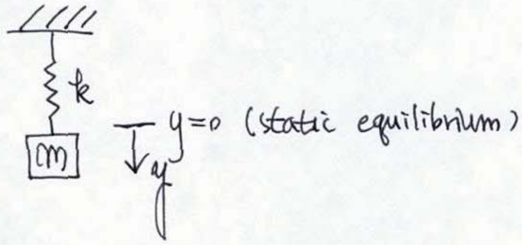
2.3 omitted

2.4 Modeling : Free oscillations

§ Undamped System



$$\downarrow \ \Sigma F = -ky = m\ddot{y} = m\frac{d^2y}{dt^2}$$

$$\therefore \ m\frac{d^2y}{dt^2} + ky = 0$$

$$\frac{d^2y}{dt^2} + \underbrace{\left(\frac{k}{m}\right)}_{=\omega_0^2 > 0} y = 0 \qquad \omega_0 = \sqrt{\frac{k}{m}}$$

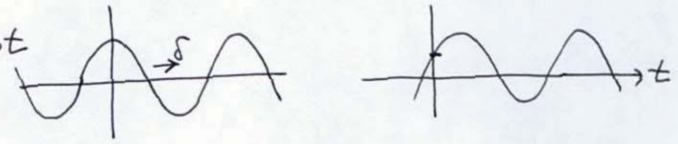
$$y = A\cos\omega_0 t + B\sin\omega_0 t \quad : \text{Harmonic oscillation}$$

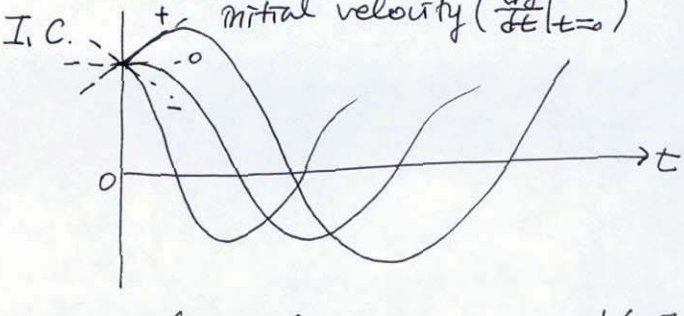
$$A, B \leftarrow \text{I.C.}$$

$$y = \sqrt{A^2+B^2}\cos(\omega_0 t - \delta) \qquad \tan\delta = \frac{B}{A}$$

$\uparrow$

plot : $y = a\cos\omega_0 t$



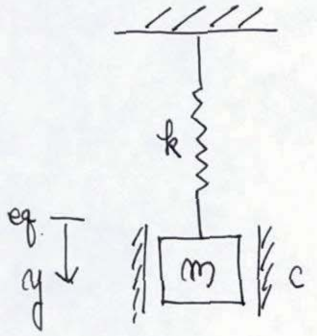


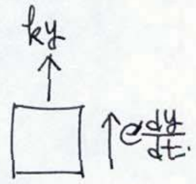
ω_0 : angular frequency. [rad/s]

f : freq [cyc/s = Hz] $= \frac{\omega_0}{2\pi}$ $\qquad T = \frac{2\pi}{\omega_0}$

T : period [s]. $\quad fT = 1. \quad \frac{\omega_0}{2\pi} = \frac{1}{T}. \quad \omega_0 = \frac{2\pi}{T}.$

§ Damped System





$$\downarrow \ \Sigma F = -ky - c\frac{dy}{dt} = m\ddot{y}$$

$$m\frac{d^2y}{dt^2} + c\frac{dy}{dt} + ky = 0.$$

$$y = e^{\lambda t}$$

$$m\lambda^2 + c\lambda + k = 0.$$

$$\lambda = -\frac{c}{2m} \pm \frac{1}{2m}\sqrt{c^2-4mk}$$

set $\alpha = \frac{c}{2m}$. $\beta = \frac{1}{2m}\sqrt{c^2-4mk}$

$\lambda_1 = -\alpha + \beta$. $\lambda_2 = -\alpha - \beta$. ~~$(\alpha>0, \beta>0)$~~

case 1.	$c^2 > 4mk$	: distinct real λ_1, λ_2	: overdamping
case 2.	$c^2 = 4mk$	: real double	: critical damping
case 3.	$c^2 < 4mk$	: compl. conj.	: under damping

