Ex.3* $x^2y''+7xy'+13y=0$

$m(m-1)+7m+13=0$

$m^2+6m+13=0$

$m=-3\pm\sqrt{9-13} = -3\pm 2i$

$y_1=x^{m_1} = x^{-3+2i} = x^{-3}\cdot x^{2i} = x^{-3}\, e^{i2\ln x}$

$= x^{-3}\left[\cos(2\ln x) + i\sin(2\ln x)\right]$. $\qquad y_2 = x^{m_2}$

basis: $x^{-3}\cos(2\ln x)$

$x^{-3}\sin(2\ln x)$

$\therefore$ G.S. $\Rightarrow$ $y = x^{-3}\left[A\cos(2\ln x) + B\sin(2\ln x)\right]$.

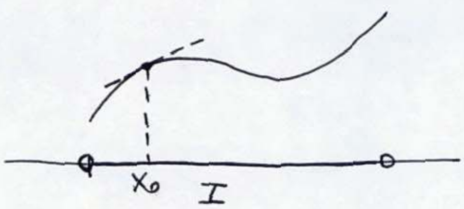
2.6. Existence and Uniqueness ~~Theory~~ of solutions

$y''+p(x)y'+q(x)y=0$; homogeneous, linear eq. $\cdots(*)$

I.C. $y(x_0)=K_0$, $y'(x_0)=K_1$ $\cdots(\S)$

Theorem1: Existence and Uniqueness theorem for IVP

If $p(x)$ and $q(x)$ are continuous functions on some open interval I and x_0 is in I, then the initial value problem consisting of $(*)$ and $(\S)$ has a unique solution $y(x)$ on the interval I.



Theorem 2 : Linear dependence and independence of solutions

Suppose that (*) has continuous coefficients $p(x)$ and $q(x)$ on an open interval I. Then two solutions y_1 and y_2 of (*) on I are linearly dependent on I if and only if their Wronskian W is zero at some x_0 in I. Furthermore, if $W=0$ for $x=x_0$, then $W \equiv 0$ on I ; hence if there is an x_1 in I at which W is not zero, then y_1, y_2 are linearly independent on I.

· Wronskian (of two solutions y_1 & y_2)

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

Proof (a) y_1, y_2 linearly dependent $\Rightarrow$ $W=0$.

"$y_1 = k y_2$" or $y_2 = l y_1$ on I.

then $W(y_1, y_2) = W(ky_2, y_2) = \begin{vmatrix} ky_2 & y_2 \\ ky_2' & y_2' \end{vmatrix} = 0.$

(b) $W=0$ (at x_0). $\Rightarrow$ y_1, y_2 linearly dependent.

Consider
$$\left.\begin{aligned} k_1 y_1(x_0) + k_2 y_2(x_0) &= 0 \\ k_1 y_1'(x_0) + k_2 y_2'(x_0) &= 0 \end{aligned}\right) \quad (\dagger)$$

$$\underbrace{\begin{bmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{bmatrix}}_{\text{determinant} = W = 0.} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = 0$$

$\Rightarrow$ $k_1 \neq 0$ or $k_2 \neq 0$.

We introduce

$y(x) = k_1 y_1(x) + k_2 y_2(x)$: solution of $(*)$

by $(\dagger)$ $\left\{ \begin{array}{l} y(x_0) = 0 \\ y'(x_0) = 0 \end{array} \right\}$ I.C. $\longleftarrow$ that satisfies

p, q. continuous $\rightarrow$ y : unique solution

Another solution : $y^* \equiv 0$ $\therefore y = 0$ //

$k_1 y_1 + k_2 y_2 \equiv 0.$ on I.

$k_1 \neq 0$ or $k_2 \neq 0$ $\therefore$ y_1, y_2 linearly dependent

(c) • $W = 0$ for $x = x_0$ $\Rightarrow$ $W \equiv 0$ on I

$W = 0$ for $x = x_0$ $\xrightarrow{(b)}$ y_1, y_2 linearly dependent

$\xrightarrow{(a)}$ $W \equiv 0$.

• $W \neq 0$ at x_1 $\Rightarrow$ y_1, y_2 linearly independent.

$W \neq 0$ cannot happen at an x_1 on I in the case of linear dependence, so that $W \neq 0$ at x_1 implies linear independence.

Theorem 3 : Existence of a general solution

If $p(x)$ and $q(x)$ are continuous on an open interval I, then $(*)$ has a general solution on I.

