What if $\det(A-\lambda I) = (\lambda-\mu)^3 \cdots = 0$ : triple eigenvalue

and only one eigenvector $\bar{x}$

$$\bar{y}^{(1)} = \bar{x}e^{\mu t}$$

$$\bar{y}^{(2)} = \bar{x}te^{\mu t} + \bar{u}e^{\mu t} \quad : (A-\mu I)\bar{u} = \bar{x}$$

$$\bar{y}^{(3)} = \frac{1}{2}\bar{x}t^2e^{\mu t} + \bar{u}te^{\mu t} + \bar{v}e^{\mu t} \quad : (A-\mu I)\bar{v} = \bar{u}$$

what if $\det(A-\lambda I) = (\lambda-\mu)^3 \cdots = 0$

↳ two linearly indep. eigenvectors $\bar{x}^{(1)}, \bar{x}^{(2)}$

$$\bar{y}^{(1)} = \bar{x}^{(1)}e^{\mu t}$$

$$\bar{y}^{(2)} = \bar{x}^{(2)}e^{\mu t}$$

$$\bar{y}^{(3)} = \bar{x}te^{\mu t} + \bar{u}e^{\mu t}. \qquad \bar{x} = c_1\bar{x}^{(1)} + c_2\bar{x}^{(2)}$$

$$(A-\mu I)\bar{u} = \bar{x}$$

## 4.4. Criteria for Critical Points. Stability

* Criteria for types of critical points

↳ determined by eigenvalues

$$\det(A-\lambda I) = \begin{vmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{vmatrix}$$

$$= \lambda^2 - (a_{11}+a_{22})\lambda + \det A = 0$$

$p = a_{11}+a_{22}$, $\qquad q = \det A = a_{11}a_{22} - a_{12}a_{21}$

$$\Rightarrow \lambda^2 - p\lambda + q = 0. \quad \left[\begin{array}{l} \lambda_1+\lambda_2 = p \\ \lambda_1\lambda_2 = q \end{array}\right] \quad \Delta = p^2 - 4q.$$

Criteria for Critical Points

(a) node if $q > 0$ and $\Delta \geq 0$

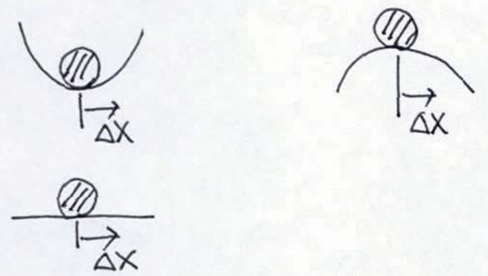
(b) saddle pt if $q < 0$

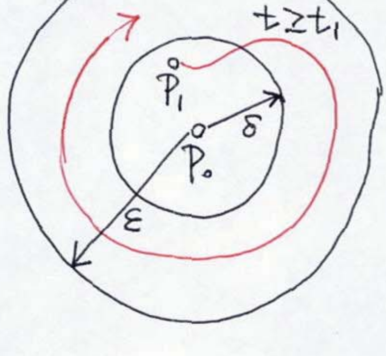
(c) center if $p = 0$ and $q > 0$ $\lambda^2 + q = 0$

(d) spiral pt if $p \neq 0$ and $\Delta < 0$

* Stability.

stable vs unstable

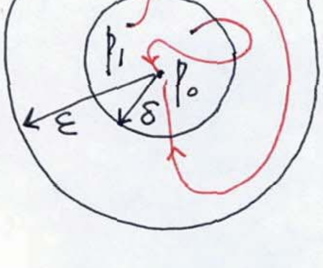



$P_0$: stable critical pt.

$P_0$: unstable


$P_0$: stable & attractive $\equiv$ asymptotically stable




every trajectory $\to P_0$.
as $t \to \infty$

Stability Criteria for Critical points

(a) stable and attractive if $p < 0$ and $q > 0$

(b) stable if $p \le 0$ and $q > 0$

(c) unstable if $p > 0$ or $q < 0$

- Supplementary note.