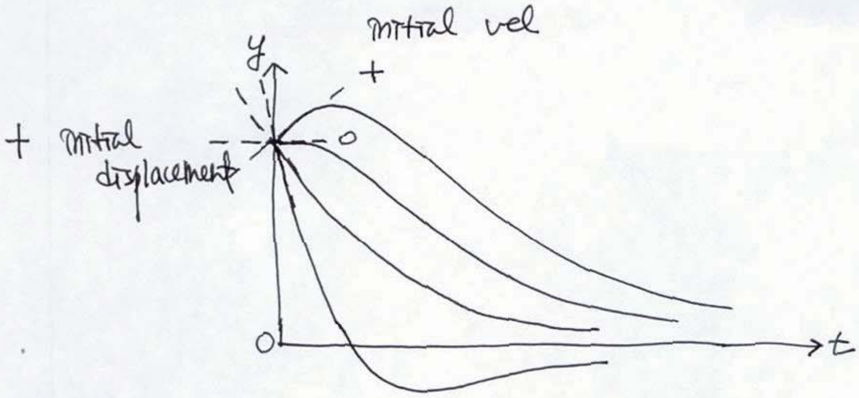
CASE I overdamping $c^2 > 4mk$.

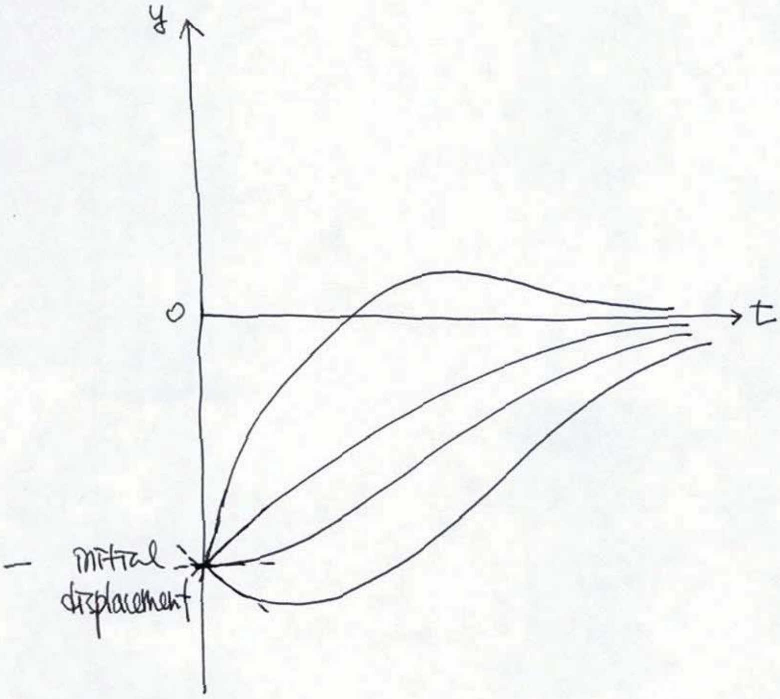
λ_1, λ_2 : real distinct

$$y(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$
$$= c_1 e^{(-\alpha+\beta)t} + c_2 e^{(-\alpha-\beta)t}$$
$$= c_1 e^{-(\alpha-\beta)t} + c_2 e^{-(\alpha+\beta)t}$$: nonoscillatory

$\alpha > \beta$. $\alpha - \beta > 0$

as $t \to \infty$, $y \to 0$.

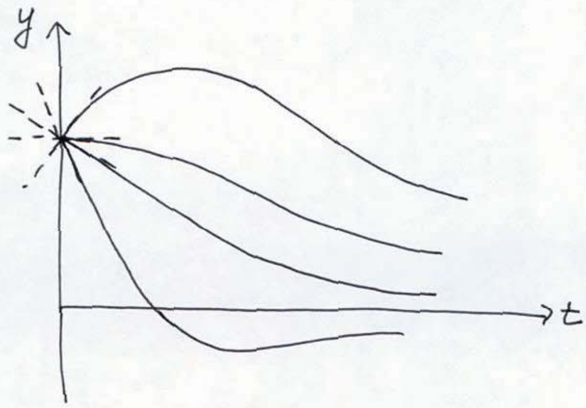




CASE II Critical damping $c^2 = 4mk$

a real double root : $\lambda = -\alpha$.

$y = (c_1 + c_2 t) e^{-\alpha t}$. as $t \to \infty$. $y \to 0$.