Theorem 4 : General solution

Suppose that $(*)$ has continuous coefficients $p(x)$ and $q(x)$ on some open interval I. Then every solution $y = Y(x)$ of $(*)$ on I is of the form

$$Y(x) = c_1 y_1(x) + c_2 y_2(x)$$

where y_1, y_2 form a basis of solutions of $(*)$ on I and c_1, c_2 are suitable constants. Hence $(*)$ does not have singular solutions (i.e. solutions not obtainable from a general solution) $\therefore$ A general solution of $(*)$ includes all solutions!

2.7. Nonhomogeneous ODEs

$$y'' + p(x)y' + q(x)y = r(x) \quad \cdots (1)$$

$$r(x) \neq 0.$$

We proceed from

$$y'' + p(x)y' + q(x)y = 0 \quad \cdots (2)$$

Theorem 1: Relations between solutions of (1) and (2)

(a) $y, \tilde{y}$: two solutions of (1) $\Longrightarrow$ $Y = y - \tilde{y}$: solution of (2)

$$y'' + py' + qy = r$$

$$-)\ \tilde{y}'' + p\tilde{y}' + q\tilde{y} = r$$

$$Y'' + pY' + qY = 0$$

(b) $y + Y$: solution of (1)

$$y'' + py' + qy = r$$

$$Y'' + pY' + qY = 0$$

$$\mathcal{L}(y+Y) = r$$

Definition

Gen. sol of nonhomogeneous eq. (1) :

$$\boxed{y(x) = y_h(x) + y_p(x)}$$

$\left[\begin{array}{l} y_h = c_1y_1 + c_2y_2 : \text{ G.S. of homogeneous eq} \\ y_p : \text{ any solution of (1) containing no arbitrary constants.} \end{array}\right.$

Theorem 2 : General solution

$p(x), q(x), r(x)$: continuous $\longrightarrow$ G.S. of (1) : $y = y_h + y_p$

(every possible sol. by assigning suitable c_1 / c_2)

Practical conclusion

To solve $y'' + p(x)y' + q(x)y = r(x)$,

① Solve homogeneous eq : $y'' + p(x)y' + q(x)y = 0$

② find any y_p of (1)

~~2.9. Solution by~~ §Method of Undetermined Coefficients.

: how to obtain y_p

$\boxed{\text{Method of undetermined coefficients}}$

$$y'' + ay' + by = r(x)$$

when? (i) a, b : const.

(ii) $r(x)$: special functions

(A) Basic Rule : Follow the table

(B) Modification Rule : first choice for $\tilde{y}_p \Rightarrow y_h$

then try $y_p = x\tilde{y}_p$ or $x^2\tilde{y}_p$ (double root)

(C) Sum Rule : $r(x) = A(x) + B(x) \Rightarrow y_p = \hat{A}(x) + \hat{B}(x)$

left column right column

Term in $r(x)$	Choice for y_p
$ke^{\gamma x}$	$Ce^{\gamma x}$
$kx^n \ (n=0,1,\cdots)$	$K_n x^n + K_{n-1}x^{n-1} + \cdots + K_1 x + K_0$
$k\cos\omega x$, $k\sin\omega x$	$K\cos\omega x + M\sin\omega x$
$ke^{\alpha x}\cos\omega x$, $ke^{\alpha x}\sin\omega x$	$e^{\alpha x}(K\cos\omega x + M\sin\omega x)$

Ex. 1.* Rule A.

$$y'' + 4y = 8x^2$$

(1) homogeneous solution:

$$y_h'' + 4y_h = 0$$

$$y_h = A\cos 2x + B\sin 2x$$

(2) $y_p = K_2x^2 + K_1x + K_0.$

$y = y_h + y_p \longrightarrow$ substitute $\quad y_p'' + 4y_p = 8x^2$

$$K_2 \cdot 2 + 4(K_2x^2 + K_1x + K_0) = 8x^2$$

$$4K_2x^2 + 4K_1x + (4K_0 + 2K_2) = 8x^2$$

$$\therefore K_2 = 2, \quad K_1 = 0, \quad K_0 = -1.$$

$$\therefore y_p = 2x^2 - 1.$$

$\therefore y = A\cos 2x + B\sin 2x + 2x^2 - 1.$: general sol.

Particular sol. : I.C.'s $\rightarrow$ A, B.