Ex. 1. Application of the criteria

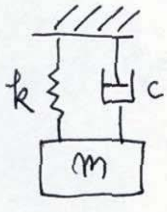
$$y' = Ay = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} \bar{y}$$

$$\det(A - \lambda I) = \begin{vmatrix} -3-\lambda & 1 \\ 1 & -3-\lambda \end{vmatrix} = (\lambda+3)^2 - 1 = \lambda^2 + 6\lambda + 8$$

$\lambda_1 = -2, \quad \lambda_2 = -4$ . $\quad \lambda_1 < 0, \ \lambda_2 < 0.$

stable, attractive

Ex. 2.




type of critical pt ?

$$my'' + cy' + ky = 0.$$

$$y'' + \frac{c}{m} y' + \frac{k}{m} y = 0 \qquad y \to y_1$$

$$\cdot\ y_1' = y_2$$

$$y_2' + \frac{c}{m} y_2 + \frac{k}{m} y_1 = 0$$

$$\cdot\ y_2' = -ky_1 - \frac{c}{m} y_2$$

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ -\frac{k}{m} & -\frac{c}{m} - \lambda \end{vmatrix} = \lambda^2 + \frac{c}{m}\lambda + \frac{k}{m} = 0.$$

## SYSTEMS OF DIFFERENTIAL EQUATIONS: SOLUTIONS AND CLASSIFICATION OF THE ORIGIN

Consider the system $\mathbf{x}' = A\mathbf{x}$, where $\mathbf{x} = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$, $A$ is the matrix of constants $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, and $\det(A) \neq 0$.

Find the eigenvalues and eigenvectors of A. Then the critical point at the origin can be classified as follows:

| | | | |
|---|---|---|---|
| 1. | $\lambda_1 < \lambda_2 < 0$ | Improper Node | Asymptotically Stable |
| 2. | $\lambda_1 > \lambda_2 > 0$ | Improper Node | Unstable |
| 3. | $\lambda_1 < 0 < \lambda_2$ | Saddle Point | Unstable |
| 4. | $\lambda = \alpha \pm i\beta,\ \alpha < 0$ | Spiral Point | Asymptotically Stable |
| 5. | $\lambda = \alpha \pm i\beta,\ \alpha > 0$ | Spiral Point | Unstable |
| 6. | $\lambda = \alpha \pm i\beta,\ \alpha = 0$ | Center | Stable |
| 7. | $\lambda_1 = \lambda_2 < 0$, e-vtrs indep. | Proper Node | Asymptotically Stable |
| 8. | $\lambda_1 = \lambda_2 > 0$, e-vtrs indep. | Proper Node | Unstable |
| 9. | $\lambda_1 = \lambda_2 < 0$, e-vtrs dep. | Improper Node | Asymptotically Stable |
| 10. | $\lambda_1 = \lambda_2 > 0$, e-vtrs dep. | Improper Node | Unstable |

In cases 1, 2, 3, 7, 8, the general solution is

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{E}^{(1)} + c_2 e^{\lambda_2 t}\mathbf{E}^{(2)}.$$

In cases 4, 5, 6, the eigenvalue-eigenvector pairs are $(\alpha \pm i\beta, \mathbf{u} \pm i\mathbf{v})$. The real-valued general solution is

$$\mathbf{x}(t) = e^{\alpha t}[c_1(\mathbf{u}\cos\beta t - \mathbf{v}\sin\beta t) + c_2(\mathbf{u}\sin\beta t + \mathbf{v}\cos\beta t)].$$

In cases 9 and 10, the only eigenvalue-eigenvector pair is $(\lambda, [E_1\ E_2]^T)$. The general solution is

$$\mathbf{x}(t) = e^{\lambda t}\left(c_1\begin{bmatrix} E_1 \\ E_2 \end{bmatrix} + c_2\left(t\begin{bmatrix} E_1 \\ E_2 \end{bmatrix} + \begin{bmatrix} 0 \\ E_1/a_{12} \end{bmatrix}\right)\right).$$

discriminant $= c^2 - 4mk$

(i) $c^2 > 4mk$ $\quad \lambda_1, \lambda_2$ : real (overdamping)

$\lambda_1 + \lambda_2 = -\frac{c}{m}$

$\lambda_1 \lambda_2 = \frac{k}{m}$.

$\lambda_1 < \lambda_2 < 0$ : stable and attractive node

(ii) $c^2 < 4mk$ $\quad \lambda_1 = \alpha + i\beta$
(underdamping) $\quad \lambda_2 = \alpha - i\beta$.