CASE III. Underdamping $\quad c^2 < 4mk$

$$\lambda = -\alpha \pm \beta. \qquad \beta : \text{imaginary}$$

$$\beta = i\omega^*. \qquad \omega^* = \frac{1}{2m}\sqrt{4mk - c^2} = \sqrt{\frac{k}{m} - \frac{c^2}{4m^2}}$$

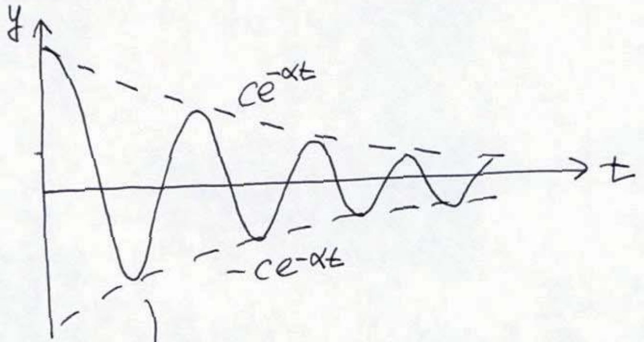
~~basis~~ $\lambda_1 = -\alpha + i\omega^*, \quad \lambda_2 = -\alpha - i\omega^*$

G.S.: $y(t) = e^{-\alpha t}(A\cos\omega^* t + B\sin\omega^* t)$

$$= Ce^{-\alpha t}\cos(\omega^* t - \delta)$$

$$C = \sqrt{A^2 + B^2}. \qquad \delta = \tan^{-1}\left(\frac{B}{A}\right).$$

damped oscillation



angular freq $\quad \omega^* < \omega_0.$

$$T^* = \frac{2\pi}{\omega^*} > T$$

$$c \uparrow \; - \; \omega^* \downarrow$$

$$c \to 0 \; - \; \omega^* \rightsquigarrow \omega_0.$$

2. ~~6~~ 5. Euler-Cauchy ~~Equation~~

$$\boxed{x^2 y'' + axy' + by = 0.} \qquad a, b : \text{const.}$$

equidimensional.

Try $\boxed{y = x^m}$

$$x^2 m(m-1) x^{m-2} + a x m x^{m-1} + b x^m = 0$$

$$[m(m-1) + am + b]\, x^m = 0$$

$$m^2 + (a-1)m + b = 0.$$

$$m = \frac{-(a-1) \pm \sqrt{(a-1)^2 - 4b}}{2}.$$

CASE I. Distinct real roots

$m_1,\ m_2$

basis: $x^{m_1},\ x^{m_2}$

g.s. $y = c_1 x^{m_1} + c_2 x^{m_2}$

EX. 1* $\quad x^2 y'' - 2.5 x y' - 2.0 y = 0$

$$y = x^m$$

$$m(m-1) - 2.5m - 2.0 = 0$$

$$m^2 - 3.5m - 2.0 = 0$$

$$(m-4)(m+0.5) = 0$$

$$m = 4,\ -0.5$$

$$y = c_1 x^4 + c_2 x^{-0.5} = c_1 x^4 + \frac{c_2}{\sqrt{x}} \qquad (x > 0)$$

$\overrightarrow{\text{next page}}$

CASE II. Double root.

$$m = \tfrac{1}{2}(1-a).$$

$$y_1 = x^{(1-a)/2}$$

$y_2 = ?$ reduction of order: $\quad y_2 = u(x) y_1$

$$x^2y'' - 2.5xy' - 2.0y = 0 \qquad (\text{if } x<0)$$

$$x = -t \qquad \frac{dy}{dx} = \frac{dt}{dx}\left(\frac{dy}{dt}\right) = -\frac{dy}{dt}$$

$$dx = -dt \,. \qquad \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{dt}{dx}\frac{d}{dt}\left(\frac{dy}{dx}\right) = -\frac{d}{dt}\left(-\frac{dy}{dt}\right) = \frac{d^2y}{dt^2}$$

