§ Critical points

$$\bar{y}' = Ay$$

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = A \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\frac{dy_2}{dy_1} = \frac{dy_2/dt}{dy_1/dt} = \frac{y_2'}{y_1'} = \frac{a_{21}y_1 + a_{22}y_2}{a_{11}y_1 + a_{12}y_2}$$: tangent direction at $P(y_1, y_2)$

What about $\overset{P(y_1,y_2)=}{P(0,0)}$: $\frac{dy_2}{dy_1} = \frac{0}{0}$: undetermined

· Critical point : a point at which $\frac{dy_2}{dy_1}$ becomes undetermined.

§ Five types of Critical Points

(1) Improper node

: Allmost all the trajectories have the same limiting direction of the tangent.

Ex. 1 : As $t \to \infty$. $\bar{y} \to c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t}$

(0.0)

↑

Improper node

limiting direction $= \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

except when $c_1 = 0$: $\bar{y} \to \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

(2) Proper node

: Every trajectory has a definite limiting direction & for any given direction d at P_0, there is a trajectory having d as its limiting direction

Ex. 2 $\quad \bar{y}' = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \bar{y}$. $\quad \bar{y} = \bar{x} e^{\lambda t}$ $\quad \begin{cases} y_1' = y_1 \\ y_2' = y_2 \end{cases}$

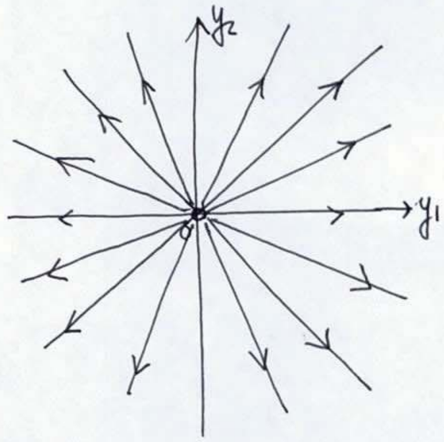
$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 = 0$$

$$\lambda_1 = \lambda_2 = 1.$$

$$(A - \lambda I)\bar{x} = \begin{bmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0.$$

$$\bar{y} = c_1 \bar{x}^{(1)} e^t + c_2 \bar{x}^{(2)} e^t. \quad \begin{cases} \bar{x}^{(1)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ \bar{x}^{(2)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{cases}$$

$$\begin{Bmatrix} y_1 = c_1 e^t \\ y_2 = c_2 e^t \end{Bmatrix} \qquad \frac{y_2}{y_1} = c$$



(3) Saddle point

: There are two incoming trajectories, two outgoing trajectories, and all the other trajectories in a neighborhood of P_0 bypass P_0

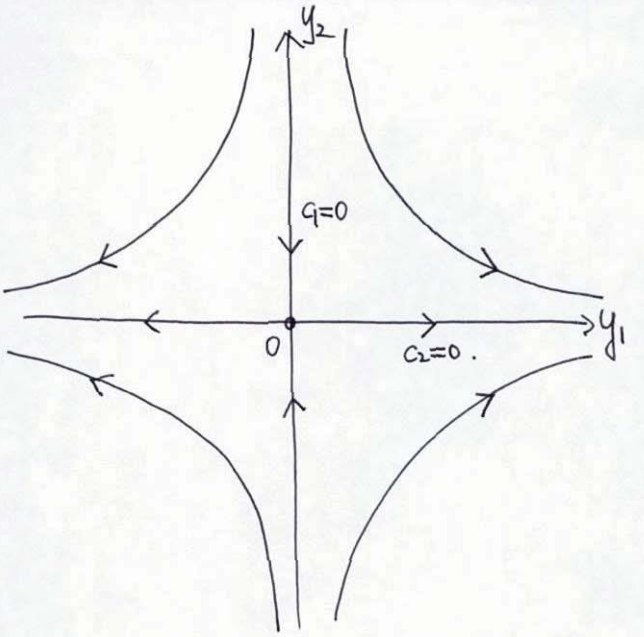
Ex. 3. $\quad \bar{y}' = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \bar{y}$. $\quad \bar{y} = \bar{x} e^{\lambda t}$

