cdot \overline{V} = 0$

N.S. : $\rho \frac{d\overline{V}}{dt} = -\nabla p + \mu \nabla^2 V + \rho \overline{g}$

B.C. fixed solid surface $z=0$: $\overline{V} = 0$

free surface $z = \eta$: $w = \frac{d\eta}{dt}$

$$p = p_a - \sigma\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$



· voice generation

§ Other dimensionless parameters

from energy eq.

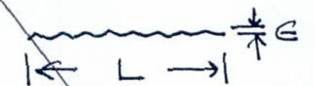
Prandtl number $Pr = \frac{\mu c_p}{k} = \frac{\nu}{\alpha}$

Eckert number $Ec = \frac{U^2}{c_p T_0}$

Grashof number $Gr = \frac{\beta \Delta T g L^3 \rho^2}{\mu^2}$

wall temp. ratio $\frac{T_w}{T_0}$

· roughness ratio $\frac{\epsilon}{L}$

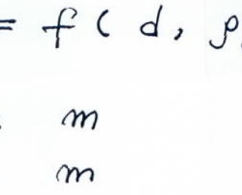


· Drag coefficient $C_D = \frac{D}{\frac{1}{2}\rho U^2 A} = \frac{\text{drag force}}{\text{dynamic force}}$

· Life coeffcient $C_L = \frac{L}{\frac{1}{2}\rho U^2 A} = \frac{\text{life force}}{\text{dynamic force}}$.

⋮

~~Ex. 5.6.~~ e.g. capillary rise problem



(a) $h = f(d, \rho g, \sigma, \theta)$

h : m

d : m

ρg : $\frac{kg}{m^3} \cdot \frac{m}{s^2} = \frac{kg}{m^2 s^2}$

σ : $\frac{kg}{s^2}$

θ : dimensionless

fund dim	quant
m	d ✓
s	σ ✓
kg	ρg

$k = 2$ not 3

$h^* = \frac{h}{d}$

$\rho g : \frac{kg}{m^2 s^2} = \frac{kg}{s^2} \cdot \frac{1}{m^2} \rightarrow \frac{\sigma}{d^2}$

$\rho g^* = \frac{\rho g d^2}{\sigma}$

$\therefore \frac{h}{d} = f\left(\frac{\rho g d^2}{\sigma}, \theta\right)$: end of dimensional analysis

+ physical argument. $h \sim \sigma$

$\frac{h}{d} = F(\theta) \frac{\sigma}{\rho g d^2}$

or $\frac{h \rho g d}{\sigma} = F(\theta)$

(b) $h = 3$ cm.

$$\begin{cases} d' = \frac{1}{2} d \\ \sigma' = \frac{1}{2} \sigma \\ \rho' = 2\rho \end{cases}$$

θ same. $h' = ?$

$\frac{h \rho g d}{\sigma} = \frac{h' \rho' g' d'}{\sigma'}$

$h' = h\left(\frac{\rho}{\rho'}\right)\left(\frac{d}{d'}\right)\left(\frac{\sigma'}{\sigma}\right) = h\left(\frac{1}{2}\right)(2)\left(\frac{1}{2}\right) = \frac{1}{2} h = 1.5$ cm

* Bond number. $Bo = \frac{\rho g d^2}{\sigma} = \frac{\text{gravity}}{\text{surface tension}}$.

§ Re and drag coefficient.

$$C_D = \frac{F_D}{\frac{1}{2}\rho S U^2}$$

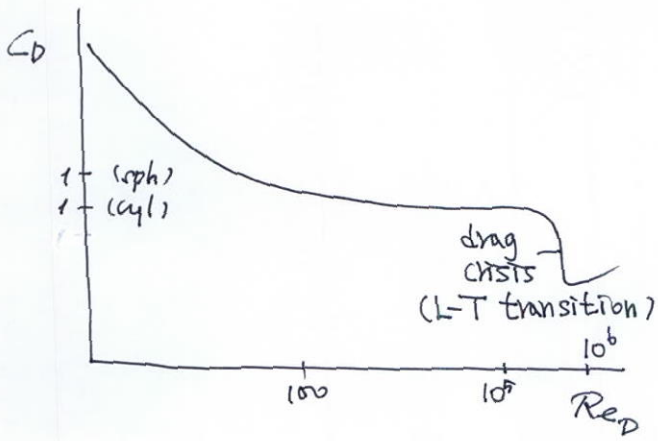
S (area)

