1.7. ~~1.9.~~ Existence and Uniqueness of solutions.

Consider IVP

$$y' = f(x, y), \quad y(x_0) = y_0. \qquad (1)$$

Case 1.

$$|y'| + |y| = 0. \quad y(0) = 1$$

i) $y > 0$, $y' > 0$

$y' + y = 0$. $y = ce^{-x}$. $\underset{>0}{y = e^{-x}}$. $\underset{<0}{y' = -e^{-x}}$ (X)

ii) :

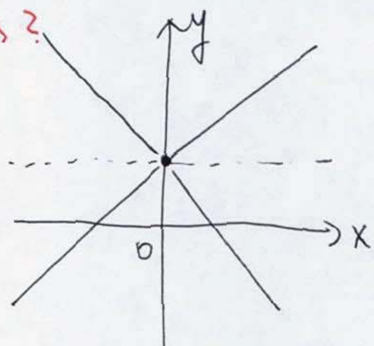
no solution except $y \equiv 0$. → existence ?

Case 2.

$$y' = x, \quad y(0) = 1$$

$y = \frac{1}{2}x^2 + c$. $y = \frac{1}{2}x^2 + 1$. : one solution

Case 3. → uniqueness ?



$$xy' = y - 1 \quad . \quad y(0) = 1$$

$x\frac{dy}{dx} = y - 1$. $\frac{dy}{y-1} = \frac{dx}{x}$

$\ln|y-1| = \ln|x| + c^*$

$|y-1| = c|x|$. $(c > 0)$

$\boxed{y = cx + 1}$

i) $x > 0$. $y > 1$: $y - 1 = cx$. $y = cx + 1$

ii) $x > 0$. $y < 1$: $-y + 1 = cx$. $y = -cx + 1$

iii) $x < 0$ $y > 1$: $y - 1 = -cx$. $y = -cx + 1$

iv) $x < 0$. $y < 1$: $-y + 1 = -cx$. $y = cx + 1$

. Infinitely many solutions

Problem of existence : Under what conditions does an IVP have at least one solution?

" uniqueness : " have at most one solution?

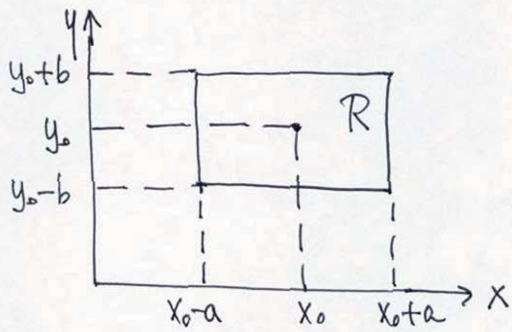
Theorem 1 Existence Theorem $y' = f(x,y)$. $y(x_0) = y_0$

If $f(x,y)$ continuous at all pts (x,y) in some rectangle

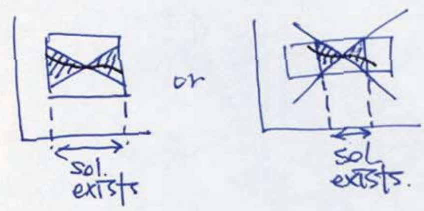
$$R: \ |x - x_0| < a, \quad |y - y_0| < b$$

and bounded in R, say, $|f(x,y)| \le K$,

then IVP (1) has at least one solution.



f: slope, if bounded



Theorem 2 Uniqueness Thm

If $f(x,y)$ and $\frac{\partial f}{\partial y}$ are continuous for all (x,y) in R and bounded, then IVP (1) has at most one solution $y(x)$.

+ by Thm 1 $\rightarrow$ IVP has precisely one solution.

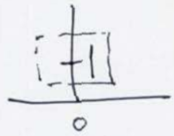
case 1.

$|y'| = -|y|.$

case 2.

$y' = x = f$ $\qquad$ f: continuous, bounded

$\begin{cases} x_0 = 0 \\ y_0 = 1 \end{cases}$ $\qquad$ $\frac{\partial f}{\partial y} = 0$ ∴ continuous, bounded



case 3.

$y' = \frac{y-1}{x} = f(x,y)$ $\qquad (0,1)$