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 0 \\ 0 & -1-\lambda \end{vmatrix} = (1-\lambda)(-1-\lambda) = -1 + \lambda^2 = 0$$

$$\lambda_1 = +1, \quad \lambda_2 = -1$$

$$\begin{bmatrix} 1-\lambda & 0 \\ 0 & -1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0. \quad \bar{x}^{(1)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \bar{x}^{(2)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\bar{y} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-t}. \quad \begin{cases} y_1 = c_1 e^t \\ y_2 = c_2 e^{-t}. \end{cases} \quad y_1 y_2 = \text{const.}$$



(4) Center

, Enclosed by infinitely many closed trajectories

$$\bar{y}' = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix} \bar{y}. \qquad \begin{cases} y_1' = y_2 \\ y_2' = -4y_1 \end{cases}$$

$$\det(A-\lambda I) = \begin{vmatrix} -\lambda & 1 \\ -4 & -\lambda \end{vmatrix} = \lambda^2 + 4 = 0. \qquad \lambda_1 = -2i, \ \lambda_2 = 2i$$

$$\begin{bmatrix} -\lambda & 1 \\ -4 & -\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\lambda x_1 + x_2 \\ -4x_1 - \lambda_2 x_2 \end{bmatrix} = 0. \qquad \begin{matrix} \lambda_1: \ 2ix_1 + x_2 = 0. \\ \lambda_2: \ -2ix_1 + x_2 = 0. \end{matrix}$$

$x_1 = 1. \quad x_2 = -2i$

$x_1 = 1. \quad x_2 = 2i$

$$\bar{x}^{(1)} = \begin{bmatrix} 1 \\ -2i \end{bmatrix}, \quad \bar{x}^{(2)} = \begin{bmatrix} 1 \\ 2i \end{bmatrix}$$

$$\bar{y} = c_1 \bar{x}^{(1)} e^{\lambda_1 t} + c_2 \bar{x}^{(2)} e^{\lambda_2 t}$$

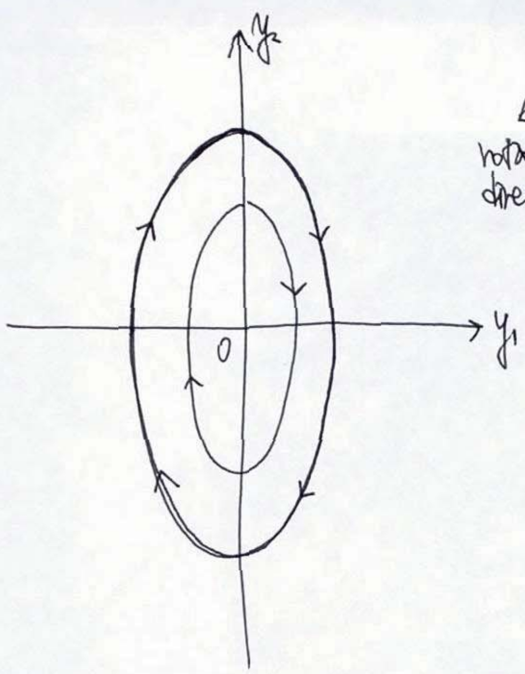
$$= c_1 \begin{bmatrix} 1 \\ -2i \end{bmatrix} e^{-2it} + c_2 \begin{bmatrix} 1 \\ 2i \end{bmatrix} e^{2it}$$

$$y_1 = c_1 e^{-2it} + c_2 e^{2it}$$
$$y_2 = c_1(-2i)e^{-2it} + c_2(2i)e^{2it}$$

Shortcut: $-4y_1 y_1' = y_2 y_2'$

integrating, $-2y_1^2 = \frac{1}{2} y_2^2 + c^*$

$2y_1^2 + \frac{1}{2} y_2^2 = \tilde{c}. \qquad y_1^2 + \frac{1}{4} y_2^2 = c^*$



$$\begin{cases} y' = y_2 \\ y_2' = -4y_1 \end{cases}$$

(5) Spiral point

: about which the trajectories spiral, approaching P_0 as $t \to \infty$ (or in the opposite sense)

Ex. 5.