$\lambda_1 + \lambda_2 = 2\alpha = -\frac{c}{m}$. $\quad \alpha < 0$ : stable and attractive spiral pt.

(iii) $c^2 = 4mk$ : critical damping

$\lambda_1 = \lambda_2 < 0$.

stable and attractive ~~damping~~ node

(iv) $c = 0$ : no damping

$\lambda = \pm i\beta$

center

4.5. Qualitative Methods for Nonlinear Systems

when ; analytical methods too difficult or impossible

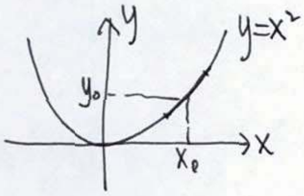
First-order nonlinear systems

$$\bar{y}' = \bar{f}(\bar{y}) \equiv \begin{cases} y_1' = f_1(y_1, y_2) \\ y_2' = f_2(y_1, y_2) \end{cases}$$

- autonomous system : independent variable $t$ does not occur explicitly

# Linearization of Nonlinear Systems

* Simple concept of linearization




$$(y-y_0)=\frac{dy}{dx}(x-x_0)+H.O.T$$

Taylor series expansion

$$y=y_0+\frac{dy}{dx}(x-x_0)+\frac{1}{2!}\frac{d^2y}{dx^2}(x-x_0)^2$$
$$+\frac{1}{3!}\frac{d^3y}{dx^3}(x-x_0)^3+\cdots$$

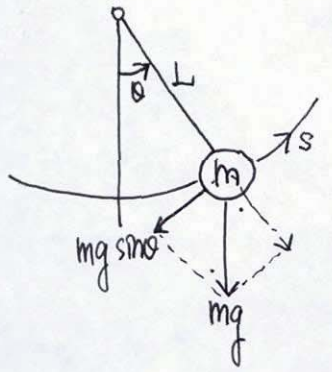
$$y=y_0+\frac{dy}{dx}(x-x_0)+H.O.T.$$ (0,0) : Maclaurin series

$$\bar{y}'=F(\bar{y}) \xrightarrow{\text{Linearize}} \bar{y}'=A\bar{y}+\cancel{h(\bar{y})}^{\,H.O.T.}$$

$$=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\begin{cases} y_1'=a_{11}y_1+a_{12}y_2 \\ y_2'=a_{21}y_1+a_{22}y_2 \end{cases}$$

Ex. 1. Free undamped pendulum




$$ms''=-mg\sin\theta$$

$$mL\theta''+mg\sin\theta=0$$

$$\theta''+k\sin\theta=0. \quad (k=\frac{g}{L})$$

$$\theta''=-k\sin\theta$$

conventional $\Rightarrow$ for small $\theta$, $\sin\theta\approx\theta$

$$\theta''+k\theta=0$$

$$\theta=A\cos\sqrt{k}t+B\sin\sqrt{k}t$$

· Discussion of critical points by linearization

$$\begin{cases} y_1 = \theta \\ y_2 = \theta' \end{cases}$$

$$y_1' = y_2$$
$$y_2' = -k\sin y_1$$

critical pt: $\frac{y_2'}{y_1'} = \frac{0}{0}$ : $y_2 = 0$, $\sin y_1 = 0$.

$(y_1, y_2) = (n\pi, 0)$. $n = 0, \pm 1, \pm 2, \dots$

in the vicinity of

@ $(0,0)$ : $\sin y_1 = y_1 - \frac{1}{3!}y_1^3 + - \cdots \approx y_1$

$$y_1' = y_2$$
$$y_2' = -ky_1$$

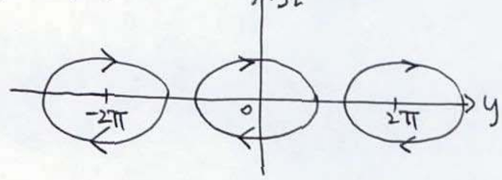
$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' = \bar{y}' = \begin{bmatrix} 0 & 1 \\ -k & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = A\bar{y}.$$

$$A = \begin{bmatrix} 0 & 1 \\ -k & 0 \end{bmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ -k & -\lambda \end{vmatrix} = \lambda^2 + k = 0. \quad \lambda = \pm i\sqrt{k}$$