$$t^2\frac{d^2y}{dt^2} - 2.5(-t)\left(-\frac{dy}{dt}\right) - 2.0y = 0$$

$$t^2\frac{d^2y}{dt^2} - 2.5t\frac{dy}{dt} - 2.0y = 0.$$

$$y = c_1t^4 + c_2t^{-0.5}$$

$$t = -x$$

$$y = c_1x^4 + c_2(-x)^{-0.5}$$

$$y_2' = u'y_1 + uy_1'$$

$$y_2'' = u''y_1 + 2u'y_1' + uy_1''$$

$$x^2y_2'' + axy_2' + by_2 = 0$$

$$u''x^2y_1 + u'x\underbrace{(2xy_1' + ay_1)} + u(\underbrace{x^2y_1'' + axy_1' + by_1}_{=0}) = 0.$$

$$\cancel{2}x\frac{(1-a)}{\cancel{2}}x^{-\frac{a}{2}-1} + ax^{\frac{1}{2}-\frac{a}{2}}$$

$$= (1-\cancel{a}+\cancel{a})x^{+\frac{1}{2}-\frac{a}{2}}$$

$$= x^{(1-a)/2} = y_1$$

$$u''x^2y_1 + u'xy_1 = 0$$

$$(u''x^2 + u'x)y_1 = 0. \qquad y_1 \neq 0.$$

$$x(u''x + u') = 0.$$

$$u' = U$$

$$\frac{dU}{dx}x = -U. \qquad \frac{dU}{U} = -\frac{dx}{x}$$

$$\ln|U| = -\ln|x|.$$

$x>0$: $\ln|U| = +\ln\frac{1}{x}$

$$U = u' = \frac{1}{x}. \qquad u = \ln x.$$

$$\therefore y_2 = y_1\ln x. \qquad y_1 = x^{(1-a)/2}$$

$\therefore$ g.s. $y = (c_1 + c_2\ln x)x^{(1-a)/2}$

Ex. 2* $x^2y'' - 3xy' + 4y = 0.$

$y = x^m$

$m(m-1) - 3m + 4 = 0$

$m^2 - 4m + 4 = (m-2)^2 = 0$

$m = 2$

basis: $y_1 = x^2$. $y_2 = x^2 \ln x$

G.S. $y = (c_1 + c_2 \ln x)x^2$

<u>CASE III. Complex conjugate roots</u>

(no great practical importance)

$m_1 = \mu + i\nu$, $m_2 = \mu - i\nu$

$y_1 = x^{m_1}$; $y_2 = x^{m_2}$

$$y_1 = x^{m_1} = x^{\mu + i\nu} = x^\mu \cdot x^{i\nu} = x^\mu \cdot \underbrace{e^{\ln x^{i\nu}}}_{= e^{i\nu \ln x}} = x^\mu[\cos(\nu \ln x) + i\sin(\nu \ln x)]$$

$(x > 0)$

$$y_2 = x^{m_2} = x^{\mu - i\nu} = x^\mu[\cos(\nu \ln x) - i\sin(\nu \ln x)]$$

basis:
$$\begin{cases} \dfrac{y_1 + y_2}{2} = x^\mu \cos(\nu \ln x) \\[2ex] \dfrac{y_1 - y_2}{2i} = x^\mu \sin(\nu \ln x) \end{cases}$$

G.S. $y = x^\mu [A\cos(\nu \ln x) + B\sin(\nu \ln x)]$

Ex.3.* $x^2y''+7xy'+13y=0$

$m(m-1)+7m+13=0$

$m^2+6m+13=0$

$m=-3\pm\sqrt{9-13} = -3\pm 2i$

$y_1=x^{m_1} = x^{-3+2i} = x^{-3}\cdot x^{2i} = x^{-3}\, e^{i2\ln x}$

$= x^{-3}\left[\cos(2\ln x)+i\sin(2\ln x)\right]$. $\quad y_2=x^{m_2}$

basis: $x^{-3}\cos(2\ln x)$

$x^{-3}\sin(2\ln x)$

$\therefore$ G.S. $\Rightarrow$ $y=x^{-3}\left[A\cos(2\ln x)+B\sin(2\ln x)\right]$.

2.6. Existence and Uniqueness ~~Theory~~ of solutions

$y''+p(x)y'+q(x)y=0$: homogeneous, linear eq. ⋯ $(*)$

I.C. $y(x_0)=K_0,\quad y'(x_0)=K_1$ ⋯ (§)

Theorem1: Existence and Uniqueness theorem for IVP

If $p(x)$ and $q(x)$ are continuous functions on some open interval I and x_0 is in I, then the initial value problem consisting of $(*)$ and (§) has a unique solution $y(x)$ on the interval I.