$$\bar{y}' = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} \bar{y}$$

$$\det(A - \lambda I) = \lambda^2 + 2\lambda + 2 = 0$$

$$\lambda_1 = -1 + i, \quad \lambda_2 = -1 - i$$

$$(A - \lambda I)\bar{x} = 0 \Rightarrow (-1-\lambda)x_1 + x_2 = 0$$

$$\lambda_1 \Rightarrow \bar{x}^{(1)} = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$\lambda_2 \Rightarrow \bar{x}^{(2)} = \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

$$\bar{y} = c_1 \begin{bmatrix} 1 \\ i \end{bmatrix} e^{(-1+i)t} + c_2 \begin{bmatrix} 1 \\ -i \end{bmatrix} e^{(-1-i)t}$$

shortcut:

$$y_1' = -y_1 + y_2 \quad \times y_1 \Rightarrow y_1 y_1' = -y_1^2 + y_1 y_2$$

$$y_2' = -y_1 - y_2 \quad \times y_2 \Rightarrow y_2 y_2' = -y_1 y_2 - y_2^2$$

$$y_1 y_1' + y_2 y_2' = -(y_1^2 + y_2^2)$$

Let $r^2 = y_1^2 + y_2^2$

$$(r^2)' = 2y_1y_1' + 2y_2y_2'$$

$$\therefore \frac{1}{2}(r^2)' = -r^2$$

by the way, $\frac{1}{2}(r^2)' = rr'$

$$rr' = -r^2$$

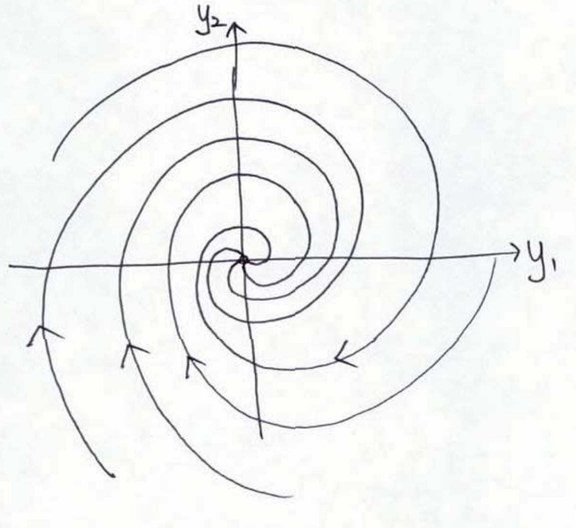
$$r' = -r$$

$$\frac{dr}{dt} = -r$$

$$\frac{dr}{r} = -dt$$

$$\ln r = -t + \tilde{c}$$

$$r = ce^{-t}$$



* No Basis of Eigenvectors Available

· Eigenvectors always independent → "basis"

if A : symmetric ($a_{jk} = a_{kj}$)

skew-symmetric ($a_{jk} = -a_{kj}$. $a_{jj} = 0$)

· Suppose $n \times n$ matrix A

$$\det(A - \lambda I) = (\lambda - \mu)^2 \cdots = 0.$$

$\lambda = \mu$: double eigenvalue

↳ one eigenvector

first solution $\bar{y}^{(1)} = \bar{x} e^{\mu t}$

second indep. solution

$$\boxed{\bar{y}^{(2)} = \bar{x} t e^{\mu t} + \bar{u} e^{\mu t}}$$

$$\bar{y}^{(2)'} = \bar{x} e^{\mu t} + \mu \bar{x} t e^{\mu t} + \mu \bar{u} e^{\mu t}$$

$$= A\bar{y}^{(2)} = A(\bar{x} t e^{\mu t} + \bar{u} e^{\mu t}) = A\bar{x} t e^{\mu t} + A\bar{u} e^{\mu t}$$

Recall $A\bar{x} = \mu \bar{x}$ (μ : eigenvalue)

$$\bar{x} e^{\mu t} + \mu \bar{u} e^{\mu t} = A\bar{u} e^{\mu t}$$

$$\bar{x} + \mu \bar{u} = A\bar{u}$$

$$\boxed{(A - \mu I)\bar{u} = \bar{x}}$$

Solve for $\bar{u}$ → $\bar{y}^{(2)}$ obtained

Ex. 6.
$$\bar{y}' = A\bar{y} = \begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix} \bar{y}$$