: center, stable

→ periodic with $T = 2\pi$ $\left[\begin{array}{l} (n\pi, 0) : \text{center, stable} \\ n = 0, \pm 2, \pm 4, \dots \end{array}\right.$




$(\pi, 0)$ : let $y_1 = \theta - \pi$

$y_2 = y_1' = (\theta - \pi)' = \theta'$

$\sin\theta = \sin(y_1 + \pi) = -\sin y_1 = -(y_1 - \frac{1}{6}\cancel{y_1^3} + \cdots)$

$\cong -y_1$

$$\begin{cases} y_1' = y_2 \\ y_2' = -k(-y_1) = +ky_1 \end{cases}$$

$ky_1y_1' = y_2y_2'$

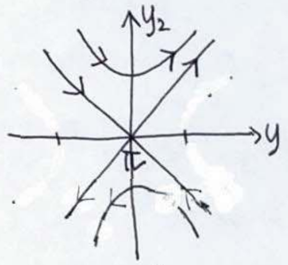
$ky_1^2 = y_2^2 + c$

$$\bar{y}' = \begin{bmatrix} 0 & 1 \\ k & 0 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ k & -\lambda \end{vmatrix} = \lambda^2 - k = 0. \qquad \lambda = \pm\sqrt{k}$$

critical point $(n\pi, 0)$. $n = \pm 1, \pm 3, \cdots$

: saddle pt unstable




Ex. 2. Linearization of the damped pendulum equation

$$\theta'' + c\theta' + k\sin\theta = 0$$

set $y_1 = \theta$

$$\begin{cases} y_1' = y_2 \\ y_2' = -cy_2 - k\sin y_1 \end{cases}$$

critical pt. $\begin{cases} y_2 = 0 \\ \sin y_1 = 0. \end{cases} \quad y_1 = 0, \pm\pi, \pm 2\pi, \cdots$

$: (n\pi, 0), \quad n = 0, \pm 1, \pm 2, \cdots$

$(0,0)$: linearized:

$$y_2' = -cy_2 - ky_1$$

$$\bar{y}' = \begin{bmatrix} 0 & 1 \\ -k & -c \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ -k & -c-\lambda \end{vmatrix} = \lambda(c+\lambda) + k$$
$$= \lambda^2 + c\lambda + k$$

Same as Ex. 2 of §4.4.

$(\pi, 0)$:

$$y_1 = \theta - \pi$$

$$y_2 = \theta' = (y_1 + \pi)' = y_1'$$

$$\sin\theta = \sin(\pi + y_1) = -\sin y_1 \approx -y_1$$

$$\begin{cases} y_1' = y_2 \\ y_2' = -cy_2 + ky_1 \end{cases}$$

$$\bar{y}' = \begin{bmatrix} 0 & 1 \\ k & -c \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ k & -c-\lambda \end{vmatrix} = \lambda(\lambda + c) - k = \lambda^2 + c\lambda - k$$

$\Delta = c^2 + 4k > 0$

$\lambda_1 + \lambda_2 = -c$.

$\lambda_1 \lambda_2 = -k$.

$\lambda_1 < 0 < \lambda_2$

saddle pt. unstable

*Fig. 93 (p. 154) phase portrait*

Ex.3. Lotka–Volterra Population Model (predator–prey population model)

$$\begin{cases} y_1: \# \text{ of rabbits} \\ y_2: \# \text{ of foxes} \end{cases}$$

$$\begin{cases} y_1' = ay_1 - by_1y_2 \\ y_2' = ky_1y_2 - ly_2 \end{cases}$$

critical pt: $y_1' = 0$. $y_2' = 0$

$y_1 = y_2 = 0$. $(0, 0)$ | $y_1 = \frac{l}{k}$, $y_2 = \frac{a}{b}$

$\left(\frac{l}{k}, \frac{a}{b}\right)$

(1) Linearized system at (0,0)

$$y_1' = ay_1$$
$$y_2' = -ly_2$$

$$\bar{y}' = \begin{bmatrix} a & 0 \\ 0 & -l \end{bmatrix} \bar{y}$$

$$\det(A-\lambda I) = \begin{vmatrix} a-\lambda & 0 \\ 0 & -l-\lambda \end{vmatrix} = (a-\lambda)(-l-\lambda) = 0$$