① frontal area

② wetted area : total surface exposed to flow

③ plan form area

④ $V^{2/3}$



for cylinder

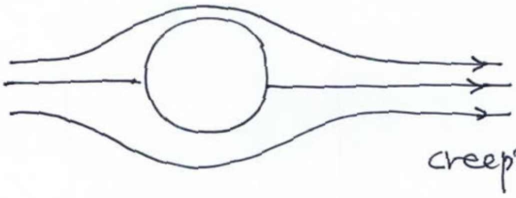
$S = \pi D L$

for sphere

$S = \frac{\pi}{4} D^2$

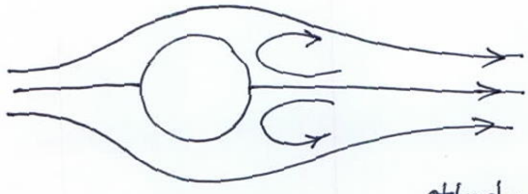
§ Flow around a cylinder

$Re < 10$



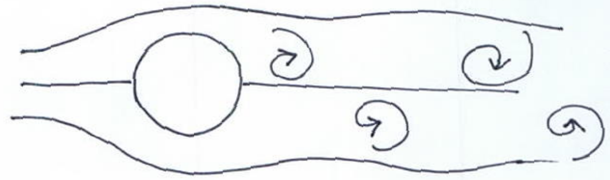
creeping flow

$10 < Re < 40$



attached vortices

$40 < Re < 200{,}000$

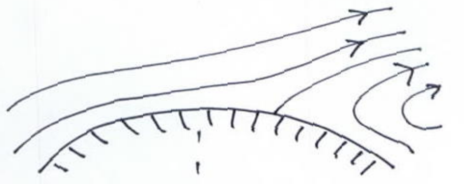


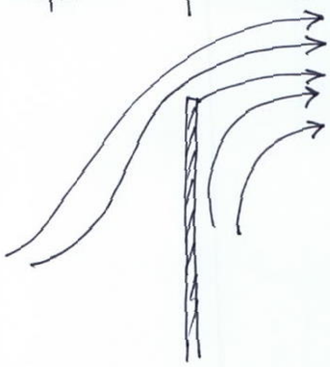
wide wake. $\theta_{sep} = 82°$
von Karman vortex trail

$Re > 200{,}000$

fully turbulent wake
$\theta_{sep} = 120°$.

· separation of flow (boundary-layer separation)



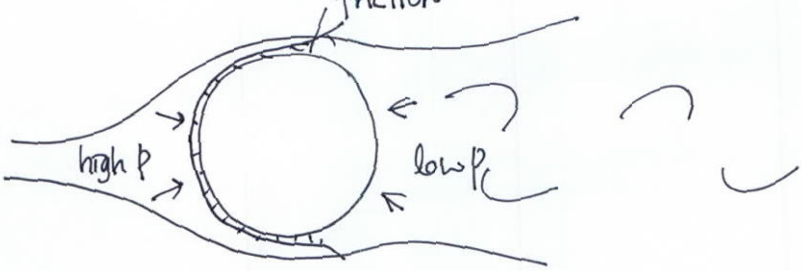


favorable pressure gradient | adverse pressure gradient

high p   low p   high p

§ Shape and two kinds of drag

· overall drag = skin friction + pressure drag



if pressure drag dominates, (high Re)

$F_D \sim \rho U^2 S$.

$C_D = \dfrac{F_D}{\frac{1}{2}\rho U^2 S} \sim k$ (const.)

if skin friction dominates, (low Re)

$$F_D \sim \mu \frac{U}{L} \cdot L^2 \sim \mu U L$$

$$C_D = \frac{F_D}{\frac{1}{2}\rho U^2 L^2} \sim \frac{\mu U L}{\rho U^2 L^2} \sim \frac{\mu}{\rho U L} \sim \frac{1}{Re}$$

· read discussion on Table 5.2 (p. 97 ~ 98)

* How to reduce drag (C_D)

· high Re — pressure drag dominates

→ "streamlining"

(forward-directed pressure from the rear nearly counterbalances the dynamic pressure on the front)

vs. blunt body (bluff body)

(energy lost in the wake ~ pressure decreases)

$C_{D,\,airplane} \sim 0.02\ C_{D\ sphere}$ (same frontal area)

· main disadvantage of streamlining as a scheme for drag reduction

~ it assumes a particular orientation to the flow.

∴ minor change in wind direction ~ wipe out the benefits of a streamlined (stationary) body.

· low Re — streamlining increases surface area

∴ skin friction ↑

— may outweigh reduction in pressure drag.

- roughness

e.g. dimples on golf balls

only effective { for bluff bodies
within a range of Re (25,000 ~ 150,000)

promoting turbulence → narrowing wake region
~ decreasing pressure drag.

Fig. 5.8.