Try $\bar{y} = \bar{x} e^{\lambda t}$

$$\det(A - \lambda I) = \begin{vmatrix} 4-\lambda & 1 \\ -1 & 2-\lambda \end{vmatrix} = (4-\lambda)(2-\lambda) + 1$$

$$= \lambda^2 - 6\lambda + 9 = (\lambda - 3)^2 = 0$$

$\lambda = 3 = \mu$

$$\begin{bmatrix} 4-\lambda & 1 \\ -1 & 2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \quad : \quad x_1 + x_2 = 0$$

$$\bar{x}^{(1)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

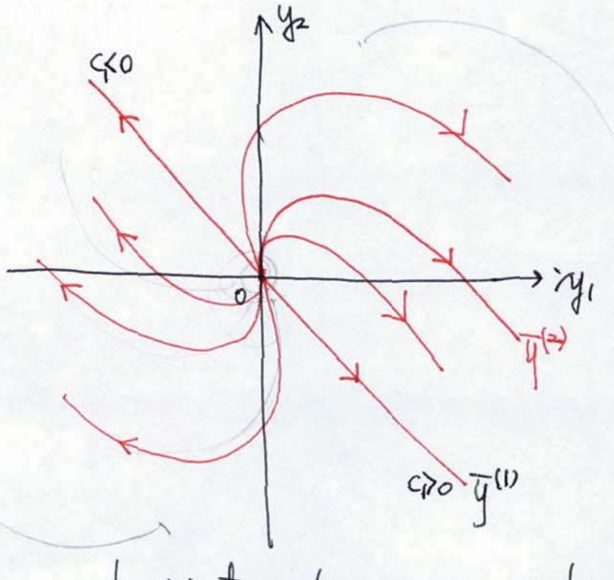
second indep. sol. $\bar{y}^{(2)} = \bar{x} t e^{\mu t} + \bar{u} e^{\mu t}$

$$(A - \mu I)\bar{u} = \bar{x}$$

$$(A - 3I)\bar{u} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\begin{aligned} u_1 + u_2 &= 1 \\ -u_1 - u_2 &= -1 \end{aligned} \qquad \therefore \bar{u} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\bar{y} = c_1 \bar{y}^{(1)} + c_2 \bar{y}^{(2)} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{3t} + c_2 \left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) e^{3t}$$



$c_1 = 0 \Rightarrow \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}^{(2)} = \begin{bmatrix} t \\ -t+1 \end{bmatrix} e^{3t}$

$y_2 = (-1 + \frac{1}{t}) y_1$

$c_1, c_2 \neq 0$: $t \to 0$

$y_1 = c_1$

$y_2 = -c_1 + c_2$

$\to -y_1 + c_2$

slope $\to -1$ as $t \to 0$

degenerate (improper) node

What if $\det(A-\lambda I) = (\lambda-\mu)^3 \cdots = 0$: triple eigenvalue and only one eigenvector $\bar{x}$

$$\bar{y}^{(1)} = \bar{x}e^{\mu t}$$

$$\bar{y}^{(2)} = \bar{x}te^{\mu t} + \bar{u}e^{\mu t} \quad : (A-\mu I)\bar{u} = \bar{x}$$

$$\bar{y}^{(3)} = \frac{1}{2}\bar{x}t^2e^{\mu t} + \bar{u}te^{\mu t} + \bar{v}e^{\mu t} \quad : (A-\mu I)\bar{v} = \bar{u}$$

what if $\det(A-\lambda I) = (\lambda-\mu)^3 \cdots = 0$

↳ two linearly indep. eigenvectors $\bar{x}^{(1)}, \bar{x}^{(2)}$

$$\bar{y}^{(1)} = \bar{x}^{(1)}e^{\mu t}$$

$$\bar{y}^{(2)} = \bar{x}^{(2)}e^{\mu t}$$

$$\bar{y}^{(3)} = \bar{x}te^{\mu t} + \bar{u}e^{\mu t}. \quad \bar{x} = c_1\bar{x}^{(1)} + c_2\bar{x}^{(2)}$$

$$(A-\mu I)\bar{u} = \bar{x}$$

4.4. Criteria for Critical Points. Stability

* Criteria for types of critical points

↳ determined by eigenvalues

$$\det(A-\lambda I) = \begin{vmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{vmatrix}$$

$$= \lambda^2 - (a_{11}+a_{22})\lambda + \det A = 0$$

$p = a_{11}+a_{22}$, $\quad q = \det A = a_{11}a_{22} - a_{12}a_{21}$

$$\Rightarrow \lambda^2 - p\lambda + q = 0. \quad \begin{bmatrix} \lambda_1+\lambda_2 = p \\ \lambda_1\lambda_2 = q \end{bmatrix} \quad \Delta = p^2 - 4q.$$