$$\lambda = a,\ -l \qquad \text{: saddle pt.}$$

(2) Linearization at $\left(\frac{l}{k}, \frac{a}{b}\right)$

$$\begin{cases} y_1 = \frac{l}{k} + \tilde{y}_1 \\ y_2 = \frac{a}{b} + \tilde{y}_2 \end{cases}$$

$$\cdot\ y_1' = \tilde{y}_1' = \cancel{a\left(\frac{l}{k} + \tilde{y}_1\right) - b\left(\frac{l}{k} + \tilde{y}_1\right)\left(\frac{a}{b} + \tilde{y}_2\right)}$$
$$= y_1(a - by_2)$$
$$= \left(\frac{l}{k} + \tilde{y}_1\right)\left[a - b\left(\frac{a}{b} + \tilde{y}_2\right)\right]$$
$$= \left(\frac{l}{k} + \tilde{y}_1\right)(-b\tilde{y}_2) \cong -\frac{lb}{k}\tilde{y}_2$$

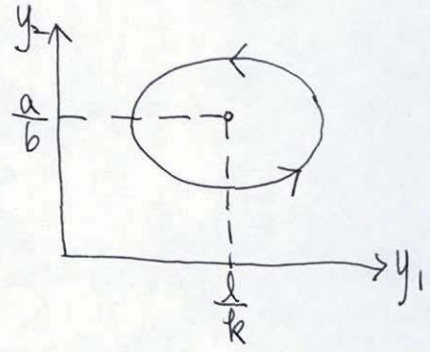
$$\cdot\ y_2' = \tilde{y}_2' = y_2(ky_1 - l)$$
$$= \left(\frac{a}{b} + \tilde{y}_2\right)\left[k\left(\frac{l}{k} + \tilde{y}_1\right) - l\right]$$
$$= \left(\frac{a}{b} + \tilde{y}_2\right)(k\tilde{y}_1)$$
$$\approx \frac{ak}{b}\tilde{y}_1$$

$$\begin{cases} \tilde{y}_1' = -\frac{lb}{k}\tilde{y}_2 \\ \tilde{y}_2' = \frac{ak}{b}\tilde{y}_1 \end{cases}$$

$$\frac{ak}{b}\tilde{y}_1\tilde{y}_1' = -\frac{lb}{k}\tilde{y}_2\tilde{y}_2'$$

$$\frac{ak}{b}\tilde{y}_1^2 = -\frac{lb}{k}\tilde{y}_2^2 + c$$

$$\frac{ak}{b}\tilde{y}_1^2 + \frac{lb}{k}\tilde{y}_2^2 = c$$




center

## 4.6. Nonhomogeneous Linear Systems

$$\bar{y}' = A\bar{y} + \bar{g}$$

Gen. sol.: $\bar{y} = \bar{y}^{(h)} + \bar{y}^{(p)}$

$$\bar{y}^{(h)\prime} = A\bar{y}^{(h)}$$

* Method of Undetermined Coefficients

Ex. 1. $\bar{y}' = A\bar{y} + \bar{g} = \begin{bmatrix} 2 & -4 \\ 1 & -3 \end{bmatrix}\bar{y} + \begin{bmatrix} 2t^2 + 10t \\ t^2 + 9t + 3 \end{bmatrix}$

(1) $\bar{y}^{(h)\prime} = A\bar{y}^{(h)}$

$\bar{y}^{(h)} = \bar{x}e^{\lambda t}$. $\quad \det(A-\lambda I) = \begin{vmatrix} 2-\lambda & -4 \\ 1 & -3-\lambda \end{vmatrix} = (\lambda+3)(\lambda-2)+4$

$$= \lambda^2+\lambda-2$$

$$= (\lambda+2)(\lambda-1) = 0$$

$\lambda_1 = -2$: $\begin{bmatrix} 2-\lambda & -4 \\ 1 & -3-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4x_1 - 4x_2 \\ x_1 - x_2 \end{bmatrix} = 0$. $\quad x_1 = x_2 \quad \bar{x}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda_2 = 1$ $\quad \begin{bmatrix} x_1 - 4x_2 \\ x_1 - 4x_2 \end{bmatrix} = 0$. $\quad x_1 = 4x_2$. $\quad \bar{x}^{(2)} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$