~~1.2. Geometric meaning of $y' = f(x, y)$~~

skipped

1.3. Separable Differential Equations

ODEs.

Consider $\quad g(y)\,y' = f(x)$

more rigourously

$\int g(y)\underbrace{y'\,dx}_{= dy} = \int f(x)\,dx$

$$g(y)\,\frac{dy}{dx} = f(x)$$

$$\boxed{\left(\frac{dy}{dx}\right)dx = dy}$$

$$\boxed{g(y)\,dy = f(x)\,dx}$$

y only : x only $\rightarrow$ separated !

$$\int g(y)\,dy = \int f(x)\,dx + C$$

Ex. 4. $\quad 9yy' + 4x = 0 . \qquad y(x) = ?$

$$9y\,dy = -4x\,dx$$

$$\frac{9}{2}y^2 = -2x^2 + C$$

$$2x^2 + \frac{9}{2}y^2 = c^*$$

$$\frac{x^2}{9} + \frac{y^2}{4} = c \,//$$

ellipses

Ex. 1. $\frac{dy}{dx} = 1+y^2$. $y(x) = ?$

$$\frac{dy}{1+y^2} = dx$$

$$\int \frac{1}{1+y^2}\,dy = \int dx + c$$

$$\tan^{-1} y = x + c$$

$$y = \tan(x+c)$$

Ex. 3. $y' = ky$. HW.

~~Ex.~~ 4. $\frac{dy}{dx} = -\frac{y}{x}$, $y(1)=1$. ← I.C.

$$\frac{dy}{y} = -\frac{dx}{x}$$

$$\ln|y| = -\ln|x| + c^*$$

i) x

$|y| = \frac{c}{|x|}$. $y = \frac{c}{x}$

$y(1) = c = 1$

$\therefore y = \frac{1}{x}$

$= \frac{c}{x}$: $\left(\begin{array}{l} c=1 \\ \therefore y = \frac{1}{x} \end{array}\right)$

ii) $x>0$, $y<0$

$$\ln(-y) = -\ln x + c^*$$

$$-y = \frac{c}{x}$$

$*$ $\int \frac{dy}{y} = \ln|y| + c$ (?)

$\frac{d}{dx}(\ln|y|) = \frac{y'}{y}$

$\int \frac{\left(\frac{dy}{dx}\right)}{y}\,dx = \ln|y| + c$

i) $y>0$: $(\ln y)' = \frac{y'}{y}$

ii) $y<0$ $(\ln(-y))' = \frac{-y'}{-y} = \frac{y'}{y}$

iii) $x<0$, $y>0$

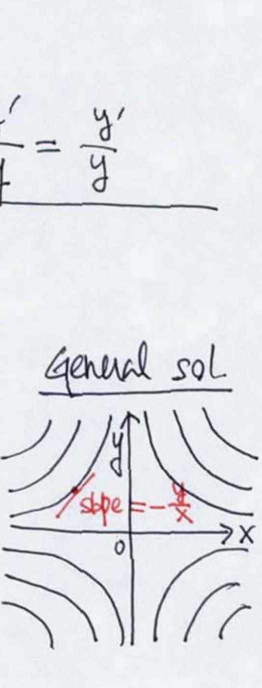
$$\ln y = -\ln(-x) + c^*$$

$$y = \frac{c}{-x}$$

iv) $x<0$, $y<0$

$-y = \frac{c}{-x}$. $y = \frac{c}{x}$

General sol.



~~Ex. 5.~~ $\frac{dy}{dx} = -2xy \qquad y(0)=1$ H.W.

Reduction to Separable form

Consider $\boxed{y' = g\left(\frac{y}{x}\right).}$: separable

→ p. 9.

Ex. 6. $2xy\frac{dy}{dx} = y^2 - x^2.$

$$\frac{dy}{dx} = \frac{y^2-x^2}{2xy}$$

$$\frac{dy}{dx} = \frac{1}{2}\left(\frac{y}{x} - \frac{x}{y}\right)$$

$\frac{y}{x} = u. \quad y = ux. \quad$ then $\frac{dy}{dx} = u'x + u$

$$u'x + u = \frac{1}{2}\left(u - \frac{1}{u}\right)$$

$$u'x = -\frac{1}{2}\left(u + \frac{1}{u}\right) = -\frac{u^2+1}{2u}$$

$$\frac{2u}{u^2+1}\,du = -\frac{dx}{x}$$

$$\ln(1+u^2) = -\ln|x| + c^*$$

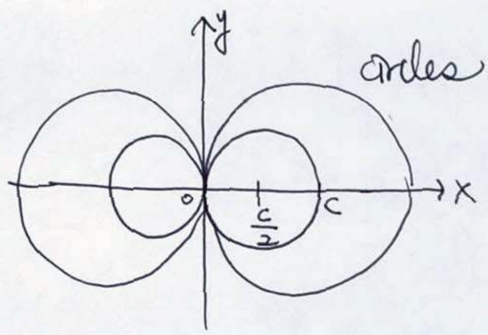
$$1+u^2 = \frac{c}{x}$$

$1 + \left(\frac{y}{x}\right)^2 = \frac{c}{x}$. multiply by x^2

$$x^2 + y^2 = cx$$

$$x^2 - cx + \frac{c^2}{4} + y^2 = \frac{c^2}{4}$$

$$\left(x - \frac{c}{2}\right)^2 + y^2 = \left(\frac{c}{2}\right)^2$$



Generalizing,

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

set $\frac{y}{x} = u$ $\quad y = ux. \quad y' = u'x + u$

$$u'x + u = f(u)$$

$$\frac{du}{dx}x = f(u) - u$$

$$\frac{du}{f(u) - u} = \frac{dx}{x}$$

Skip. p.10

Another technique

$$v = ay + bx + k$$

Ex. 7. $\underbrace{(2x-4y}_{=2(x-2y)}+5)\,y' + x - 2y + 3 = 0$ $\qquad \therefore \dfrac{dy}{dx} = -\dfrac{(x-2y)+3}{2(x-2y)+5}.$

Set $x - 2y = v$

$$2y = x - v. \qquad y = \frac{1}{2}(x-v). \qquad y' = \frac{1}{2}(1-v')$$

$$\frac{1}{2}(1-v') = -\frac{v+3}{2v+5}$$

$$1 - v' = -\frac{2v+6}{\underbrace{2v+5}_{v \text{ only}}}$$

$$v' = 1 + \frac{2v+6}{2v+5} = \frac{4v+11}{2v+5}$$

$$\left(\frac{2v+5}{4v+11}\right) dv = dx$$

$$\frac{4v+11-1}{4v+11}\, dv = 2\,dx$$

$$\left(1 - \frac{1}{4v+11}\right) dv = 2\,dx$$

$$v - \frac{1}{4}\ln|4v+11| = 2x + c^*$$

$$x - 2y - \frac{1}{4}\ln|4(x-2y)+11| = 2x + c^*$$

$$4x - 8y - \ln|4x-8y+11| = 8x + c$$

$$4x + 8y + \ln|4x-8y+11| + c = 0$$