Ex. 2* Rule B.

$$y'' - 3y' + 2y = e^x$$

(1) hom. sol. $\quad y_h'' - 3y_h' + 2y_h = 0$

$$y_h = e^{\lambda x}$$

$$\lambda^2 - 3\lambda + 2 = 0$$

$$(\lambda - 2)(\lambda - 1) = 0 \qquad \lambda = 1, 2$$

$$y_h = c_1 e^x + c_2 e^{2x}$$

(2) y_p: $\quad$ Try $y_p = Ce^x$ (no!!)

$$\rightarrow \; y_p = Cxe^x$$

$$y_p' = C(1+x)e^x$$

$$y_p'' = C(2+x)e^x$$

$$C(2+x)e^x - 3C(1+x)e^x + 2Cxe^x = e^x$$

$$(2C - 3C - 1)e^x + (C - 3C)xe^x = 0$$

$$+2C$$

$$-C - 1 = 0 \quad \therefore \; C = -1.$$

$$\therefore \; y = y_h + y_p = c_1 e^x + c_2 e^{2x} - xe^x$$

Ex. 2b. Rule B

$$y'' + 2y' + y = e^{-x}, \qquad y(0) = -1, \quad y'(0) = 1.$$

(1) y_h: $\quad y_h'' + 2y_h' + y_h = 0$

$$y_h = e^{\lambda x}$$

$$\lambda^2 + 2\lambda + 1 = 0. \qquad (\lambda+1)^2 = 0. \qquad \lambda = -1$$

$$y_h = c_1 e^{\lambda x} + c_2 x e^{\lambda x}$$

$$= c_1 e^{-x} + c_2 x e^{-x}$$

(2) y_p: $\quad y_p = cx^2 e^{-x}$

$$y_p' = c(2x - x^2)e^{-x}$$

$$y_p'' = c(2 - 2x - 2x + x^2)e^{-x} = c(2 - 4x + x^2)e^{-x}$$

$$[c(2-4x+x^2) + 2c(2x-x^2) + cx^2]\, e^{-x} = e^{-x}$$

$$[2c + (-4c+4c)x + (c-2c+c)x^2]\, e^{-x} = e^{-x}$$

$$2c = 1. \qquad \therefore c = 1/2$$

$$\underset{g.s.}{y} = y_h + y_p = (c_1 + c_2 x)e^{-x} + \frac{1}{2}x^2 e^{-x}$$

(3) I.C

$$y(0) = c_1 = -1$$

$$y' = (-c_1 + c_2 - c_2 x)e^{-x} + \frac{1}{2}(2x - x^2)e^{-x}$$

$$y'(0) = -c_1 + c_2 = 1 \quad . \qquad c_2 = 0$$

$$\therefore y = -e^{-x} + \frac{1}{2}x^2 e^{-x}$$

$$= \left(\frac{1}{2}x^2 - 1\right)e^{-x}$$

Ex. ~~#~~ 3* Rule c

$$y'' + 2y' + 5y = 1.25\, e^{0.5x} + 40\cos 4x - 55 \sin 4x$$

(1) y_h: $\quad y_h'' + 2y_h' + 5y_h = 0$

$$y_h = e^{\lambda x}$$

$$\lambda^2 + 2\lambda + 5 = 0$$

$$\lambda = -1 \pm \sqrt{1-5} = -1 \pm 2i$$

$$y_h = e^{-x}(A\cos 2x + B\sin 2x)$$

(2) y_p: $y_p = Ce^{0.5x} + K\cos 4x + M\sin 4x$

$$y_p' = 0.5Ce^{0.5x} - 4K\sin 4x + 4M\cos 4x$$

$$y_p'' = 0.25Ce^{0.5x} - 16K\cos 4x - 16M\sin 4x$$

$$(0.25+1+5)Ce^{0.5x} + (-16K+8M+5K)\cos 4x + (-16M-8K+5M)\sin 4x$$

$$= 1.25e^{0.5x} + 40\cos 4x - 55\sin 4x$$

$$\therefore C = 0.2, \quad K = 0, \quad M = 5$$

$$y = y_h + y_p$$

$$= e^{-x}(A\cos 2x + B\sin 2x) + 0.2e^{0.5x} + 5\sin 4x$$

→ p.56

2.10. Solution by Variation of Parameters.

· previous method (undetermined coeff.)

$$y'' + \underbrace{ay' + by}_{①\ \text{const.}} = \underbrace{r(x)}_{②\ \text{special fn}}$$

· General: $y'' + p(x)y' + q(x)y = r(x)$

p, q, r: continuous on I

$$\boxed{y_p(x) = -y_1\int\frac{y_2 r}{W}dx + y_2\int\frac{y_1 r}{W}dx}$$

$$\begin{cases} y_1, y_2: \text{basis of solutions of homo. eq.} \\ W\ (\text{Wronskian}) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_2y_1' \end{cases}$$